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DENOMINATOR SEQUENCES FOR CONTINUED FRACTIONS, III

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To Professor K. Mahler on his seventy-fifth birthday

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Abstract

It is shown that for every irrational α the set of α', for which α and α' have infinitely many convergents with the same denominator, has the cardinality of the continuum.

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For given real irrational α, let the sequence of the denominators of the convergents of the simple continued fraction for α be called the denominator sequence of α. By a result of Schmidt (1967) it follows that if α and α' are two algebraic irrationals whose denominator sequences have a common subsequence then α, α' and 1 are linearly dependent over the rationals. In a recent paper (Worley, 1973) it was shown that the same conclusion held for α and α' not restricted to be algebraic provided that the common subsequence (B_n) is the denominator sequence for a third number α'' and $B_n^2 > \frac{1}{2}B_{n+1}$ for infinitely many n. The main result of this paper is in the opposite direction, namely:

THEOREM. *For every real irrational α the set of α' for which α and α' have a common denominator subsequence has the cardinality of the continuum.*

From this result it trivially follows that there is an α' such that α', α and 1 are not linearly dependent over the rationals. The proof of the theorem is based on the following lemma which in itself is of interest.

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LEMMA 1. *Let a and b be natural numbers with $(a,b)=1$. Then there exists an integer $n_0 = n_0(a,b)$ such that every integer $n>n_0$ can be written as $n = ax+by$ with x,y integers such that $x>y>0$ and $(x,y)=1$.*

PROOF. Considering the congruence classes mod a of yb for $0 \leqslant y < a$ it is clear that $n = ax_0+by_0$ where $0 \leqslant y_0 < a$. We let r denote the greatest integer less than $(x_0-y_0)/(a+b)$, and take the representations $n = ax_l+by_l$ for $l = 1,2,\ldots,r$, where $x_l = x_0 - lb$, $y_l = y_0+la$. The lemma is equivalent to the statement that if n is sufficiently large the sum

$$N = \sum_{\substack{1\leqslant l\leqslant r\\ \delta_l=1}} 1,$$

where $\delta_l = (x_l, y_l)$, is non-zero. We show by a simple sieve argument that $N>0$ for $n > c[a(a+b)]^{1+\varepsilon}$, where c is a constant depending on the choice of $\varepsilon>0$.

We have

$$\begin{aligned} N &= \sum_{1\leqslant l\leqslant r} \sum_{d\mid\delta_l} \mu(d) \\ &= \sum_{d\mid n} \mu(d) \sum_{\substack{1\leqslant l\leqslant r\\ d\mid\delta_l}} \\ &= \sum_{d\mid n} \mu(d)\,(rd^{-1}+O(1)) \end{aligned}$$

since the l for which $d\mid\delta_l$ form a unique congruence class mod d. The main term is clearly $r\varphi(n)/n$ and the error term is certainly less than $d(n) = \sum_{d\mid n} 1$.

Take an arbitrary $\varepsilon>0$ and let ρ be such that $1/(1-2\rho) = 1+\varepsilon$. By well-known estimates for $\varphi(n)$ and $d(n)$ we know there exist constants c' and c'' such that $\varphi(n) > c'n^{1-\rho}$ and $d(n) < c''n^{\rho}$. Furthermore, if we assume $n > 4a(a+b)$ we have $r > n/2a(a+b)$. Hence

$$N > c'n^{1-\rho}/2a(a+b) - c''n^{\rho}$$

and so $N>0$ provided $n^{1-2\rho} > 2c''a(a+b)/c'$, that is, $n > c[a(a+b)]^{1+\varepsilon}$ for a suitable constant c. It should be noted that this is compatible with our assumption $n > 4a(a+b)$ providing $c>4$.

LEMMA 2. *Let S denote an infinite subset of the natural numbers, containing at least two relatively prime integers. Then there exist uncountably many distinct real numbers β for which the denominator sequence contains infinitely many elements of S.*

PROOF. We first note that if $(a,b)=1$ and $a>b>0$ then there exists a rational number $r(a,b) = \langle 0, c_1, \ldots, c_m\rangle$ which has b and a as its final two denominators, where $\langle c_0, \ldots, c_m \rangle$ denotes the simple continued fraction

$$c_0 + \frac{1}{c_1+} \frac{1}{c_2+} \cdots \frac{1}{+c_m}.$$

Secondly we note that if n can be expressed as $xa+yb$ where $(x,y)=1$ and $x>y>0$ and if $r(x,y)=\langle 0, d_1, \ldots, d_p \rangle$ then n is the final convergent of

$$\langle 0, c_1, \ldots, c_m, d_1, \ldots, d_p \rangle.$$

The proof is basically just the following procedure. Take two relatively prime elements $a, b \in S$. Using Lemma 1 there exists $n_3 \in S$ with $n_3 = ax+yb$ where $(x,y)=1$ and $x>y>0$. Hence there is a rational number $r_3 = r(a,b,n_3)$ with a, b and n_3 included among its denominators. Now let b_2, a_2 denote the final two denominators of $r(a,b,n_3)$, and repeat the above procedure to get $r_4 = r(a,b,n_3,n_4)$ a rational number for which the continued fraction is an extension of the continued fraction for $r(a,b,n_3)$ and such that $r(a,b,n_3,n_4)$ has four elements a, b, n_3 and n_4 of S among its denominators. If this procedure is repeated indefinitely the sequence $r_3, r_4, \ldots$ clearly converges to a number β which has the desired property. A slight modification of this procedure ensures a choice at each step, so leading to uncountably many β with the desired property. The exact details are as follows.

Let $a, b \in S$ be such that $(a,b)=1$ and $a>b$. We choose from S a sequence $s_0, s_1, \ldots$ with the property $s_0 = a$, $s_{m+1} > \max\{n_0(c,d)\colon c<d \leqslant s_m+1\}$ for $m \geqslant 1$.

Choose an arbitrary sequence $(\varepsilon_i)_{i \in N}$ with the following properties.

(i) $\varepsilon_1 = \varepsilon_2 = 1$,

(ii) $\varepsilon_i = 0$ for infinitely many i, and

(iii) for each i, $\varepsilon_i = 0$ or 1.

The set of such sequences clearly has the cardinality of the continuum. For each such sequence we construct a number β such that s_i is a denominator of β if $\varepsilon_i = 0$ and $s_i + 1$ is a denominator of β if $\varepsilon_i = 1$. This ensures that for $i \geqslant 3$, s_i is a denominator of β if and only if $\varepsilon_i = 0$ and so the β corresponding to distinct sequences are distinct. To see that s_i is not a denominator of β if $\varepsilon_i = 1$ and $i \geqslant 3$ it is necessary to observe that consecutive integers are denominators only for $\beta = \langle 0, 2, \ldots \rangle$ which has consecutive denominators 1,2 and for $\beta = \langle 0, 1, q, \ldots \rangle$ which has consecutive denominators $q, q+1$.

We define β to be the limit of the sequence $r_0, r_1, r_2, \ldots$ of rationals constructed inductively as follows. Firstly, we set $r_0 = b/a$. Secondly, if r_j has been constructed we let $r_j = \langle 0, c_1, \ldots, c_q \rangle$ have last two convergents e and f, where $f \leqslant s_j + 1$. If $\varepsilon_{j+1} = 0$ we set $v_{j+1} = s_{j+1}$, while if $\varepsilon_{j+1} = 1$ we set $v_{j+1} = s_{j+1} + 1$. By the choice of s_{j+1} we can write

$$v_{j+1} = fy + ex$$

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where $x > y > 0$ and $(x, y) = 1$. We now set

$$r_{j+1} = \langle 0, c_1, \ldots, c_q, d_1, \ldots, d_k \rangle$$

where $y/x = \langle 0, d_1, \ldots, d_k \rangle$, so that the final denominator of r_{j+1} is v_{j+1}. Plainly the sequence (r_j) converges to a number β with the desired property. As noted above, the numbers β corresponding to different sequences (ε_i) are distinct and so there are uncountably many such β.

Proof of the theorem. The theorem follows from Lemma 2 on taking S to be the denominator sequence of the given number α.

It will be noted that if (B_n) denotes the common denominator sequence to α, and α' is constructed as in Lemma 2, then we have $B_{n+1} > c[B_n(B_n + b)]^{1+\varepsilon}$ where $b < B_n$. Hence, regardless of the values chosen for ε and c, we must have $B_{n+1} \geqslant 2B_n^2$ for all but finitely many n. In other words, the condition $B_n^2 > \frac{1}{2}B_{n+1}$ for finitely many n cannot hold.

It will also be noted that the exact choice of r_0 in the proof of Lemma 2 is not critical to the conclusion of the theorem. Hence we can modify the proofs to show the existence of uncountably many α', satisfying the conditions of the theorem, within any internal of the form $(a/b, (a+1)/b)$.

References

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