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ON THE NON-VANISHING OF POINCARÉ SERIES

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R. A. Rankin [2] and J. Lehner [1] considered the non-vanishing of Poincaré series for the classical modular matrix group and for an arbitrary fuchsian group, respectively.

In this paper we consider the non-vanishing of Poincaré series for the congruence group

$$\Gamma=\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\equiv\begin{pmatrix} * & * \\ 0 & *\end{pmatrix}(\operatorname{mod} N)\right\};\ N\geqq 1.$$

For $k>2, k\equiv 0 \pmod 2$, let $\mathcal{M}_k^0(\Gamma)$ be the space of cusp forms for Γ of weight k. Let μ_k be the dimension of $\mathcal{M}_k^0(\Gamma)$. Let

$$P_m(z,k)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(j(\gamma,z))^{-k}e(m\gamma z),$$

where

$$j(\gamma,z)=cz+d \quad \text{if} \quad \gamma=\begin{pmatrix} * & * \\ c & d\end{pmatrix},$$

and

$$e(z)=e^{2\pi i z}$$

be the Poincaré series of weight k attached to Γ. The space $\mathcal{M}_k^0(\Gamma)$ is spanned by $P_m(z,k)$. Since $\mathcal{M}_k^0(\Gamma)$ is finite dimensional, there must be many linear relations between $P_m(z,k)$. Very little is known about these relations. In particular one does not know which $P_m(z,k)$ do not vanish identically.

In the case of full modular group $\Gamma=\Gamma_0(1)$, when $k=4,6,8,10$ and 14, $\mathcal{M}_k^0(\Gamma)$ has dimension zero; so that $P_m(z,k)$ vanishes identically for all positive integers m. We have $\mu_k>0$ for $k=12$ and all $k\geqq 16$. Indeed by Theorem 6.1.2 in [3] we have for $k\geqq 4$,

$$\mu_k=\begin{cases}\left[\dfrac{k}{12}\right] & \text{if } k\not\equiv 2 \pmod{12},\\[2ex] \left[\dfrac{k}{12}\right]-1 & \text{if } k\equiv 2 \pmod{12}.\end{cases}$$

Clearly, since $P_m(z,k)$ $(1 \leqq m \leqq \mu_k)$ span the space $\mathcal{M}_k^0(\Gamma)$, we have $P_m(z,k) \not\equiv 0$ for $1 \leqq m \leqq \mu_k$. Rankin [2] was able to show that many more Poincaré series do not vanish.

In this paper we extend the arguments of Rankin to establish:

Theorem 1. *For $\Gamma = \Gamma_0(N)$; $N \geqq 1$, we have $P_m(z,k) \equiv 0$ if*

$$m(m,N)\alpha^2(m) \leqq \frac{1}{2^{15}\pi^3}\left(\frac{N}{\tau(N)\log 2N}\right)^2$$

where

$$\alpha(m) = \sum_{d \mid m} \frac{\tau(d)}{\sqrt{d}}$$

and $\tau(N)$ is the number of positive divisors of N.

Remarks. Stripped of factors of lower order, Theorem 1 states essentially that

$$m(m,N) \leqq K(N/\log N)^2,$$

where K is an explicitly defined numerical constant and $K<1$. Thus, for small values of N, where the right hand side is less than 1, this tells us nothing. Even for large N it is vacuous in some cases, e.g., when N divides m, as it then gives $m/N < K(\log N)^{-2}$. However, in other cases it will give information. For example, whenever

$$N/\log N > K^{-1/2}$$

it tells us that, for all $k>2$, the first Poincaré series does not vanish.

Note also that, unlike the results of Rankin and Lehner, the upper bound does not depend on the weight k. However, in Theorem 2 and Theorem 3, which follow, the upper bound does depend on the weight k.

Let $S(m,m;c)$ be the Kloosterman sum defined

$$S(m,m;c) = \sum_{d \,(\mathrm{mod}\, c)}^{*} e\left(m\frac{d+\bar{d}}{c}\right);\ d\bar{d} \equiv 1 (\mathrm{mod}\, c).$$

Let $J_{k-1}(y)$ be the Bessel function of order $k-1$.

Lemma 1. (A. Weil cf. [4]). *We have*

$$|S(m,m;c)| \leqq (m,c)^{1/2}c^{1/2}\tau(c).$$

Lemma 2. (cf. [5]). *We have*

$$|J_{k-1}(y)| \leqq \min\left\{1, \frac{1}{(k-1)!}\left(\frac{y}{2}\right)^{k-1}\right\} \leqq \min\left\{1, \frac{y}{2}\right\}.$$

Proof (of Theorem 1).

By the argument presented in Section 2 of [2], and in Chapter 5 of [3], we have

$$P_m(z,k) \equiv 0 \quad \text{if} \quad |S_m| < \frac{1}{2\pi}$$

where

$$S_m = \sum_{r=1}^{\infty} (rN)^{-1} S(m,m;rN) J_{k-1}\left(\frac{4\pi m}{rN}\right).$$

Clearly, by Lemma 1 and Lemma 2, we have

$$|S_m| \leqq \frac{(m,N)^{1/2}\tau(N)}{N^{1/2}} \sum_{r=1}^{\infty} \frac{(m,r)^{1/2}\tau(r)}{r^{1/2}} \min\left\{1, \frac{2\pi m}{rN}\right\}$$

$$\leqq \frac{(m,N)^{1/2}\tau(N)}{N^{1/2}} \sum_{d|m} \tau(d) \sum_{r=1}^{\infty} \frac{\tau(r)}{r^{1/2}} \min\left\{1, \frac{2\pi m}{r\, dN}\right\}$$

$$\leqq \frac{(m,N)^{1/2}\tau(N)}{N^{1/2}} \sum_{d|m} \tau(d) \sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \frac{1}{(r_1 r_2)^{1/2}} \min\left\{1, \frac{2\pi m}{r_1 r_2\, dN}\right\}.$$

Let

$$R = \left(\frac{2\pi m}{dN}\right).$$

Then

$$\sum_{r_1=1}^{\infty} \sum_{r_2=1}^{\infty} \frac{1}{(r_1 r_2)^{1/2}} \min\left\{1, \frac{R}{r_1 r_2}\right\} = 2\sum_{r=1}^{\infty} \frac{1}{r^{1/2}} \min\left\{1, \frac{R}{r}\right\} + \sum_{r_1=2}^{\infty} \sum_{r_2=2}^{\infty} \frac{1}{(r_1 r_2)^{1/2}} \min\left\{1, \frac{R}{r_1 r_2}\right\}$$

$$= 2S_1 + S_2.$$

Case I: $R > 1$.

$$S_1 \leqq 1 + \int_1^R t^{-1/2}\, dt + R \int_R^{\infty} t^{-3/2}\, dt; \text{ so that}$$

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$$S_1 \leqq 4R^{1/2} - 1 \leqq 4R^{1/2}(1+\log(R+1)).$$

$$S_2 \leqq \int_1^R \left(\int_1^{R/t_1} (t_1t_2)^{-1/2}\, dt_2 \right) dt_1 + R\int_1^R \left(\int_{R/t_1}^{\infty} (t_1t_2)^{-3/2}\, dt_2 \right) dt_1$$

$$+R\int_R^{\infty} \left(\int_1^{\infty} (t_1t_2)^{-3/2}\, dt_2 \right) dt_1; \text{ so that}$$

$$S_2 \leqq 4R^{1/2}(1+\log(R+1)).$$

Case II: $0<R\leqq 1$.

$$S_1 \leqq R + R\int_1^{\infty} t^{-3/2}\, dt; \text{ so that}$$

$$S_1 \leqq R + 2R \leqq 3R^{1/2}(1+\log(R+1)); \text{ since } R \leqq 1.$$

$$S_2 \leqq \int_1^{\infty} \left(\int_1^{\infty} \frac{R}{(t_1t_2)^{3/2}}\, dt_2 \right) dt_1; \text{ so that}$$

$$S_2 \leqq 4R \leqq 4R^{1/2}(1+\log(R+1)); \text{ since } R \leqq 1.$$

By combining both cases with the earlier calculations, the proof is completed. □

Theorem 2. *Let $\Gamma=\Gamma_0(N); N\geqq 1$. There exist positive constants k_0 and B (both independent of N), where $B>4\log 2$ such that, for all $k\geqq k_0$ and all positive integers m such that*

$$k \leqq m \leqq k^2 \exp(-B\log k/\log\log k),$$

$P_m(z,k)\not\equiv 0$.

Proof. Let $Q^* = (4\pi m/vN)$.

$$|S_m| \leqq \sum_{1\leqq q<Q^*} \frac{|S(m,m;qN)|}{qN} \left| J_{k-1}\left(\frac{4\pi m}{qN}\right)\right| + \sum_{q\geqq Q^*} \frac{|S(m,m;qN|}{qN} \left| J_{k-1}\left(\frac{4\pi m}{qN}\right)\right|.$$

$|S_m| \leqq S'_m + S''_m$.

Clearly,

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$$S'_m \leqq \sum_{1 \leqq q < Q^* N = Q} \frac{|S(m, m; q)|}{q} \left| J_{k-1}\left(\frac{4\pi m}{q}\right) \right|,$$

where Q is defined in [2]. Hence by exactly the same argument presented in [2], we have

$$S'_m \leqq A_6 M(m)\{\sigma^6 m^{1/2} \sigma_{-1/2}(m) + (4\pi)^{1/2} \sigma^2 \sigma_0(m)\}.$$

Clearly, by the argument presented in [2],

$$S''_m \leqq \sum_{q \geqq Q^*} \left| J_v\left(\frac{vQ^*}{q}\right) \right| \leqq A_5 \sum_{q \geqq Q^*} f\left(\frac{Q^*}{q}\right) \leqq A_5 \left\{ Q^* \int_0^1 x^{-2} F(x)\, dx + \sigma^2 \right\};$$

so that, since $Q^* N = Q$,

$$S''_m \leqq A_5 \sigma^2 + \frac{A_7 m \sigma^{15}}{N} \left(\frac{1}{2} e x_0\right)^v + \frac{A_8 m \sigma^{12}}{N} \leqq A_5 \sigma^2 + A_9 m \sigma^{12}.$$

Hence $|S_m| \leqq A_6 m^{1/2} \sigma^6 M(m) \sigma_{-1/2}(m) + A_{10} \sigma^2 M^2(m) + A_5 \sigma^2 + A_9 m \sigma^{12}$, and the result follows by the argument presented in [2] with the obseration that $A_5 \sigma^2 = o(1)$. □

Theorem 3. *For $\Gamma = \Gamma_0(N)$; $N \geqq 1$, we have $P_m(z, k) \not\equiv 0$ if $k_0(N) \leqq k$ and for any $\varepsilon > 0$*

$$m^{1+\varepsilon}(m, N)\alpha^2(m) \ll \left(\frac{Nk}{\tau(N)}\right)^2.$$

Proof. By Lemma 1 we have

$$S'_m \leqq \frac{(m, N)^{1/2} \tau(N)}{N^{1/2}} \sum_{1 \leqq q < Q^*} \frac{(m, q)^{1/2} \tau(q)}{q^{1/2}} \left| J_{k-1}\left(\frac{4\pi m}{qN}\right) \right|.$$

Clearly,

$$S'_m \leqq \frac{(m, N)^{1/2} \tau(N)}{N^{1/2}} \sum_{\substack{d \mid m \\ d \leqq Q^*}} \tau(d) \sum_{1 \leqq r < (Q^*/d)} \frac{\tau(r)}{r^{1/2}} \left| J_{k-1}\left(\frac{4\pi m}{r d N}\right) \right|$$

$$S'_m \ll \frac{(m, N)^{1/2} \tau(N) m^{\varepsilon}}{N^{1/2} Q^{*1/2}} \sum_{\substack{d \mid m \\ d < Q^*}} \tau(d)\, d^{1/2} S_d, \text{ where}$$

$$S_d = \sum_{1 \leqq r < (Q^*/d)} \left(\frac{Q^*}{rd}\right)^{1/2} \left| J_v\left(\frac{vQ^*}{rd}\right) \right|.$$

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By the same argument presented in [2], we have

$$S_d \ll \left(\frac{m\sigma^9}{Nd} + \sigma^2\right).$$

Hence

$$S'_m \ll \frac{(m,N)^{1/2}\tau(N)m^{1/2+\varepsilon}}{N(k-1)}\left(\sum_{\substack{d\mid m\\ d<Q^*}} \frac{\tau(d)}{d^{1/2}}\right) + (m,N)^{1/2}\tau(N)m^{\varepsilon}(k-1)^{1/6}\sum_{\substack{d\mid m\\ d<Q^*}} \tau(d)\left(\frac{d}{m}\right)^{1/2}.$$

But

$$\sum_{\substack{d\mid m\\ d<Q^*}} \frac{\tau(d)}{d^{1/2}} \leqq \alpha(m),$$

and since $d<Q^*$ we have

$$\sum_{\substack{d\mid m\\ d<Q^*}} \tau(d)\left(\frac{d}{m}\right)^{1/2} \ll \sum_{d\mid m} \tau(d)\left(\frac{1}{(vN)^{1/2}}\right) \ll \frac{m^{2\varepsilon}}{(vN)^{1/2}}.$$

Hence

$$S'_m \ll \frac{(m,N)^{1/2}\tau(N)m^{1/2+\varepsilon}\alpha(m)}{N(k-1)} + \frac{(m,N)^{1/2}\tau(N)m^{3\varepsilon}(k-1)^{1/6}}{N^{1/2}(k-1)^{1/2}}.$$

By the argument given in the proof of Theorem 2, we have $S''_m \leqq A_5\sigma^2 + A_9 m\sigma^{12}$. Hence

$$S_m \ll \frac{(m,N)^{1/2}\tau(N)m^{1/2+\varepsilon}\alpha(m)}{N(k-1)} + \frac{(m,N)^{1/2}\tau(N)m^{3\varepsilon}}{N^{1/2}(k-1)^{1/3}} + \frac{1}{(k-1)^{1/3}} + \frac{m}{(k-1)^2},$$

and the result follows from the hypothesis; since the last three terms are sufficiently small for $k>k_0(N)$. □

Note added in proof. All of the results in this paper are also true for the principal congruence groups $\Gamma(N); N \geqq 1$.

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