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THE ALMOST LINDELÖF DEGREE

BY

S. WILLARD AND U. N. B. DISSANAYAKE

ABSTRACT. In [A], Arhangel'skii showed that for any T_2 space X, $|X| \le 2^{L(X)\chi(X)}$, where $L(X)$ is the Lindelöf degree of X and $\chi(X)$ is the character of X.

In [B], Bell, Ginsburg and Woods improved this result, assuming normality, by showing that for T_4 spaces X, $|X| \le 2^{wL(X)\chi(X)}$, where $wL(X)$ is the weak Lindelöf degree of X.

We introduce below a new cardinal function $aL(X)$, the *almost Lindelöf degree* of X, which agrees with $L(X)$ on T_3 spaces, but which is often smaller than $L(X)$ on T_2 spaces, and show that for T_2 spaces X,

$$|X| \le 2^{aL(X)\chi(X)}.$$

1. **Introduction.** Our undefined notation follows that in [J]. Briefly, then, for a topological space X,

$$L(X) = \text{the } \textit{Lindelöf degree} \text{ of } X,$$

$$\chi(X) = \text{the } \textit{character} \text{ of } X,$$

$$\partial(X) = \text{the } \textit{tightness} \text{ of } X,$$

$$\pi\chi(X) = \text{the } \pi\textit{-character} \text{ of } X,$$

$$\psi_c(X) = \text{the } \textit{closed character} \text{ of } X.$$

In addition, a subset E of a topological space X will be *almost* κ*-Lindelöf* iff every X-open cover $\mathcal{U}$ of E has a subsystem $\mathcal{U}'$ with $|\mathcal{U}'| \le \kappa$ and $E \subset \bigcup \{Cl_X U \mid U \in \mathcal{U}'\}$. We then define

$$aL(E, X) = \min\{\kappa \mid E \text{ is almost } \kappa\text{-Lindelöf}\},$$

$$aL(X) = \omega + \sup\{aL(E, X) \mid E \text{ closed in } X\},$$

and refer to $aL(X)$ as the *almost Lindelöf degree* of X.

Similarly, the *weak Lindelöf degree* $wL(X)$ of X is by definition the least $\kappa \ge \omega$ such that every open cover $\mathcal{U}$ of x has a subsystem $\mathcal{U}'$ with $|\mathcal{U}'| \le \kappa$ and $X = \overline{\bigcup \mathcal{U}'}$.

Note that $wL(X) \le aL(X, X) + \omega$.

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1.1 THEOREM. *For any topological space X,*
(i) $wL(X) \le aL(X) \le L(X)$.
(ii) $aL(X) = L(X)$ *if X is* T_3.

Proof. Follows from the definitions.

1.2 REMARK. Let κK be the Katěov H-closed extension of the infinite discrete space X. A typical basic open set B in κX takes the form $B = G \cup \{\mathcal{P}\}$ where $\mathcal{P}$ is a free ultrafilter on X and $G \in \mathcal{P}$ or B is a subset of X. Since $\kappa X - X$ is the set of all free ultrafilters on X, it is clear that $\kappa X - X$ is a closed discrete subset of κX and $|\kappa X - X| = 2^{2^{|X|}}$. Thus $L(\kappa X) = 2^{2^{|X|}}$. We denote the Stone-Cêch compactifiaction of X by βX. Since the weight of βX is $2^{|X|}$, and by considering basic open neighbourhoods at $\mathcal{P} \in \beta X - X$ in κX, it follows that $aL(\kappa X) = 2^{|X|}$ and $aL(\kappa X, \kappa X) < \aleph_0$.

This standard example (19N of [W]) shows that in general $wL(X)$, $aL(X)$ and $L(X)$ are distinct for T_2 spaces. Next we shall give a more general example since not every infinite cardinal is of the form 2^κ.

1.3 EXAMPLE. For any cardinal $\kappa \ge \omega$, there exists a T_2-space X with $aL(X) \le \kappa$ and $L(X) = 2^\kappa$.

Proof. Let $T = I^\kappa \times I$, and let E be the subspace $I^\kappa \times \{0\}$ of T. Note that T is hereditarily κ-Lindelöf (since $w(T) = \kappa$).

Let X be the set T with the following topology: neighbourhoods of points $p \in T - E$ will be unchanged in X, while neighbourhoods of points $p \in E$ will take the form

$$U_p^* = (U - E) \cup \{p\}$$

where U is a neighbourhood of p in T.

Certainly $L(X) = 2^\kappa$ since E is a closed relatively discrete subspace of X of cardinality 2^κ.

Moreover, $aL(X) = \kappa$. This follows from the fact that if U is open in T and $p \in U \cap E$, then

$$Cl_X U_p^* \supset U.$$

To see this, let $x \in U$. If $x \notin E$, then $x \in U_p^*$, so assume $x \in U \cap E$. Let V_x^* be a neighbourhood of x in X. Then

$$V_x^* = (V - E) \cup \{x\}$$

where V is a neighbourhood of x in T. Since E is nowhere dense, it follows that $V_x^* \cap U_p^* \ne \Phi$. Thus $x \in Cl_X U_p^*$.

2. **Main theorem.** Here, our terminology follows that of [J]. We require the following lemma.

2.1 LEMMA. (a) *Let X be a* T_2 *space. Then* $|X| \le d(X)^{\chi(X)}$.

(b) *Let X be a space with $\partial(X) \le \beta$. Let G: $P(X)_{\le\beta} \to P_{\le\kappa}(X)$ be a set mapping with the property that $G(A) \supset \bar{A}$ for every $A \in P_{\le\beta}(X)$. Suppose there exists $B \subset X$ such that $B \supset G(A)$ for every $A \in P_{\le\beta}(B)$. Then $B = \bar{B}$.*

Proof. (a) See [J], 2.5.

(b) Let $x \in \bar{B}$. Then, since $\partial(X) \le \beta$, there is an $A \in P_{\le\beta}(B)$ with $x \in \bar{A}$. Thus $x \in G(A) \subset B$.

2.2 Theorem. *Let X be a T_2 space. Then $|X| \le 2^{\chi(X)aL(X)}$.*

Proof. Set $\beta = \chi(X)aL(X)$ and $\kappa = 2^\beta$. For each $x \in X$ let $\mathcal{W}_x$ be a collection of neighbourhoods of x such that $|\mathcal{W}_x| \le \beta$ and

$$\{x\} = \bigcap \{\bar{W} : W \in \mathcal{W}_x\}.$$

For $A \subset X$, we write

$$\mathcal{W}_A = \bigcup \{\mathcal{W}_X : x \in A\}.$$

Suppose $A \subset X$ with $|A| \le \beta$. Let V_A consist of all $\mathcal{U} \subset \mathcal{W}_A$ such that $|\mathcal{U}| \le \beta$ and $X - \bigcup \{\bar{U} : U \in \mathcal{U}\} \ne \Phi$. Since $|\mathcal{W}_A| \le \beta$, we have $|V_A| \le \beta^\beta = 2^\beta = \kappa$. For each $\mathcal{V} \in V_A$, choose $p(\mathcal{V}) \in X - \bigcup \{\bar{U} : U \in \mathcal{V}\}$ and set

$$G(A) = \overline{A \cup \{p(\mathcal{V}) : \mathcal{V} \in V_A\}}.$$

Then, by 2.1 (a), $|G(A)| \le \kappa^\beta = 2^\beta = \kappa$.

Hence $G : P_{\le\beta}(X) \to p_{\le\kappa}(X)$.

Now by 2.24 (a) in [J], there is some $B \subset X$ with $|B| = \kappa$ such that $B \supset G(A)$ for every $A \in P_{\le\beta}(B)$.

We claim $X = B$.

First, by 2.1 (b), B is a closed subset of X. Suppose $q \in X - B$. For each $y \in B$, choose $V_y \in \mathcal{W}_y$ such that $q \notin \bar{V}_y$. Since $aL(B, X) \le \kappa$, there is some $D \in P_{\le\beta}(B)$ such that $B \subset \bigcup \{\bar{V}_y : y \in D\} \subset X - \{q\}$. Thus $\mathcal{V} = \{V_y : y \in D\}$ belongs to V_D and by construction, $p(\mathcal{V}) \in G(D) \subset B$. But $p(\mathcal{V}) \in X - \bigcup \{\bar{U} : U \in \mathcal{V}\} \subset X - B$, a contradiction. Thus, it follows that $|X| \le \kappa$.

The following theorem generalizes 2.1 (a) and the proof is similar to 2.5 of [J].

2.3 Theorem. *Let X be a T_2 space. Then $|X| \le d(X)^{\pi\chi(X)\psi_c(X)}$.*

Following the main lines of the proof of the Theorem 2.2 and applying 2.3 instead of 2.1 (a), one can easily obtain the following generalization of 2.2.

2.4 Theorem. *Let X be a T_2 space. Then $|X| \le 2^{aL(X)\psi_c(X)\pi\chi(X)\partial(X)}$.*

2.5 Example. The space constructed in 1.3 illustrates that the bound provided by Theorem 2.2 is sharper for (non-regular) T_2 spaces than Arhangel'sklii's famous

$$|X| \le 2^{L(X)\chi(X)}$$

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That example can be modified to show that our 2.2 differs also from the following result of Hajnal and Juhasz:

$$|X| \leq 2^{c(X)\chi(X)}$$

(for T_2 spaces; cf [J], 2.15(B)). To do this, let X be the space constructed in 1.3 and let Y be the Alexandroff double of the space T in 1.3. Then the disjoint union Z of X and Y has $aL(Z) = \chi(Z) = \kappa$ while $c(Z) = L(Z) = 2^\kappa$.

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References

[A] A. V. Arhangel'skii, *On the cardinality of bicompacta satisfying the first axiom of countability*, Soviet Math. Dok. **10** (1969), 951–955.

[B] M. J. Bell, J. N. Ginsburg and G. Woods, *Cardinal inequalities for topological spaces involving the weak Lindelöf number*, Pacific J. Math. **79** (1978), 37–45.

[J] I. Juhasz, *Cardinal functions in Topology – ten years later*, Math. Centre Tract #123, Math. centrum, Amsterdam, 1980.

[W] S. Willard, *General Topology*, Addison-Wesley, Reading, Mass., 1968.

The University of Alberta
Edmonton, Alberta