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THE MOMENTS OF THE ZETA-FUNCTION ON THE LINE $\sigma = 1$

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1. Introduction

The evaluation of the integral

$$\int_1^T |\zeta(\sigma + it)|^{2k}\, dt \quad (\sigma \in \mathbf{R},\ k \in \mathbf{R}^+ \text{ fixed}) \tag{1.1}$$

represents one of the fundamental problems of the theory of the Riemann zeta-function (see [4] for a comprehensive account). In view of the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s),\ \chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \asymp |t|^{\frac{1}{2}-\sigma} \quad (s = \sigma + it)$$

it is clear that one has to distinguish between the following three principal cases:

a) $\sigma = 1/2$ ("the critical line"),
b) $1/2 < \sigma < 1$ ("the critical strip"),
c) $\sigma = 1$.

Although the cases a) and b), which have countless applications to various branches of number theory, have been extensively studied in the literature, it seems that the case c) has been somewhat neglected. Thus it is only recently that R. Balasubramanian et al. [1] have obtained precise results for the case c) if $k = 1$, which is the most important case. If, for $T > 3$, one defines the function $R(T)$ by the formula

$$\int_1^T |\zeta(1 + it)|^2\, dt = \zeta(2)T - \pi\log T + R(T), \tag{1.2}$$

then it was proved in [1] that

$$R(T) = O(\log^{2/3} T(\log\log T)^{1/3}), \tag{1.3}$$

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(1.4) $$\int_1^T R(t)\,dt = O(T),$$

and

(1.5) $$\int_1^T R^2(t)\,dt = O(T(\log\log T)^4).$$

The starting point for the evaluation of the integral in (1.2) was the approximate functional equation (see Theorem 1.8 of [3])

(1.6) $$\zeta(s) = \sum_{n\le x} n^{-s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}),$$

which holds for $0 < \sigma_0 \le \sigma \le 2$, $x \ge |t|/\pi$, $s = \sigma + it$. In case $\sigma = 1$ and $k = 1$, (1.6) is sufficient to yield (1.3)-(1.5). For $\sigma = 1$ and $k \ne 1$ one cannot hope for an approximate functional equation which is as simple and precise as (1.6) is.

2. Formulation of the Theorem

The aim of this note is to study the integral in (1.1) when $\sigma = 1$ and $k > 0$ is arbitrary, but fixed. To this end we define the function $R_k(T)$ by the formula

(2.1) $$\int_1^T |\zeta(1+it)|^{2k}\,dt = T\sum_{n=1}^{\infty} d_k^2(n)n^{-2} + R_k(T) \quad (T > 3),$$

so that a comparison with (1.2) shows that

(2.2) $$R_1(T) = -\pi\log T + R(T).$$

The function $d_k(n)$, which appears in (2.1), is commonly called the generalized divisor function. When k (≥ 1) is an integer, $d_k(n)$ denotes the number of ways in which n may be written as a product of k fixed factors, so that $d_1(n) = 1$ and $d_2(n) = d(n) = \sum_{\delta|n} 1$ is the number of divisors of n. In the general case (k may be supposed to be even an arbitrary complex number) one defines $d_k(n)$ by

(2.3) $$\zeta^k(s) = \prod_p (1-p^{-s})^{-k} = \sum_{n=1}^{\infty} d_k(n)n^{-s} \quad (\mathrm{Re}\, s > 1).$$

Here a branch of $\zeta^k(s)$ is defined by the relation

$$\zeta^k(s) = \exp(k\log\zeta(s)) = \exp(-k\prod_p \sum_{j=1}^{\infty} j^{-1}p^{-js}) \quad (\mathrm{Re}\, s > 1).$$

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From this definition it follows that $d_k(n)$ is a multiplicative function of n for a given k, and that if p^α is an arbitrary prime power, one has

$$d_k(p^\alpha) = (-1)^\alpha \binom{-k}{\alpha} = \frac{k(k+1)\cdots(k+\alpha-1)}{\alpha!} = \frac{\Gamma(k+\alpha)}{\Gamma(k)\alpha!}.$$

Our result is then the following

THEOREM. *Let $k > 0$ be given and let $R_k(T)$ be defined by* (2.1). Then

$$(2.4)\qquad R_k(T) \ll_k (\log T)^{\frac{5k^2}{3}} (\log\log T)^{\frac{k^2}{3}} + (\log T)^{\frac{10k-2}{3}} (\log\log T)^{\frac{2k-1}{3}} + (\log T)^{\frac{5k+3}{3}} (\log\log T)^{\frac{k}{3}},$$

while if $k \geq 2$ is an integer, then

$$(2.5)\qquad R_k(T) \ll_k \log^{k^2} T.$$

In conjunction with (1.2) and (1.3) the Theorem provides then fairly sharp estimates for $R_k(T)$ for each given $k > 0$. Note that the first term in (2.4) is the largest one for $k > 1 + \sqrt{0.6} = 1.77459\ldots$. One could also, in analogy with (1.4) and (1.5), consider the integrals of $R_k(t)$ and $R_k^2(t)$. However, the presence of the divisor function $d_k(n)$ in the relevant expressions would cause considerable difficulties, and the results would be far poorer than those furnished by (1.4) and (1.5).

3. The approximate functional equation

The key to proof of the Theorem is the existence of an approximate functional equation for $\zeta^k(1+it)$. The main difficulty is that $k > 0$ is not neccessarily an integer, so that $\zeta^k(s)$ has branch points at the zeros of $\zeta(s)$, which complicates the use of analytic methods suitable for dealing with this problem. A natural tool is the asymptotic formula for the summatory function

$$(3.1)\qquad D_k(x) : r = \sum_{n \leq x} d_k(n).$$

Sharpening a result of A. Selberg, R.D. Dixon [2] proved

$$(3.2)\quad D_z(x) = c_1(z) x \log^{z-1} x + \cdots + c_N(z) x \log^{z-N} x + \cdots + O_N(x \log^{\mathrm{Re} z - N - 1}),$$

where $N \geq 1$ is an arbitrary but fixed integer, $|z| \leq A$ for any fixed $A > 0$, $c_j(z) = B_j(z)/\Gamma(z-j+1)$ $(j = 1,\ldots,N)$, and each $B_j(z)$ is regular for

$|z| \le A$. Since the error term in (3.2) is only by a log-factor of a smaller order of magnitude than the main term, the use of (3.2) is insufficient to produce an approximate functional equation for $\zeta^k(1+it)$ capable of proving (2.4). Instead of (3.2) we shall use a recent result of H. Nakaya [5], who proved, uniformly for $|z| \le A$ and $x \ge 3$,

$$D_z(x) = \Phi_z(x) + \Delta_z(x), \tag{3.3}$$

$$\Phi_z(x) = \frac{1}{2\pi i}\int_{\mathscr{L}} \zeta^z(w) x^w \frac{dw}{w}, \tag{3.4}$$

$$\Delta_z(x) \ll x\exp(-c\log^{3/5}x(\log\log x)^{-1/5}) \quad (c>0). \tag{3.5}$$

For any $0 < r < \frac{1}{2}$, $\mathscr{L} = \mathscr{L}(r)$ is the contour which begins at $w = \frac{1}{2}$, moves to $w = 1 - r$ along the real axis, encircles the point $w = 1$ with radius r in the counterclockwise direction, and returns to $w = \frac{1}{2}$ along the real axis. The advantage of (3.3)-(3.5) over (3.2) lies in the fact that the error term in (3.5) is sufficiently sharp, while at the same time an asymptotic evaluation of the integral in (3.4) leads to Dixon's formula (3.2) for $D_z(x)$. The error term in (3.5) is actually the same as in the sharpest known form of the prime number theorem. Both are a consequence of the best zero-free region for $\zeta(s)$, namely

$$\zeta(s) \ne 0 \text{ for } \sigma \ge 1 - C(\log t)^{-2/3}(\log\log t)^{-1/3} \ (C>0,\ s=\sigma+it,\ t\ge t_0). \tag{3.6}$$

A proof of this result, due essentially to I. M. Vinogradov, may be found in Ch. 6 of [3]. Thus an improvement of the bound in (3.6) would give a better estimate in (3.5), and consequently lead to a better result in (2.4).

Now we may derive the desired approximate functional equation for $\zeta^k(1+it)$. Suppose $k > 0$, $\sigma = \operatorname{Re} s > 1$, $1 \le \operatorname{Im} s = t \le T$, $X = N + \frac{1}{2}$, and N is a large natural number. From (2.3) and (3.3) we have

$$\begin{aligned} \zeta^k(s) &= \sum_{n\le X} d_k(n)n^{-s} + \int_X^\infty x^{-s}\,dD_k(x) \\ &= \sum_{n\le X} d_k(n)n^{-s} + \int_X^\infty x^{-s}\,d\Phi_k(x) + \int_X^\infty x^{-s}\,d\Delta_k(x). \end{aligned} \tag{3.7}$$

Absolute covergence and (3.4) give

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$$(3.8)\quad \int_X^\infty x^{-s}\, d\Phi_k(x) = \frac{1}{2\pi i}\int_{\mathscr{L}} \zeta^k(w)\Big(\int_X^\infty x^{w-s-1}\,dx\Big)dw = \frac{1}{2\pi i}\int_{\mathscr{L}} \frac{\zeta^k(w)X^{w-s}\,dw}{s-w}.$$

In $\mathscr{L}$ we take $r = 1/\log X$, so that the integrand of the last integral is regular in $\mathscr{L}$ and on its boundary. In view of (3.5) it follows that the integral converges absolutely for $\sigma \geq 1$, which means that we have obtained analytic continuation of the left-hand side of (3.8) that is valid for $\sigma = 1$. Thus for $k \neq 1$

$$(3.9)\quad \int_X^\infty x^{-1-it}\, d\Phi_k(x) = \frac{1}{2\pi i}\int_{\mathscr{L}} \frac{\zeta^k(w)X^{w-1-it}}{1+it-w}\,dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\zeta^k(1+re^{i\theta})re^{i\theta}X^{re^{i\theta}-it}d\theta}{it-re^{i\theta}} + O\Big(\frac{1}{t}\int_{\frac{1}{2}}^{1-r} |\,\zeta^k(u)\,|\, X^{u-1}du\Big)$$

$$\ll t^{-1}r^{1-k}X^r + t^{-1}\int_{\frac{1}{2}}^{1-r}(1-u)^{-k}du \quad \ll_k t^{-1}(\log^{k-1}X + 1),$$

since $\zeta(w) = 1/(w-1) + O(1)$ in the neighborhood of $w = 1$. Furthermore, for $\sigma > 1$,

$$(3.10)\quad \int_X^\infty x^{-s}d\Delta_k(x) = -\Delta_k(X)X^{-s} + s\int_X^\infty \Delta_k(x)x^{-s-1}\,dx,$$

and in view of (3.5) the integral on the right-hand side converges absolutely for $\sigma \geq 1$. More precisely

$$(3.11)\quad \int_X^\infty \Delta_k(x)x^{-it-2}dx \ll \int_X^\infty \exp(-c\log^{3/5}x(\log\log x)^{-1/5})\,\frac{dx}{x}$$

$$= \int_{\log X}^\infty \exp(-c\,y^{3/5}(\log y)^{-1/5})\,dy \ll \exp\Big(-\frac{c}{2}\log^{3/5}X(\log\log X)^{-1/5}\Big).$$

Hence if we take

$$(3.12)\quad X = [\exp\{D(\log T)^{5/3}(\log\log T)^{1/3}\}] + \frac{1}{2}$$

with a sufficiently large constant $D = D(c) > 0$, then we shall have

$$\exp\Big(-\frac{c}{2}\log^{3/5}X(\log\log X)^{-1/5}\Big) \leq T^{-2},$$

and (3.7)-(3.11) give, for $1 \leq t \leq T$ and $k > 0,\ k \neq 1$,

$$(3.13)\quad \zeta^k(1+it) = \sum_{n\leq X} d_k(n)n^{-1-it} + S_k(t)$$

where

$$S_k(t) \ll_k (\log^{k-1} X + 1)t^{-1}. \tag{3.14}$$

This is the desired approximate functional equation in the general case when $k > 0$. If k is additionally an integer, then we can use the well-known elementary result (see (12.1.4) of E. C. Titchmarsh [6])

$$\sum_{n\le x} d_k(n) = xP_{k-1}(\log x) + \Delta_k(x),\ \Delta_k(x) \ll_k x^{(k-1)/k}\log^{k-2} x, \tag{3.15}$$

where $k \ge 2$ and $P_{k-1}(y)$ is a polynomial of degree $k-1$ in y. With the aid of (3.15), we shall obtain by the method used in deriving (3.13) and (3.14) that

$$\zeta^k(1+it) = \sum_{n\le T^{2k}} d_k(n)n^{-1-it} + O_k(t^{-1}\log^{k-1} T) \quad (1 \le t \le T). \tag{3.16}$$

4. Proof of the Theorem

To prove (2.4) we shall use (3.13) and (3.14), while for the proof of (2.5) we need (3.16). Since both proofs are similar, only the proof of (2.4) will be given in detail. We have, for $k \ne 1$,

$$\int_1^T |\zeta(1+it)|^{2k}\,dt = O_k(\log^{2k-2} X + 1) + \tag{4.1}$$

$$+ \int_1^T |\sum_{n\le X} d_k(n)n^{-1-it}|^2 dt + 2\,\mathrm{Re}\left\{\int_1^T \sum_{n\le X} d_k(n)n^{-1-it}\,\overline{S_k(t)}\,dt\right\},$$

where X is given by (3.12). To evaluate the first integral on the right-hand side of (4.1) we use the well-known Montgomery-Vaughan mean value theorem (see Ch. 4 of [3])

$$\int_1^T |\sum_{n\le N} a_n n^{it}|^2 dt = T\sum_{n\le N} |a_n|^2 + O\Big(\sum_{n\le N} n|a_n|^2\Big),$$

which is valid for arbitrary complex numbers a_n. This gives

$$\int_1^T |\sum_{n\le X} d_k(n)n^{-1-it}|^2 dt = (T-1)\sum_{n\le X} d_k^2(n)n^{-2} + O\Big(\sum_{n\le X} d_k^2(n)n^{-1}\Big) \tag{4.2}$$

$$= T\sum_{n=1}^{\infty} d_k^2(n)n^{-2} + O_k(\log^{k^2} X),$$

since

$$\sum_{n\le X} d_k^2(n) \ll_k X\log^{k^2-1} X, \quad \sum_{n\le X} d_k^2(n)n^{-1} \ll_k \log^{k^2} X. \tag{4.3}$$

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It suffices to indicate now the first bound in (4.3) is obtained, since the second one follows easily from the first one by partial summation. Note that for $\sigma > 1$

$$\sum_{n=1}^{\infty} d_k^2(n) n^{-s} = \zeta^{k^2}(s) G_k(s) \tag{4.4}$$

with

$$G_k(s) = \sum_{n=1}^{\infty} g_k(n) n^{-s} = \prod_p (1 - p^{-s})^{k^2} (1 + d_k^2(p) p^{-s} + d_k^2(p^2) p^{-2s} + \cdots). \tag{4.5}$$

But since $d_k^2(p) = k^2$ and $d_k(n) \ll_{k,\varepsilon} n^{\varepsilon}$ it is easily seen that the Dirichlet series in (4.5) converges uniformly and absolutely for $\sigma \geq \frac{1}{2} + \delta$ and any $\delta > 0$. From (3.2) with $z = k^2$ we have

$$\sum_{n \leq X} d_{k^2}(n) \ll_k X \log^{k^2-1} X, \tag{4.6}$$

and since (4.4) implies that $d_k^2(n)$ is the convolution of $d_{k^2}(n)$ and $g_k(n)$, it follows from (4.6) that (4.3) holds.

Finally, by using (3.14) and trivial estimation we have

$$\int_1^T \sum_{n \leq X} d_k(n) n^{-1-it} \overline{S_k(t)}\, dt \ll_k (\log^{2k-1} X + \log^k X) \log T, \tag{4.7}$$

and (2.4) follows from (4.1)-(4.3), (4.7) and the definition (3.12) of X.

Note added in Proof. After this work was submitted R. Balasulramanian et al. [Acta Arith. 65 (1993), 45–51] proved a general result which sharpens (2.4).

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