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ON THE SPLITTING OF MODULES AND ABELIAN GROUPS

PAUL HILL

In a fundamental paper on torsion-free abelian groups, R. Baer [**1**] proved that the group P of all sequences of integers with respect to componentwise addition is not free. This means precisely that P is not a direct *sum* of infinite cyclic groups. However, E. Specker proved in [**9**] that P has the property that any countable subgroup is free. Since an infinite abelian group G is called $\aleph_\sigma$-free if each subgroup of rank less than $\aleph_\sigma$ is free, these results are equivalent to: P is $\aleph_1$-free but not free. Specker also proved in [**9**] that the subgroup F of P consisting of all *bounded* sequences of integers is, in fact, $\aleph_2$-free. Thus the continuum hypothesis, together with Specker's result, implies that F is free. However, without the continuum hypothesis, the freeness of F remained an open question until it was proved to be free by G. Nöbeling in [**8**]. Actually, Nöbeling proved the corresponding result for an arbitrary product of integers, not just for a countable product, and Nöbeling proved that certain subgroups (that he called Specker subgroups) always split out of one another. Nöbeling's results were generalized to modules by L. Kaup and M. Keane in [**7**] and independently by L. Fuchs and K. Rangaswamy in [**2**]. The following theorem contains an abstract version of these results and has the advantage that its proof is simpler than the original proof of the special case due to Nöbeling [**8**]. As will soon become apparent, our approach is different.

THEOREM 1. *Let $\mathscr{F}$ be a family of R-modules* (membership of which is independent of notation) *closed with respect to arbitrary direct sums including the vacuous sum representing* 0. *For a Boolean ring $\mathscr{B}$, suppose that $A \to M(A)$ is a function* (not necessarily onto) *from the subrings of $\mathscr{B}$ to the submodules of an R-module P that satisfies the following conditions:*

(1) *$M(A) \subseteq M(B)$ if $A \subseteq B$;*

(2) *$M(A \cap B) = M(A) \cap M(B)$;*

(3) *$M(\{A, B\}) = \langle M(A), M(B)\rangle$ whenever $AB \subseteq B$;* the module generated by X and Y is denoted by $\langle X, Y\rangle$, whereas, the ring generated by A and B is denoted by $\{A, B\}$;

(4) *$M(\{x\}) = M(0) \oplus C_x$ for some $C_x \in \mathscr{F}$;*

(5) *$M(\bigcup A_\alpha) = (\bigcup M(A_\alpha)) \oplus X$, where $X \in \mathscr{F}$, whenever $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_\alpha \subseteq \ldots$ is a chain of subrings of A leading up to $A = \bigcup A_\alpha$.*

Then $M(B) = M(A) \oplus X$ for some $X \in \mathscr{F}$ if $A \subseteq B$. In particular, $M(\mathscr{B}) = M(0) \oplus K$ where $K \in \mathscr{F}$. Thus $M(\mathscr{B}) \in \mathscr{F}$ if $M(0) \in \mathscr{F}$.

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Proof. The proof is by induction on $|B|$, the cardinality of B. It follows from condition (5) that it suffices to prove the theorem in case B is a simple extension of A since we can ascend from A to an arbitrary $B \supseteq A$ with a chain of simple extensions taking unions at limit ordinals. Thus suppose that $B = \{A, x\}$ is a simple extension of A. Next, we observe that there is no loss of generality in assuming that $B = \{A, Ax\}$. This, of course, is the case if $1 \in A$ or, more generally, if $x \in \{A, Ax\}$. But if $x \notin \{A, Ax\}$, the reason that there is no loss of generality in assuming that $B = \{A, Ax\}$ is based on the fact that $M(\{A, x\}) = M(\{A, Ax\}) \oplus K$ where $K \in \mathscr{F}$. In order to show that $M(\{A, x\}) = M(\{A, Ax\}) \oplus K$, we observe that $x\{A, Ax\} \subseteq \{A, Ax\}$. Hence by condition (3),

$$M(\{A, x\}) = M(\{\{A, Ax\}, x\}) = \langle M(\{A, Ax\}), M(\{x\})\rangle$$

and

$$M(\{A, x\})/M(\{A, Ax\}) \cong M(\{x\})/M(\{x\} \cap \{A, Ax\}) = M(\{x\})/M(0).$$

Since $M(\{x\}) = M(0) \oplus C_x$ by condition (4), we have that

$$M(\{A, x\}) = M(\{A, Ax\}) \oplus C_x,$$

where $C_x \in \mathscr{F}$. Thus the problem is reduced to the case that $B = \{A, Ax\}$, so we shall assume that $B = \{A, Ax\}$. The next step is the main reduction. Since $M(B) = M(\{A, Ax\}) = \langle M(A), M(Ax)\rangle$,

$$(*) \qquad M(B)/M(A) \cong M(Ax)/M(A \cap Ax)$$

and

$$M(B) = M(A) \oplus X \quad \text{if} \quad M(Ax) = M(A \cap Ax) \oplus X.$$

Thus the theorem is already proved (inductively) for finite B since $|Ax| \leqq |A| \leqq |B|$ and $|A| < |B|$ unless $A = B$. However, the reduction $(*)$ is more significant than merely proving the theorem in case B is finite, for $A \cap Ax$ is an ideal of Ax not simply a subring. As we have already observed from the reduction $(*)$, the conclusion of the theorem holds if Ax is finite. Thus we assume that Ax is infinite. Since Ax is a Boolean ring, there exists an ascending chain

$$0 = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_\alpha \subseteq \ldots, \qquad \alpha < \lambda,$$

of subrings of Ax such that:

(i) $|C_\alpha| < |Ax|$ for each $\alpha < \lambda$;

(ii) $C_\beta = \bigcup_{\alpha<\beta} C_\alpha$ if β is a limit ordinal less than λ;

(iii) $Ax = \bigcup_{\alpha<\lambda} C_\alpha$.

In other words, we can reach the infinite Boolean ring Ax with a chain of smaller subrings.

Since $A \cap Ax$ is an ideal of Ax, we observe that $C_\alpha(A \cap Ax) \subseteq A \cap Ax$ and $M(\{A \cap Ax, C_\alpha\}) = \langle M(A \cap Ax), M(C_\alpha)\rangle$ for each α. Therefore,

$$M(\{A \cap Ax, C_{\alpha+1}\})/M(\{A \cap Ax, C_\alpha\}) \cong M(C_{\alpha+1})/M(C_{\alpha+1} \cap \{A \cap Ax, C_\alpha\}),$$

and recall that $|C_\alpha| < |Ax| \leqq |A| \leqq |B|$. Since

$$M(C_{\alpha+1}) = M(C_{\alpha+1} \cap \{A \cap Ax, C_\alpha\}) \oplus X$$

for some $X \in \mathscr{F}$ by the induction hypothesis, we have that

$$M(\{A \cap Ax, C_{\alpha+1}\}) = M(\{A \cap Ax, C_\alpha\}) \oplus X.$$

Since $C_0 = 0$ and since $Ax = \bigcup_{\alpha<\lambda} C_\alpha$, we conclude (because we have hypothesized, through condition (5), that nothing goes wrong at limits) that

$$M(Ax) = M(A \cap Ax) \oplus Y,$$

where $Y \in \mathscr{F}$. In view of the reduction (*), this completes the proof of the theorem.

The conclusion of the following corollary is *essentially* contained in [**2**] and [**7**]; for Z-modules and for $G = Z$, it is Nöbeling's result.

COROLLARY 1. *Let G be an R-module and, for an arbitrary index set I, let $P = \prod_{i \in I} G_i$ where $G_i = G$ for each $i \in I$. Denote by $\mathscr{B}$ the Boolean ring of all subsets of I. For each subring A of $\mathscr{B}$, define the submodule $M(A)$ of P by:*

$$M(A) = \langle g_i \operatorname{ch}(a_i) : g_i \in G \text{ and } a_i \in A \rangle,$$

where $g_i \operatorname{ch}(a_i)$ denotes the function whose value is g_i on a_i and zero elsewhere. If $A \subseteq B$, then $M(B) = M(A) \oplus N$ where $N = \sum \oplus G$. In particular, $M(\mathscr{B}) = \sum \oplus G$, that is, the module of all finite valued functions (with pointwise addition) from I to G is isomorphic to a direct sum of the module G.

Proof. The R-modules that can be expressed as a direct sum of the R-module G will constitute the family $\mathscr{F}$. Using Theorem 1, we need to show only that the function M that we have defined satisfies condition (1)–(5), for it is clear that $M(0) = 0$. Likewise, it is obvious that $M(B) \supseteq M(A)$ if $B \supseteq A$. Since $M(\bigcup A_\alpha) = \bigcup M(A_\alpha)$ for a chain $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_\alpha \ldots$, condition (5) is also trivial. Condition (4) is equally trivial, for $M(\{x\}) \cong G$ since $M(\{x\})$ represents the constant functions from the set x to G. This leaves the verification of conditions (2) and (3). If $AB \subseteq B$, then $\{A, B\} = A + B$. Thus in order to show that $M(\{A, B\}) = \langle M(A), M(B)\rangle$, it suffices to show that $g \operatorname{ch}(a + b) \in \langle M(A), M(B)\rangle$ if $a \in A$, $b \in B$ and $g \in G$. However,

$$g(\operatorname{ch}(a + b) = g \operatorname{ch}(a) + g \operatorname{ch}(b) - 2g \operatorname{ch}(ab)$$

shows that $g \operatorname{ch}(a + b) \in \langle M(A), M(B)\rangle$. Finally, the condition that $M(A \cap B) = M(A) \cap M(B)$ follows from the fact an element $x \in M(A)$ has a unique representation, for some positive integer n, of the form $x = \sum_{i=1}^{n} g_i \operatorname{ch}(a_i)$ where the elements g_i are distinct in G, $a_i \in A$ and $a_i a_j = 0$ if $i \neq j$. By definition of $M(A)$, we know that $x = \sum_{i=1}^{m} g_i \operatorname{ch}(a_i)$.

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The representation with $a_i a_j = 0$ if $i \neq j$ can easily be established by induction on m. For example,

$$g_1 \operatorname{ch}(a_1) + g_2 \operatorname{ch}(a_2) = g_1 \operatorname{ch}(a_1 + a_1 a_2) + (g_1 + g_2) \operatorname{ch}(a_1 a_2) + g_2 \operatorname{ch}(a_2 + a_1 a_2),$$

and terms having equal coefficients can then be collected to obtain the desired representation.

Remark. A brief analysis of the submodule $M(\{x\})$ and of the reduction $(*)$ in the proof of Theorem 1 makes it apparent that the condition that N, the complement of $M(A)$ in $M(B)$, has a characteristic G-basis can be carried along in Corollary 1 with no difficulty; compare [**7**] and [**8**].

Recall that a subgroup B of an abelian group G is called a *basic* subgroup if B is pure, B is a direct sum of cyclic groups, and G/B is divisible. Griffith [**3**, p. 103] points out that the group $P = \prod_{\aleph_0} Z$ has a basic subgroup. In the next corollary, we observe that any product of integers has, in fact, a distinguished basic subgroup.

COROLLARY 2. *If P is an arbitrary product of integers and if F denotes the subgroup consisting of those elements in P that have a finite component spectrum, then F is a basic subgroup of P.*

Proof. By Nöbeling's result (Corollary 1 with $R = Z = G$), F is free. Furthermore, it is routine to prove that F is pure in P. Thus it suffices to show that P/F is divisible. Let $x \in P$ and let d be an arbitrary positive integer. We can write $x = \sum_{n \in Z} n \operatorname{ch}(a_n)$, where a_n is the support of g at the integer n. Letting $n = dq_n - r_n$ with $|r_n| \leqq d$, we see that $x - \sum_{n \in Z} r_n \operatorname{ch}(a_n)$ is divisible in P by d. Since $\sum_{n \in Z} r_n \operatorname{ch}(a_n)$ is contained in F, the corollary is proved.

The $\aleph_1$-freeness of a product $P = \prod Z$ of integers and the freeness of F are unified in the result that follows.

COROLLARY 3. *Let $P = \prod_{i \in I} Z_i$ be a product of integers ($Z_i = Z$) and let F, as before, denote the elements of P that have a finite component spectrum. Any countable extension of F in P is free.*

Proof. We know that F, itself, is free. Write $F = \sum_{j \in J} \langle x_j \rangle$, and let $H = \langle F, C \rangle$ where C is a countable subgroup of P. Let $K_0 = F \cap C$ and choose a countable subset $J(0)$ of J so that $K_0 \subseteq \sum_{j \in J(0)} \langle x_j \rangle$. Now define

$$C_1 = \left\langle \sum_{j \in J(0)} \langle x_j \rangle, C \right\rangle.$$

Let $K_1 = F \cap C_1$, and repeat infinitely the process to obtain an ascending sequence of countable subgroups

$$C \subseteq C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n \ldots$$

and subsets $J(n)$ such that $F \cap C_n \subseteq \sum_{j \in J(n)} \langle x_j \rangle$ and $\sum_{j \in J(n)} \langle x_j \rangle \subseteq C_{n+1}$. Now, letting $D = \bigcup_{n<\omega} C_n$ and $K = \bigcup_{n<\omega} J(n)$, we observe that D is countable and $D \cap F = \sum_{j \in K} \langle x_j \rangle$. Hence

$$\langle F, D \rangle = D \oplus \sum_{j \in J-K} \langle x_j \rangle,$$

a free group. Since $H = \langle F, C \rangle \subseteq \langle F, D \rangle$, H is free.

The preceding corollary leads immediately to the following result.

COROLLARY 4. *Any countable subgroup of a product of integers is contained in a basic subgroup.*

Proof. Let C be any countable subgroup of $P = \prod Z$ and let F be the elements in P that have a finite component spectrum. Since F is pure in P, there exists a countable extension K of F containing C that is pure in P; purification over F does not alter infinite cardinals. By Corollary 3, K is free. Obviously, P/K is divisible since P/F is divisible. Thus K is a basic subgroup of P containing C.

We shall now use the freeness of F (in Corollaries 2 and 3) to obtain the solution to a problem raised by E. Weinberg [**10**] in 1963 concerning lattice ordered groups.

THEOREM 2. *The free abelian l-group over a free abelian group is free (as an unordered group).*

Proof. For an arbitrary cardinal m, let $A = \sum_m Z$ be a free abelian group and let A^* be the free abelian l-group over A. One description of A^* is the following. Let T represent the collection of all total orders on the group A, and form the cardinal product $C = \prod_{t \in T} [+] A_t$, where A_t denotes A endowed with the total order $t \in T$. Let δ be the diagonal map of A into C; $\delta(a) = (a_t)$ where $a_t = a$ for each $t \in T$. Finally, A^* is the l-subgroup of C generated by $\delta(A)$. Any element $x \in A^*$ can be written in the form

$$x = \bigvee_i \bigwedge_j \delta(a_{i,j})$$

where $a_{i,j} \in A$ and i and j range over finite sets. Since A_t is totally ordered, x_t is one of the elements $a_{i,j}$ for each t-component x_t of x. Thus the set $\{x_t\}_{t \in T}$ is finite. Moreover, $A_t \cong A = \sum_m Z$ is free, so each x_t involves only a finite number of integers. We conclude that the Z-components of a fixed element

$$x \in A^* \subseteq \prod_{t \in T} [+] A_t = \prod_{t \in T} [+] \left(\sum_m Z\right)$$

is a finite set. Therefore, A^* is free as an unordered abelian group.

As we have mentioned, a torsion-free abelian group is called $\aleph_\sigma$-free if each subgroup of rank less than $\aleph_\sigma$ is free. By a theorem of Pontryagin, $\aleph_0$-freeness implies $\aleph_1$-freeness. However, it is well-known that $\aleph_1$-freeness does not imply

$\aleph_2$-freeness. In particular, a group can be $\aleph_1$-free without being free. Indeed, it is known (see, for example, [**3**, p. 101]) that any product of integers is $\aleph_1$-free, but an infinite product of integers is never $\aleph_2$-free.

Little, if anything, seems to have been established concerning the implications of $\aleph_2$-freeness. In fact, we have been without an example of an $\aleph_2$-free group that is not free. (*Added in proof*: Phillip Griffith has now constructed such groups.) The question of the existence of such a group is of particular interest in connection with the discussion of the result of Nöbeling, for Specker [**9**] proved in 1950 that the group of bounded sequences of integers is $\aleph_2$-free. We shall present a general method of constructing $\aleph_2$-free groups that are not free. The key to the construction of the desired $\aleph_2$-free groups is a recent generalization of Pontryagin's theorem for countable torsion-free groups. We shall state and refer to this result as Theorem A.

THEOREM A (Hill [**6**]). *If the torsion-free abelian group G is the union of a countable ascending sequence*

$$H_1 \subseteq H_2 \subseteq \ldots \subseteq H_n \subseteq \ldots$$

of pure subgroups H_n, then G is free provided that H_n is free for each n.

The preceding theorem leads to the following result, which is of considerable interest in its own right.

THEOREM 3. *Let μ be a limit ordinal of cardinality not exceeding $\aleph_1$. If*

$$F_0 \subseteq F_1 \subseteq \ldots \subseteq F_\alpha \subseteq \ldots, \qquad \alpha < \mu,$$

is an ascending chain of free subgroups of G, indexed by the ordinals less than μ, such that

(i) $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$ *if β is a limit less than μ,*
(ii) $G = \bigcup_{\alpha<\mu} F_\alpha$,
(iii) $|F_\alpha| \leqq \aleph_1$ *for each α,*

then G is free provided that $F_{\alpha+1}/F_\alpha$ is $\aleph_1$-free for each $\alpha < \mu$.

Proof. If μ is cofinal with ω, it follows from Theorem A that G is free since F_α is necessarily pure in G due to the fact that the groups $F_{\alpha+1}/F_\alpha$ are torsion free. Thus, we may assume that μ is cofinal with Ω, the first uncountable ordinal. We claim that there is no loss of generality in assuming that $\mu = \Omega$. Since μ is cofinal with Ω, to validate this claim we need only show that F_β/F_α is $\aleph_1$-free whenever $\alpha < \beta < \mu$. Suppose that F_β/F_α is not $\aleph_1$-free and that β is the smallest ordinal satisfying $\alpha < \beta < \mu$ such that F_β/F_α is not $\aleph_1$-free. Since an extension of an $\aleph_1$-free group by an $\aleph_1$-free group is $\aleph_1$-free, β cannot be isolated because, in this case, F_β/F_α is an extension of $F_{\beta-1}/F_\alpha$ by $F_\beta/F_{\beta-1}$ each of which is $\aleph_1$-free. Thus suppose that β is a limit, and let A/F_α be a subgroup of F_β/F_α of finite rank. Then $A/F_\alpha \subseteq F_\gamma/F_\alpha$ for some $\gamma < \beta$, and therefore A/F_α is free since F_γ/F_α is $\aleph_1$-free. Since A/F_α is free, Pontryagin's theorem implies that F_β/F_α is $\aleph_1$-free. Since we have shown that there is no loss of generality in assuming that $\mu = \Omega$, we shall make that assumption.

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Note that under the hypotheses on the groups F_α, the cardinality of $G = \bigcup_{\alpha<\mu} F_\alpha$ does not exceed $\aleph_1$. Let

$$G = [g_0, g_1, \ldots, g_\alpha, \ldots]_{\alpha<\mu}.$$

Since F_α is free, for each $\alpha < \mu$, we can write $F_\alpha = \sum_{i\in I(\alpha)} \langle x_{\alpha,i}\rangle$.

Suppose that τ is a countable ordinal and that we already have developed a chain

$$A_0 \subseteq A_1 \subseteq \ldots \subseteq A_\alpha \subseteq \ldots, \qquad \alpha < \tau,$$

of *countable*, pure subgroups of G such that the following conditions are satisfied.

(1) $A_\alpha \subseteq F_\alpha$ for every $\alpha < \tau$.

(2) If α is isolated, $A_\alpha = \sum_{i\in J(\alpha)} \langle x_{\alpha,i}\rangle$ where $J(\alpha)$ is a subset of $I(\alpha)$.

(3) If $\beta < \tau$ is a limit, $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$.

(4) If $\gamma < \alpha < \tau$, then $A_\alpha \cap F_\gamma = \sum_{i\in J_\alpha(\gamma)} \langle x_{\gamma,i}\rangle$ with $J_\alpha(\gamma)$ a subset of $I(\gamma)$.

(5) $g_\gamma \in A_{\alpha+1}$ if $\gamma < \alpha + 1 < \tau$ and if $g_\gamma \in F_{\alpha+1}$.

(6) $\langle F_\gamma, A_\alpha\rangle$ is pure in G if $\alpha < \tau$ and $\gamma < \alpha$.

We wish to find a countable, pure subgroup A_τ of G containing A_α for all $\alpha < \tau$ such that conditions (1)–(6) remain valid when $\tau + 1$ replaces τ. There are, as usual, two cases to distinguish. We shall first dispose of the easy one.

Case 1. τ is a limit ordinal: In this case, we simply define $A_\tau = \bigcup_{\alpha<\tau} A_\alpha$ as condition (3) dictates. Since $A_\alpha \subseteq F_\alpha \subseteq F_\tau$ if $\alpha < \tau$, we observe that $A_\tau \subseteq F_\tau$. Conditions (2) and (5) are not relevant since τ is a limit ordinal. Condition (6) follows because purity is an inductive property. Furthermore, condition (4) is satisfied for $\alpha = \tau$ if we set $J_\tau(\gamma) = \bigcup_{\gamma<\beta<\tau} J_\beta(\gamma)$. Finally, A_τ is countable since τ is countable, and A_τ is pure because A_τ is the union of a chain of pure subgroups.

Case 2. $\tau - 1$ exists: For simplicity of notation, let $\sigma = \tau - 1$ and let

$$B = \langle A_\sigma, [g_\gamma]_{\gamma<\tau} \cap F_\tau\rangle.$$

Then B is a countable subgroup of F_τ. We are searching for a countable, pure subgroup C of F_τ containing B such that conditions (2), (4) and (6) are satisfied. We shall first show the existence of C (a countable, pure subgroup of F_τ containing B) that satisfies condition (6) individually. Toward this end, we recall from [**5**, Theorem 1] that given a countable group B of G and a countable collection of subgroups G_i of G that there exists a countable subgroup C of G containing B such that

$$\text{(P)} \qquad \langle G_i, C\rangle/G_i \cap \langle G_i, G_j\rangle/G_i$$

is pure in $\langle G_i, G_j\rangle/G_i$ for all i and j. For our collection $\{G_i\}$, we shall take the set

$$\{H_\gamma\}_{\gamma\leqq\sigma} \cup \{0, G\},$$

a countable collection. Upon setting $G_i = H_\gamma$ and $G_j = G$, we obtain from the purification property (P) the purity of $\langle H_\gamma, C\rangle$ in G if $\gamma \leqq \sigma$ because H_γ is pure in G. Also observe that if $G_i = 0$ and $G_j = G$, the conclusion of (P) is that C is pure in G. We have produced a countable, pure subgroup C of G containing B such that condition (6) is satisfied for $\alpha \leqq \tau$ if we set $A_\tau = C$. However, C is not our final choice of A_τ since we do not know that conditions (2) and (4) are satisfied.

We now choose, inductively, sequences $\{I_n(\tau)\}$ and $\{I_n(\gamma)\}$, for $\gamma < \tau$, of countable subsets of $I(\tau)$ and $I(\gamma)$, respectively, so that

$$C_n \subseteq \sum_{i \in I_n(\tau)} \langle x_{\tau,i} \rangle \subseteq C_{n+1}$$

and

$$C_n \cap F_\gamma \subseteq \sum_{i \in I_n(\gamma)} \langle x_{\gamma,i} \rangle \subseteq C_{n+1} \cap F_\gamma,$$

where $C = C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n \subseteq \ldots$ is an ascending sequence of countable, pure subgroups of F_τ such that condition (6) is satisfied for $A_\tau = C_n$. If we (finally) define $A_\tau = \bigcup C_n$ and let $J(\tau) = \bigcup I_n(\tau)$ and $J_\tau(\gamma) = \bigcup I_n(\gamma)$, then A_τ is a countable, pure subgroup of F_τ satisfying all the conditions (1)–(6) (when $\tau + 1$ replaces τ).

The conclusion now is that there exists a chain

$$A_0 \subseteq A_1 \subseteq \ldots \subseteq A_\alpha \subseteq \ldots, \qquad \alpha < \mu = \Omega,$$

of countable, pure subgroups of G such that conditions (1)–(6) are satisfied for any countable α, without regard to τ. Note that condition (5) implies that $G = \bigcup_{\alpha<\Omega} A_\alpha$, and condition (3) states that $G_\beta = \bigcup_{\alpha<\beta} A_\alpha$ if β is a countable limit ordinal. Since $A_0 \subseteq F_0$ and since F_0 is free, A_0 is free. Thus in order to show that G is free, it suffices to show that $A_{\alpha+1}/A_\alpha$ is free for each countable α. Furthermore, we know that

$$A_{\alpha+1}/(A_{\alpha+1} \cap F_\alpha) \cong \langle A_{\alpha+1}, F_\alpha \rangle / F_\alpha \subseteq F_{\alpha+1}/F_\alpha$$

is free because $F_{\alpha+1}/F_\alpha$ is $\aleph_1$-free. Hence, it is enough to prove that $(A_{\alpha+1} \cap F_\alpha)/A_\alpha$ is free. If α is isolated, we have that $(A_{\alpha+1} \cap F_\alpha)/A_\alpha \subseteq F_\alpha/A_\alpha$ is free because A_α is a direct summand of F_α by condition (2). Therefore, assume that α is a limit ordinal. Then the group $(A_{\alpha+1} \cap F_\alpha)/A_\alpha$ is the union of its subgroups $\langle A_{\alpha+1} \cap F_\gamma, A_\alpha \rangle / A_\alpha$ with $\gamma < \alpha$. These subgroups are pure (even in G/A_α) according to condition (6) because

$$\langle A_{\alpha+1} \cap F_\gamma, A_\alpha \rangle / A_\alpha = \langle F_\gamma, A_\alpha \rangle / A_\alpha \cap A_{\alpha+1}/A_\alpha,$$

the intersection of two pure subgroups of the torsion-free group G/A_α. Since

$$\langle A_{\alpha+1} \cap F_\gamma, A_\alpha \rangle / A_\alpha \cong (A_{\alpha+1} \cap F_\gamma)/(A_\alpha \cap F_\gamma) \subseteq F_\gamma/(A_\alpha \cap F_\gamma),$$

condition (4) implies that $\langle A_{\alpha+1} \cap F_\gamma, A_\alpha \rangle / A_\alpha$ is free if $\gamma < \alpha$. Theorem A implies that $(A_{\alpha+1} \cap F_\alpha)/A_\alpha$ is free, and the proof of Theorem 1 is finished.

COROLLARY 5. *Let G be the union of a chain*

$$F_0 \subseteq F_1 \subseteq \ldots \subseteq F_\alpha \subseteq \ldots, \qquad \alpha < \mu = \omega\gamma,$$

of free subgroups F_α, indexed by the initial segment of ordinals less than some limit ordinal μ, such that $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$ when β is a limit ordinal. If $|G| \leqq \aleph_1$ and if G/F_α is a subgroup of a product of integers for each α, then G is free.

Proof. If the chain is countable, apply Theorem A. In this case, the condition on the cardinality of G is not necessary and it suffices to assume that G/F_α is torsion free. If the chain is uncountable, there is no loss of generality in assuming that $|\mu| = \aleph_1$ since $|G| \leqq \aleph_1$. Since a product of integers is $\aleph_1$-free, it follows that $F_{\alpha+1}/F_\alpha \subseteq G/F_\alpha$ is $\aleph_1$-free. Thus the corollary follows directly now from Theorem 3.

We are now prepared to prove our existence theorem for nonfree, $\aleph_2$-free groups.

THEOREM 4. *There exist $\aleph_2$-free abelian groups that are not free of cardinality $\aleph_2$.*

Proof. Suppose that

$$F_0 \subseteq F_1 \subseteq \ldots \subseteq F_\alpha \subseteq \ldots, \qquad \alpha < \mu,$$

is a chain of abelian groups satisfying the following conditions.

(1) $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$ if $\beta < \mu$ is a limit.

(2) F_α is free of cardinality $\aleph_1$ for each $\alpha < \mu$.

(3) $|\mu| \leqq \aleph_1$.

(4) $F_{\alpha+1}/F_\alpha$ is $\aleph_1$-free but not free.

If μ is a limit ordinal, define $F_\mu = \bigcup_{\alpha<\mu} F_\alpha$. By Theorem 3, F_μ is free. Since $|\mu| \leqq \aleph_1$, F_μ retains the cardinality $\aleph_1$. Thus we can continue the development of the above chain of free groups through limit ordinals (of cardinality not exceeding $\aleph_1$). Now suppose that $\mu - 1$ exists. Choose $A = A_\mu$ to be an $\aleph_1$-free group of cardinality $\aleph_1$ that is not free; such groups reside in a countable product of integers. Let F_μ be a free abelian group of cardinality $\aleph_1$ and let $\phi : F_\mu \twoheadrightarrow A$ be an epimorphism. We can identify $F_{\mu-1}$ with the kernel of ϕ (since $\operatorname{Ker} \phi$ cannot be countable), and obtain $F_\mu/F_{\mu-1} \cong A$. Thus we can also continue to develop the above chain of free groups when μ is isolated. The conclusion is that there exists a chain of free abelian groups

$$F_0 \subseteq F_1 \subseteq \ldots \subseteq F_\alpha \subseteq \ldots,$$

indexed by the ordinals α of cardinality at most $\aleph_1$, such that

(i) $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$ if β is a limit,

(ii) $|F_\alpha| = \aleph_1$ for each α,

(iii) $F_{\alpha+1}/F_\alpha$ is $\aleph_1$-free but not free for each α.

As we shall presently see, the direct limit of the chain of free groups satisfying conditions (i)–(iii) is $\aleph_2$-free but not free.

Let $G = \bigcup_{|\alpha|<\aleph_2} F_\alpha$. The group G cannot be free, for in that case, by an argument similar to the proof of Lemma 1.45 in [**3**] (with $\aleph_1$ replacing $\aleph_0$), F_α would be a direct summand of G for some α, which is impossible since $F_{\alpha+1}/F_\alpha$

is not free. However, every subgroup H of G of cardinality $\aleph_1$ is contained in F_α for some α and, therefore, is free.

In conclusion, we mention two general problems of interest concerning $\aleph_2$-free groups. First, how close to a free group can a group (of cardinality $\aleph_2$) be without being free? For example, in the construction that we have given above, what is the significance of letting $F_{\alpha+1}/F_\alpha$ be $\aleph_1$-separable (the existence of which is given by the Hill-Griffith construction [**4**; **3**, p. 105])? Secondly how far from a free group can an $\aleph_2$-free group be? For example, is it possible for an $\aleph_2$-free group of cardinality $\aleph_2$ to have a subgroup H of cardinality $\aleph_1$ such that G/H is divisible?

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Florida State University,
Tallahassee, Florida

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