EXTREME FORMS

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Contents

1. Introduction. The two ternary quadratic forms

$$\phi = x^2 + y^2 + z^2 - yz - xy, \quad \phi' = \tfrac{3}{2}x^2 + 2y^2 + \tfrac{3}{2}z^2 + 2yz + zx + 2xy$$

are said to be *reciprocal* (to each other) since their coefficients form inverse matrices:

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}, \qquad \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix}.$$

The corresponding determinants Δ are $\frac{1}{2}$ and 2. These forms are *positive definite*, since their values are positive for all values of (x, y, z) except $(0, 0, 0)$. Their minimum values M for integers x, y, z (not all zero) are 1 and 3/2. These minima are attained for the following sets of values of (x, y, z): in the case of ϕ,

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (1, 1, 0), \quad (1, 1, 1), \quad (0, 1, 1)$$

and the same with signs reversed; and in the case of ϕ',

$$(1, 0, 0), \quad (1, -1, 0), \quad (0, 1, -1), \quad (0, 0, 1)$$

and the same with signs reversed. Accordingly we say that the number of representations for the minimum (not distinguishing opposites) is s, where

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$s = 6$ for ϕ and $s = 4$ for ϕ'. These representations are the coefficients of x, y, z in linear forms

$$x,\ y,\ z,\ x + y,\ x + y + z,\ y + z \qquad \text{(for } \phi\text{)}$$

and

$$x,\ x - y,\ y - z,\ z \qquad \text{(for } \phi'\text{)}$$

whose sums of squares are

$$x^2 + y^2 + z^2 + (x + y)^2 + (x + y + z)^2 + (y + z)^2 = 2\phi',$$

$$x^2 + (x - y)^2 + (y - z)^2 + z^2 = 2\phi.$$

Such a form, whose reciprocal is the sum of positive multiples of the squares of its "minimal" linear forms, is said to be *eutactic*.

The above form ϕ is the only quadratic form $\phi(x, y, z)$ such that

$$\phi(1, 0, 0) = \phi(0, 1, 0) = \phi(0, 0, 1) = \phi(1, 1, 0) = \phi(1, 1, 1) = \phi(0, 1, 1) = 1.$$

To see this we write

$$\phi(x, y, z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{31}zx + 2a_{12}xy,$$

and obtain for the six unknown coefficients the six equations

$$a_{11} = a_{22} = a_{33} = a_{11} + a_{22} + 2a_{12} = a_{11} + a_{22} + a_{33} + 2a_{23} + 2a_{31} + 2a_{12}$$
$$= a_{22} + a_{33} + 2a_{23} = 1,$$

which imply

$$a_{11} = a_{22} = a_{33} = 1,\ 2a_{12} = 2a_{23} = -1,\ a_{31} = 0.$$

Thus ϕ is determined by its minimum and all the representations of this minimum. Such a form is said to be *perfect*. The reciprocal form ϕ' is not perfect, since it provides only four equations for the six coefficients. In fact, a necessary condition for an n-ary form to be perfect is

1.1 $$s \geqslant \tfrac{1}{2}n(n + 1).$$

It is interesting to see how a form is affected when the coefficients are infinitesimally changed. For instance, if ϵzx is added to ϕ, the minimum remains 1 (so long as ϵ is small and positive), but Δ is increased by $\frac{1}{4}\epsilon(1 - \epsilon)$. In fact, this form has the property that *any* small change, not affecting M, will increase Δ. On the other hand, if ϵzx is added to ϕ', M remains $3/2$ while Δ is *decreased* by $\frac{1}{2}\epsilon^2$. We express this difference of behaviour by saying that ϕ is an *extreme* form, but ϕ' is not.

We do not know whether every perfect form is extreme. But Korkine and Zolotareff [**32**, pp. 248-251] proved that every extreme form is perfect, and Voronoï [**42**, pp. 126-128] proved that every extreme form is eutactic. Conversely [**42**, pp. 128-130], every perfect eutactic form is extreme. (The form ϕ' is eutactic but not perfect, and therefore not extreme.) It seems to be a reasonable conjecture that every eutactic form with $s \geqslant \frac{1}{2}n(n + 1)$ should be extreme; but this has not been proved.

The above properties of a form are not changed when we apply to the variables a linear transformation of determinant ± 1 with integral coefficients [**1**, pp. 270-273] so as to obtain an *equivalent* form; e.g., ϕ is equivalent to both

$$x^2 + (-z)^2 + y^2 - (-z)y - x(-z) = x^2 + y^2 + z^2 + yz + zx$$

and

$$(-x)^2 + (y + z)^2 + z^2 - (y + z)z - (-x)(y + z) = x^2 + y^2 + z^2 + yz + zx + xy$$

[**41**, pp. 150-151]. It is sometimes convenient to extend the definition of equivalence so as to include the result of multiplying the given form by a constant c. Then M is multiplied by c, and Δ by c^n, but the combination M^n/Δ remains invariant. In this sense, the reciprocal form is equivalent to the "adjoint" form (whose matrix is adjoint to that of the given form).

Equivalent forms are said to belong to the same *class*. Gauss [**23**] observed that every binary extreme form is equivalent to $x^2 - xy + y^2$, and that every ternary extreme form is equivalent to ϕ. Korkine and Zolotareff [**30**; **31**; **32**] proved that there are just two classes of quaternary extreme forms and three classes of quinary extreme forms.

Gauss used a lattice in Euclidean n-space to represent any positive definite form, and showed that the corresponding *point*-lattice represents the class of equivalent forms. The case of 3-dimensional lattices and ternary forms has been very beautifully developed by Niggli [**32a**], who observed that reciprocal forms are represented by reciprocal lattices [**20**]. (A similar remark about "polar" forms had already been made by Bravais [**4a**].) In particular, the ternary forms ϕ and ϕ' yield the face-centred and body-centred cubic lattices, i.e., the ordinary cubic lattice plus the centres of its square faces or its cubic cells, respectively [**5**, p. 492].

Blichfeldt [**2**] observed that spheres of diameter $M^{\frac{1}{2}}$ centred at all the points of Gauss's lattice constitute a non-overlapping system or *packing* of spheres, and that extreme forms correspond to packings that are *rigid* in the sense that the "solid" spheres cannot be displaced without increasing the total content occupied by them. In fact, the elementary cell of the lattice has content $\Delta^{\frac{1}{2}}$, and therefore the requirement for a rigid packing is that any infinitesimal variation of the coefficients (leaving M unchanged) will increase Δ.

A systematic notation for a large family of eutactic forms, mostly extreme, is suggested by the classification of compact simple Lie groups. Cartan [**9**, pp. 216-228] used a discrete group generated by reflections to represent the local structure of such a continuous group. Stiefel [**40**, p. 374] showed that the various locally isomorphic Lie groups can be distinguished by considering the point-lattices that are invariant under the discrete group. This classification is worked out in detail in §10. Since one lattice may arise from several different discrete groups (see 10.6), the list of lattices is considerably shorter; in fact, these are the Gauss lattices for the following forms:

$$\begin{aligned}
A_n &= x_1^2 - x_1x_2 + x_2^2 - x_2x_3 + \ldots - x_{n-1}x_n + x_n^2,\\
A_n^r &= A_{n-1} - x_qx_n + \tfrac{1}{2}q(1 - r^{-1})x_n^2 \qquad (n = qr - 1 > 1, r > 1),\\
C_n &= 2A_n - x_n^2 = x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \ldots + (x_{n-1} - x_n)^2,\\
C_n^2 &= x_1^2 - x_1x_n + x_2^2 - x_2x_n + \ldots - x_{n-1}x_n + \tfrac{1}{4}nx_n^2,\\
D_n &= A_{n-1} - x_{n-2}x_n + x_n^2,\\
D_n^2 &= D_{n-1} - x_{n-1}x_n + \tfrac{1}{8}nx_n^2 \qquad (n \text{ even}),\\
E_6 &= A_5 - x_3x_6 + x_6^2,\\
E_6^3 &= A_6 - \tfrac{1}{3}x_5^2.
\end{aligned}$$

We shall find that the reciprocals of these forms are equivalent to A_n^{n+1}, A_n^q, C_n, D_n, C_n^2, D_n^2, E_6^3, E_6, respectively. All of them are eutactic; all except A_n^{n+1} $(n > 2)$, A_3^2, A_5^2, C_n, C_n^2 $(n \neq 4)$, D_4^2, D_6^2 are perfect, and therefore extreme.

In particular, A_n, D_n, D_n^2, E_6, A_7^2, A_5^3 are equivalent to the

$$U_n, \quad V_n, \quad W_n, \quad X, \quad Y, \quad Z$$

of Korkine and Zolotareff. On the other hand, their T_n is of an entirely different nature; in fact, many extreme forms with $n > 7$ are beyond the scope of the present treatment.

The new senary form E_6^3 is interesting for two reasons. First, its 27 pairs of minimal vectors correspond to the 27 lines on the general cubic surface (see §14). Second, it corroborates the number of extreme senary forms as computed by Hofreiter [**28**], but reveals a serious mistake in his work. Concerning his fourth form "F_4," for which $M = 2$ and $\Delta = 3^3\, 53/2^8$, the late Professor Blichfeldt wrote (in a letter dated May 18, 1944):

> I am certain that he made an error in §4, pp. 136-9, where he attempts to extend to 6 variables the form Z given by Korkine and Zolotareff.

We now see that Blichfeldt's skepticism was justified: $3E_6^3$ (with $M = 2$) has $\Delta = 3^5/2^6$, so Hofreiter's incredible factor 53 is replaced by 36. (In fact, "F_4" is neither perfect nor eutactic.)

Chaundy [**10**] reserves the title "extreme" for those forms which give M^n/Δ its greatest possible value for each n: an absolute maximum instead of a relative maximum. When $n \leqslant 6$, so that the extreme forms are all known, these *absolutely* extreme forms can be picked out at once. Blichfeldt [**2**] and Chaundy carried the work farther, so we know now that the absolutely extreme forms for $n \leqslant 9$ are Korkine and Zolotareff's

$$U_1, \quad U_2, \quad U_3, \quad V_4, \quad V_5, \quad X, \quad Y, \quad W_8, \quad T_9$$

(with a modicum of doubt in the last case, because Chaundy's argument is not quite valid). When these are adjusted so that $M = 2$, their determinants are

$$2, \quad 3, \quad 4, \quad 4, \quad 4, \quad 3, \quad 2, \quad 1, \quad 1.$$

O'Connor and Pall [**33**, p. 329] have observed that the absolutely extreme forms for $n \leqslant 8$ (adjusted so that $M = 1$) are integral positive forms of least possible determinant for each n. Such integral forms for $n > 8$ are no longer necessarily extreme; but we shall find some extreme ones in §12.

The absolutely extreme forms provide the densest packings of equal spheres whose centres form a lattice [**2**]. But the densest packings without this restriction on the centres present a far more difficult problem, which has only been solved in two dimensions [**41a**; **39a**]. Already in three dimensions [**1a**] there is a non-lattice packing just as dense as the cubic close-packing which represents the form ϕ. It is therefore conceivable that some irregular packing might be still denser.

2. Perfect and eutactic forms. Let $\sum\sum a_{ij}x^ix^j$ be a positive definite quadratic form in n variables; let M be the smallest value taken by the form for integers x^i, not all zero; and let Δ be the determinant of the coefficients a_{ij} $(= a_{ji})$. The form is said to be *extreme* if the ratio M^n/Δ decreases (or remains constant[1]) for every small change in the coefficients [**31**, p. 368]; that is, if M is maximum for a variation keeping Δ constant, or Δ is minimum for a variation keeping M constant.

Suppose the form attains its minimum M for the $2s$ sets of values

$$(x^1, \ldots, x^n) = \pm(m^{1k}, \ldots, m^{nk}) \qquad (k = 1, \ldots, s).$$

Waiving the distinction between opposites, we call these s *representations* of the minimum. The form is said to be *perfect* if it is uniquely determined by the value of its minimum and all the representations [**42**, p. 100]. If another form $\sum\sum(a_{ij} + b_{ij})x^ix^j$ had the same minimum and representations, we would find

$$\sum_i\sum_j(a_{ij} + b_{ij})m^{ik}m^{jk} = \sum_i\sum_j a_{ij}m^{ik}m^{jk}.$$

Hence, a necessary and sufficient condition for a perfect form is that the s linear equations

$$2.1 \qquad \sum\sum m^{ik}m^{jk}b_{ij} = 0 \qquad (k = 1, \ldots, s)$$

(for the $\frac{1}{2}n(n + 1)$ unknowns $b_{ij} = b_{ji}$) imply

$$b_{ij} = 0 \qquad (i, j = 1, \ldots, n),$$

i.e., that the matrix of coefficients $m^{ik}m^{jk}$, having $\frac{1}{2}n(n + 1)$ rows and s columns, is of rank $\frac{1}{2}n(n + 1)$. Another way of putting it is that there is no quadratic equation

$$\sum\sum b_{ij}x^ix^j = 0$$

satisfied by all the s sets of values $(x^1, \ldots, x^n) = (m^{1k}, \ldots, m^{nk})$.

[1]This alternative is introduced because Theorem 2.9 makes it desirable to include the unary form $a_{11}(x^1)^2$ among the extreme forms.

Thus 1.1 is a *necessary* condition for perfection.

Along with $\sum\sum a_{ij}x^ix^j$ we consider the *reciprocal* form $\sum\sum a^{ij}x_ix_j$, whose matrix is inverse to that of the original form, so that the cofactor of a_{ij} in Δ is Δa^{ij}. The form $\sum\sum a_{ij}x^ix^j$ is said to be *eutactic* if there exists s positive numbers $\rho_1, \ldots, \rho_s$ such that

2.2 $$\sum \rho_k(\sum m^{ik}x_i)^2 = \sum\sum a^{ij}x_ix_j.$$

In other words, a necessary and sufficient condition for a eutactic form is that s positive ρ's can be found to satisfy the $\frac{1}{2}n(n+1)$ linear equations

2.3 $$\sum_{k=1}^{s} m^{ik}m^{jk}\rho_k = a^{ij}.$$

The above condition for a perfect form implies that the equations 2.3 can be solved. But this does not suffice for a proof that every perfect form is eutactic, since we have no guarantee that a *positive* solution exists. Actually, for every known perfect form in less than nine variables there is a solution with the ρ's all *equal* (and therefore certainly all positive). Nor can we conclude that every eutactic form satisfying 1.1 is perfect: we would be assuming that the equations 2.3 are independent.

The above condition for a eutactic form is not very convenient in practice, as it demands knowledge of the reciprocal form. This disadvantage can be overcome by expressing the given form as a sum of squares, say

2.4 $$(\xi^1)^2 + (\xi^2)^2 + \ldots + (\xi^n)^2,$$

where

2.5 $$\xi^i = \sum c^i_j x^j \qquad (i = 1, \ldots, n).$$

Then the reciprocal form is simply $\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$, where ξ_i is related to x_i by the equations

$$x_j = \sum c^i_j \xi_i \qquad (j = 1, \ldots, n).$$

Considering integral values of x^i, we may suppose that the form attains its minimum M for the values

$$(\xi^1, \ldots, \xi^n) = \pm(\mu^{1k}, \ldots, \mu^{nk}) \qquad (k = 1, \ldots, s).$$

Since $\mu^{ik} = \sum c^i_j m^{jk}$, we have

$$\sum \mu^{ik}\xi_i = \sum\sum m^{jk}c^i_j\xi_i = \sum m^{jk}x_j.$$

Hence, a necessary and sufficient condition for the form $\sum(\xi^i)^2$ to be eutactic is

2.6 $$\sum \rho_k(\sum \mu^{ik}\xi_i)^2 = \sum \xi_i^2, \qquad \rho_k > 0.$$

In other words, the $\frac{1}{2}n(n+1)$ equations

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2.7 $$\sum_{k=1}^{s} \mu^{ik}\mu^{jk}\rho_k = \delta^{ij}$$

must have a positive solution $\rho_1, \ldots, \rho_s$.

For instance, to verify that the form ϕ' of §1 is eutactic, we observe that the expressions

$$\xi^1 = x^1 + x^3, \quad \xi^2 = x^1 + 2x^2 + x^3, \quad \xi^3 = x^1 - x^3$$

yield integral x's whenever the ξ's are integers of like parity. Thus the ternary form $(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2$ attains its minimum $M = 3$ for the values

$$(\xi^1, \xi^2, \xi^3) = (\pm 1, \pm 1, \pm 1) \qquad (s = 4);$$

and 2.6 becomes

$$\tfrac{1}{4}\sum(\xi_1 \pm \xi_2 \pm \xi_3)^2 = \xi_1^2 + \xi_2^2 + \xi_3^2.$$

Voronoï [**42**, pp. 126-130] proved an important theorem which can now be expressed as follows:

2.8 *A form is extreme if and only if it is both perfect and eutactic.*

Here is a somewhat simpler proof [cf. **32**, p. 244] of one half of this theorem, namely:

2.9 *Every perfect eutactic form is extreme.*

Let the coefficients of the form $\sum\sum a_{ij}x^ix^j$ be slightly varied, say from a_{ij} to $a_{ij} + \epsilon b_{ij}$ $(\epsilon > 0)$, in such a way that their determinant Δ remains constant while the b's do not all vanish. The constancy of Δ implies

$$\det(a_{ij}) = \det(a_{ij} + \epsilon b_{ij}) = \det(a_{ij}) + \sum\sum \epsilon\Delta a^{ij}b_{ij} + O(\epsilon^2),$$

whence $\sum\sum a^{ij}b_{ij} = O(\epsilon)$. Since we are interested in arbitrarily small variations, we make ϵ tend to zero and conclude that

$$\sum\sum a^{ij}b_{ij} = 0.$$

Since the form is eutactic, we can use the expression 2.3 for a^{ij}, so that

$$\sum_{k=1}^{s} \left(\sum\sum m^{ik}m^{jk}b_{ij}\right)\rho_k = 0,$$

where the ρ's are all positive. Since the form is perfect, the equations 2.1 have no non-trivial solution; hence there must be at least one value of k for which

$$\sum\sum m^{ik}m^{jk}b_{ij} < 0.$$

For such a k, since $\epsilon > 0$,

$$\sum\sum(a_{ij} + \epsilon b_{ij})m^{ik}m^{jk} < \sum\sum a_{ij}m^{ik}m^{jk}.$$

Thus the modified form $\sum\sum(a_{ij} + \epsilon b_{ij})x^ix^j$ attains a value less than M. Since the variation was arbitrary, this shows that the given form $\sum\sum a_{ij}x^ix^j$ has a greater minimum than any neighbouring form, i.e., it is extreme.

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3. The point-lattice representing a class of forms. The expression 2.4 may be interpreted, in Euclidean n-space, as the square of the distance from the origin to the point with rectangular Cartesian coordinates $(\xi^1, \ldots, \xi^n)$, or as the squared magnitude of the vector

$$\sum \xi^i \mathbf{p}_i,$$

where $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are unit vectors along the Cartesian axes. The equations 2.5 enable us to express the same vector as $\sum x^j \mathbf{t}_j$, where

3.1 $$\mathbf{t}_j = \sum c_j^i \mathbf{p}_i.$$

Thus the numbers x^j are the contravariant components of this vector, referred to the n vectors $\mathbf{t}_j$ as a covariant basis [**17**, pp. 178-180]. In other words, the point having Cartesian coordinates $(\xi^1, \ldots, \xi^n)$ has affine coordinates $(x^1, \ldots, x^n)$. Since

$$\sum\sum a_{ij}x^ix^j = \sum(\xi^i)^2 = (\sum\xi^i\mathbf{p}_i)^2 = (\sum x^j\mathbf{t}_j)^2 = \sum\sum x^ix^j\mathbf{t}_i \cdot \mathbf{t}_j,$$

the coefficients of our original form are the inner products of pairs of basic vectors:

3.2 $$a_{ij} = \mathbf{t}_i \cdot \mathbf{t}_j.$$

The set of vectors $\sum x^i \mathbf{t}_i$, where the x's take all integral values, has the property that the difference of any two of its members is a member. In other words, the vectors $\mathbf{t}_i$ generate a *lattice* [**26**, p. 26]. Thus any positive definite quadratic form determines a lattice. Conversely, any lattice (with a given basis) determines a positive definite form $\sum\sum(\mathbf{t}_i \cdot \mathbf{t}_j)x^ix^j$, and different bases determine equivalent forms [**6**, p. 30]. For instance, the lattice of unit squares may be generated by perpendicular unit vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, or equally well by $\mathbf{p}_1 - \mathbf{p}_2$ and $\mathbf{p}_2$. The corresponding forms are

$$(x^1\mathbf{p}_1 + x^2\mathbf{p}_2)^2 = (x^1)^2 + (x^2)^2$$

and

$$\begin{aligned}\{x^1(\mathbf{p}_1 - \mathbf{p}_2) + x^2\mathbf{p}_2\}^2 &= \{x^1\mathbf{p}_1 - (x^1 - x^2)\mathbf{p}_2\}^2 \\ &= (x^1)^2 + (x^1 - x^2)^2.\end{aligned}$$

In other words, there is a point-lattice for each class of equivalent forms.

Along with the covariant basis $\mathbf{t}_1, \ldots, \mathbf{t}_n$, we shall find it desirable to use the contravariant basis $\mathbf{t}^1, \ldots, \mathbf{t}^n$, where

$$\mathbf{t}^i \cdot \mathbf{t}_j = \delta_j^i$$

(which means that $\mathbf{t}^i$ is perpendicular to every $\mathbf{t}_j$ except $\mathbf{t}_i$, and $\mathbf{t}^i \cdot \mathbf{t}_i = 1$; see [**27a**] or [**17**, p. 180]). The vector $\sum x^i\mathbf{t}_i$ may now be expressed as $\sum x_i\mathbf{t}^i$, and its squared magnitude is the reciprocal form

$$(\sum x_i\mathbf{t}^i)^2 = \sum\sum a^{ij}x_ix_j,$$

where

$$a^{ij} = \mathbf{t}^i \cdot \mathbf{t}^j.$$

The corresponding lattice, based on the vectors $\mathbf{t}^i$, is reciprocal to the lattice based on the vectors $\mathbf{t}_i$ [**17**, p. 181]. Hence

3.3 *Reciprocal forms are represented by reciprocal lattices.*

All bases for a given point-lattice provide elementary cells (or period parallelotopes) of the same content. This is the geometrical counterpart of the fact that equivalent forms have the same determinant. For, *the parallelotope spanned by the vectors* $\mathbf{t}_i, \ldots, \mathbf{t}_n$ *has content* $\Delta^{\frac{1}{2}}$. This can be proved by induction over n, as follows. (It is obvious when $n = 1$ and $\Delta = a_{11}$.) Assuming the same result in $n - 1$ dimensions, where Δ is replaced by the cofactor of a_{nn}, which is Δa^{nn}, we have $(\Delta a^{nn})^{\frac{1}{2}}$ for the content of the parallelotope spanned by $\mathbf{t}_1, \ldots, \mathbf{t}_{n-1}$. To obtain the content of the n-dimensional parallelotope we must multiply this by the "altitude," which is the projection of $\mathbf{t}_n$ on the direction perpendicular to all of $\mathbf{t}_1, \ldots, \mathbf{t}_{n-1}$. Since the unit vector in that direction is $(a^{nn})^{-\frac{1}{2}}\mathbf{t}^n$, the projection is

$$(a^{nn})^{-\frac{1}{2}}\mathbf{t}^n \cdot \mathbf{t}_n = (a^{nn})^{-\frac{1}{2}},$$

and the desired content is $(\Delta a^{nn})^{\frac{1}{2}}(a^{nn})^{-\frac{1}{2}} = \Delta^{\frac{1}{2}}$. (For a direct proof, using 2.5, 3.1, 3.2 and a Jacobian, see Bachmann [**1**, pp. 273-275].)

A fundamental region for the translation group T, generated by the vectors $\mathbf{t}_1, \ldots, \mathbf{t}_n$, may be chosen in various ways; e.g., it may be the parallelotope spanned by these n vectors or by the basic vectors of any equivalent lattice (determining the same point-lattice). Fricke and Klein [**22**, pp. 108, 216] constructed a unique "standard" fundamental region, symmetrical about the origin. This is a polytope consisting of all points that are at least as near to the origin as to any other lattice point $(y^1, \ldots, y^n)$. Clearly, the aggregate of such polytopes surrounding all the lattice points is a honeycomb filling the whole space without interstices. Since the distance between points $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ is

$$\{\sum(x_i - y_i)(x^i - y^i)\}^{\frac{1}{2}},$$

where $x_i = \sum a_{ij}x^j$ and $y_i = \sum a_{ij}y^j$, this class of points is given by

$$\sum(x_i - y_i)(x^i - y^i) \geqslant \sum x_i x^i,$$

i.e., since $\sum y_i x^i = \sum\sum a_{ij}x^i y^j = \sum y^i x_i$,

$$\sum\sum a_{ij}y^i y^j - 2\sum y^i x_i \geqslant 0.$$

The polytope is given (in covariant coordinates x_i) by such inequalities for all sets of integers y^i; but of this infinite set of inequalities only a finite subset is effective. We see in this manner that Fricke and Klein's standard fundamental region is the same as the "parallelohedron" described by Voronoï [**43**, p. 278; **1**, pp. 145, 334].

The lattice points nearest to the origin represent those integral values of $(x^1, \ldots, x^n)$ for which $\sum\sum a_{ij}x^ix^j$ attains its minimum value M. The geometrical figure formed by these $2s$ points

$$(m^{1k}, \ldots, m^{nk}), \quad (-m^{1k}, \ldots, -m^{nk}),$$

or by the $2s$ *minimal vectors*

$$\pm \sum m^{ik}\mathbf{t}_i \qquad (k = 1, \ldots, s),$$

is the same for the whole class of equivalent forms. Let us call it the *vertex figure* of the point-lattice. For instance, the body-centred and face-centred cubic lattices, representing the forms φ' and φ of §1, have for respective vertex figures a cube and a cuboctahedron.

A point-lattice has an infinite group of symmetry operations, in which the translation group $\mathbf{T}$ occurs as a self-conjugate abelian subgroup whose quotient group is finite. In fact, the quotient group is isomorphic to the subgroup that leaves the origin invariant [**6**, p. 103]. In terms of the corresponding form, the operations of this finite subgroup are those linear transformations of $x^1, \ldots, x^n$ which leave $\sum\sum a_{ij}x^ix^j$ invariant, i.e., they are the *automorphs* of the form. Thus every automorph is represented by a symmetry operation of the vertex figure.

If the point-lattice can be generated by n of its minimal vectors, it is determined by its vertex figure, and therefore every symmetry operation of the vertex figure is a symmetry operation of the whole point-lattice. But if the point-lattice cannot be so generated, its vertex figure may possibly have "accidental" symmetry operations which give new positions to some more distant points. Hence,

3.4 *If some n of the minimal vectors are independent (so that the vertex figure is properly n-dimensional), the group of automorphs of the form is a subgroup of the symmetry group of the vertex figure. It is the whole symmetry group if these n minimal vectors generate the point-lattice.*

These geometrical considerations make it evident that equivalent forms (represented by equivalent lattices) and their reciprocals (represented by the reciprocal lattices) have isomorphic groups of automorphs.

We saw in §2 that a positive definite form is perfect if and only if there is no quadratic equation satisfied by all the representations of its minimum. In terms of the point-lattice,

3.5 *A necessary and sufficient condition for a form to be perfect is that there be no quadric cone (degenerate or non-degenerate) whose generators include all the minimal vectors.*

In our chosen system of affine coordinates, $\sum\sum a_{ij}x^ix^j = 1$ is the equation of a sphere of unit radius. If $M \geqslant 1$, the point-lattice given by integral values of the x^j includes no interior point of the sphere except its centre (the origin).

Accordingly, we say that such a lattice is *admissible* for the unit sphere. (Clearly, a lattice having $M < 1$ is *not* admissible.) An admissible lattice whose elementary cell has the smallest possible content occurs when both M and Δ/M^n are as small as possible, viz, when $M = 1$ and the form is absolutely extreme. In other words,

The critical lattices of the unit sphere are the lattices representing absolutely extreme forms with $M = 1$.

This remark, kindly contributed by K. Mahler, supersedes the detailed argument of Mrs. Ollerenshaw [**34** and **35**].

4. Eutactic stars of vectors. Let us define a *star* to be a set of $2s$ vectors $\pm\, \mathbf{a}^1, \ldots, \pm\, \mathbf{a}^s$, issuing from a fixed origin in Euclidean n-space. Following Schläfli [**36**, p. 298] we call this a *eutactic* star if the sum of the squares of the orthogonal projections of $\mathbf{a}^1, \ldots, \mathbf{a}^s$ on a line is the same in all directions, i.e., if there is a constant λ such that

$$\sum(\mathbf{a}^k \cdot \mathbf{x})^2 = \lambda \mathbf{x}^2. \tag{4.1}$$

The simplest instance is when $s = n$ and the $\mathbf{a}^k$ are mutually orthogonal unit vectors; then $\mathbf{a}^k \cdot \mathbf{x}$ is the magnitude of the projection of $\mathbf{x}$ on $\mathbf{a}^k$, and $\lambda = 1$. In the general case

$$\lambda = \sum\, (\mathbf{a}^k)^2/n$$

[**17**, p. 261, with s and n interchanged]. The connection between eutactic stars and eutactic forms is supplied by the following theorem:

4.2 *A form is eutactic if and only if its minimal vectors are parallel to the vectors of a eutactic star.*

Proof. Let the form $\sum\sum a_{ij}x^ix^j$ have minimal vectors $\pm\,\mathbf{m}^1, \ldots, \pm\,\mathbf{m}^s$, where $\mathbf{m}^k = \sum m^{jk}\mathbf{t}_j$ $(k = 1, \ldots, s)$. Then

$$\mathbf{m}^k \cdot \mathbf{t}^i = \sum m^{jk}\mathbf{t}_j \cdot \mathbf{t}^i = \sum m^{jk}\delta^i_j = m^{ik}.$$

First, let the form be eutactic, so that, for certain positive numbers ρ_k,

$$\sum \rho_k(\sum m^{ik}x_i)^2 = \sum\sum a^{ij}x_ix_j.$$

Then the vectors $\mathbf{a}^k = \rho_k^{\frac{1}{2}}\mathbf{m}^k$ satisfy the condition

$$\begin{aligned}\sum(\mathbf{a}^k \cdot \mathbf{x})^2 &= \sum \rho_k(\mathbf{m}^k \cdot \sum x_i\mathbf{t}^i)^2 = \sum \rho_k(\sum m^{ik}x_i)^2\\ &= \sum\sum a^{ij}x_ix_j = (\sum x_i\mathbf{t}^i)^2 = \mathbf{x}^2,\end{aligned}$$

which is 4.1 with $\lambda = 1$.

Conversely, if the minimal vectors $\mathbf{m}^k$ are parallel to the vectors $\mathbf{a}^k = \sigma_k\mathbf{m}^k$ of a eutactic star, we have

$$\begin{aligned}\sum\sigma_k^2(\sum m^{ik}x_i)^2 &= \sum(\sigma_k\mathbf{m}^k \cdot \sum x_i\mathbf{t}^i)^2 = \sum(\mathbf{a}^k \cdot \mathbf{x})^2 = \lambda\mathbf{x}^2\\ &= \lambda\sum\sum a^{ij}x_ix_j,\end{aligned}$$

which is the condition for a eutactic form (with $\rho_k = \sigma_k^2/\lambda$).

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Schläfli [36, p. 302] deduced from 4.1

$$\sum(\mathbf{a}^k \cdot \mathbf{x})\,(\mathbf{a}^k \cdot \mathbf{y}) = \lambda \mathbf{x} \cdot \mathbf{y},$$

whence $\sum \mathbf{a}^k\mathbf{a}^k \cdot \mathbf{x} = \lambda\mathbf{x}$. In the notation of Hadwiger [25], this is simply

$$\mathrm{T}\mathbf{x} = \lambda\mathbf{x}.$$

Hadwiger proved that the vectors $\mathbf{a}^k$ of any eutactic star can be derived by orthogonal projection from s mutually perpendicular vectors of magnitude $\lambda^{\frac{1}{2}}$ in Euclidean s-space.

Schläfli proved that the vectors from the centre of any regular polytope to its vertices constitute a eutactic star. This was generalized by Brauer and Coxeter [4, Theorem 1 with $h = 1$] as follows:

A star is eutactic if there is an irreducible group of orthogonal transformations which is transitive on the pairs of opposite vectors.

Since any two eutactic stars with the same origin form together a eutactic star [17, p. 253], we can deduce the following still more general result:

A star is eutactic if it is transformed into itself by an irreducible group of orthogonal transformations.

Hence, by 4.2,

4.3 *A form is eutactic if its vertex figure is invariant under an irreducible group of orthogonal transformations.*

This may be proved more directly, as follows. Since the hyperplanes through the origin perpendicular to the minimal vectors are

$$\sum m^{ik}x_i = 0 \qquad (k = 1, \ldots, s),$$

and this set of hyperplanes is invariant under an irreducible group, the expression $\sum(\sum m^{ik}x_i)^2$ must be proportional to the unique quadratic invariant $\sum\sum a^{ij}x_ix_j$. Thus 2.2 is satisfied with the ρ's all equal.

It should be remembered that the above conditions are sufficient but not necessary. Korkine and Zolotareff's nonary form T_9 [31, p. 367] is extreme, and therefore eutactic, although its group of automorphs is reducible. In such cases the appropriate criterion is 2.3 or 2.7.

5. Reflexible forms. A form is said to be *disconnected* if it is a sum of two or more forms involving separate sets of variables [17, p. 175]. A form that is not disconnected is *connected.* If a definite form is disconnected, we can name its variables in such an order that x^1 is in one set and x^n in another. Then every representation of the minimum has either its first or its last coordinate zero, i.e., the minimal vectors all satisfy the quadratic equation $x^1x^n = 0$. But a form cannot be perfect if its minimal vectors all satisfy a quadratic equation. Hence a disconnected form is not perfect. In other words,

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5.1 *Every perfect form is connected.*

Let a definite form be called a *reflexible* form if its point-lattice Λ is symmetrical by the reflections that reverse the n basic vectors in turn. The reflecting hyperplane for a basic vector OP may be taken either through the origin O or through the midpoint of OP; for, if either of these reflections is a symmetry operation of the point-lattice, so also is the other. The reflections in the n hyperplanes through O generate a finite group S. By adjoining the reflections in the parallel hyperplanes through the midpoints, or the translations along the basic vectors, we obtain an infinite discrete group G. The hyperplanes and their transforms decompose the Euclidean space into an infinity of congruent (or symmetric) polytopes, any one of which may be used as a fundamental region for G. G is generated by the reflections in the bounding hyperplanes of this region, which is either a simplex or a "simplicial prism" —the rectangular product of several simplexes [**13**, p. 599]. In other words, G is either the group generated by reflections in the bounding hyperplanes of a simplex (whose dihedral angles are submultiples of π) or the direct product of several such groups.

Of course, the same point-lattice Λ represents not only the given reflexible form but also all equivalent forms, some of which may not be reflexible. Λ is derived from the origin O by applying either the whole group G or (equally well) its translation subgroup T, whose quotient group G/T is isomorphic to S. We shall choose the fundamental region for G in such a position that O is one of its vertices. (See [**17**, pp. 191, 205], where this is called a *special* vertex.)

If the finite group S is reducible, it leaves a certain subspace invariant. Each basic vector $\mathbf{t}_k$ lies either in this subspace or in the completely orthogonal subspace, and the n basic vectors fall into two sets, all of the one set being perpendicular to all of the other, so that the form is disconnected. Hence, if the form is connected the group must be irreducible. Combining this result with 4.3, we see that

5.2 *Every connected reflexible form is eutactic.*

We proceed to investigate the possible coefficients of such a form.

The projection of any vector $\mathbf{x}$ in the direction of the basic vector $\mathbf{t}_k$, or of the unit vector $a_{kk}^{-\frac{1}{2}}\mathbf{t}_k$, is $a_{kk}^{-\frac{1}{2}}\mathbf{t}_k \,.\, \mathbf{x} = a_{kk}^{-\frac{1}{2}}x_k$. Hence the reflection in the perpendicular hyperplane through the origin changes $\mathbf{x}$ into

5.3 $$\mathbf{x} - 2a_{kk}^{-1}x_k\mathbf{t}_k.$$

Since $\mathbf{x} = \sum x^i\mathbf{t}_i$, this reflection leaves invariant every contravariant component x^i except x^k, which is changed into

$$x^k - \frac{2x_k}{a_{kk}} = x^k - \sum_{i=1}^{n} \frac{2a_{ik}}{a_{kk}}x^i.$$

Since the point-lattice Λ is given by integral values of the x^i, a necessary and

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sufficient condition for the form to be reflexible [**44**, p. 369] is that $2a_{ik}/a_{kk}$ be integral for all i and k. Hence, if such a form is *connected*, its coefficients a_{ii} and $2a_{ik}(i\neq k)$ must be all commensurable, and we can suppose them to be integers with no common divisor greater than 1. Since the form must remain positive definite when all but two of the x's vanish, we have $a_{ii}a_{kk} > a_{ik}^2$, whence

$$\frac{2a_{ik}}{a_{ii}}\;\frac{2a_{ik}}{a_{kk}} < 4 \qquad (i \neq k).$$

Thus, whenever $a_{ik}\neq 0$, $|2a_{ik}/a_{ii}|$ and $|2a_{ik}/a_{kk}|$ must be two positive integers whose product is less than 4: one of them is 1 and the other 1 or 2 or 3.

The ternary forms admitted by this restriction (with x, y, z instead of x^1, x^2, x^3, for simplicity) are:

$$\begin{array}{ll} x^2 + pxy + py^2 + \;\; pyz + \;\;\; z^2, & \Delta = \tfrac{1}{2}p(2-p);\\ px^2 + pxy + \;\; y^2 + \;\; qyz + \;\; qz^2, & \Delta = \tfrac{1}{4}pq(4-p-q);\\ x^2 + pxy + py^2 + pqyz + pqz^2, & \Delta = \tfrac{1}{4}p^2q(4-p-q);\\ x^2 + \;\; xy + \;\; y^2 + \;\; pyz + \;\; pz^2 + pzx, & \Delta = \tfrac{1}{4}p(3-p);\\ x^2 + pxy + py^2 + \;\; pyz + \;\; pz^2 + pzx, & \Delta = \tfrac{1}{4}p^2(3-p); \end{array}$$

and the same with minus signs in any of the product terms. In the first three cases such changes of sign merely yield equivalent forms; thus the condition $\Delta > 0$ implies $p = 1$ in the first case, and in the next two cases $p + q < 4$: one of p and q is 1 and the other 1 or 2. In the last two cases one or three changes of sign would replace the factor $3 - p$ by $3(1 - p)$, which cannot be positive; so we may keep the positive signs and conclude that $p = 1$ or 2.

Thus coefficients 3 cannot occur in ternary or higher forms, but only in binary forms such as

$$3x^2 \pm 3xy + y^2 = x^2 + x(x \pm y) + (x \pm y)^2,$$

which are equivalent to $x^2 + xy + y^2$. Apart from this simple case, we can assume every a_{ii} to be 1 or 2, and every $2a_{ij}(i \neq j)$ to be 0 or $\pm$ 1 or $\pm$ 2, namely 0 or $\pm$ 1 whenever $a_{ii} = a_{jj} = 1$, and 0 or $\pm$ 2 otherwise.

Instead of enumerating all the possible arrangements of 1's and 2's that will make the form definite, let us simplify the discussion by taking the basic vectors $\mathbf{t}_i$ to be perpendicular to those bounding hyperplanes of the fundamental region which meet at the special vertex O. The angle between any two of these n vectors, being the supplement of a dihedral angle of the fundamental region, is greater than or equal to a right angle [**17**, p. 189]. Hence

$$a_{ij} = \mathbf{t}_i \cdot \mathbf{t}_j \leqslant 0 \qquad (i \neq j).$$

This means that any connected reflexible form is equivalent to one in which every non-vanishing a_{ij} is negative, namely

5.4 $$2a_{ij} = -\max(a_{ii}, a_{jj}) \qquad (i \neq j).$$

To determine which such forms are definite we may modify them by taking new variables

$$\mathrm{x}^i = a_{ii}^{\frac{1}{2}} x^i, \tag{5.5}$$

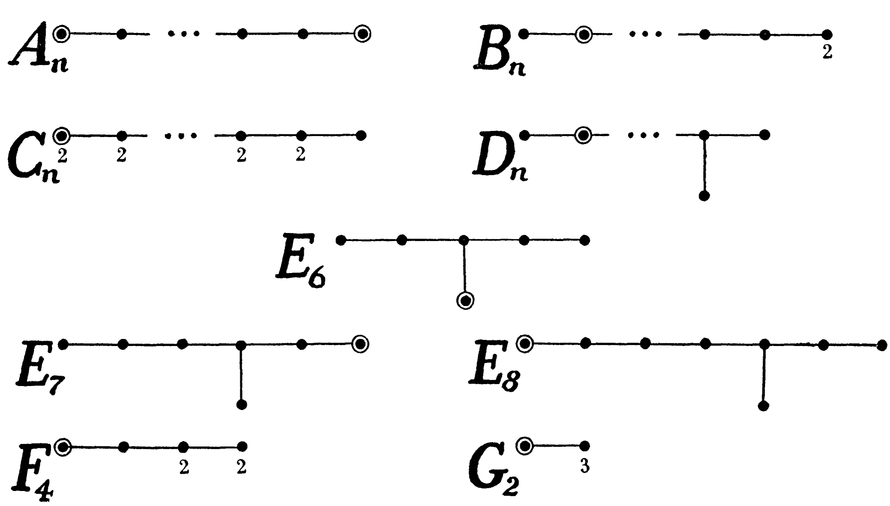
so that $\sum\sum a_{ij} x^i x^j$ becomes $\sum\sum \mathrm{a}_{ij} \mathrm{x}^i \mathrm{x}^j$ with

$$\mathrm{a}_{ii} = 1, \; 2\mathrm{a}_{ij} = -\nu^{\frac{1}{2}} \qquad (i \neq j; \; \nu = 0, 1, 2 \text{ or } 3).$$

These modified forms have been enumerated elsewhere [**45**, p. 301; **17**, p. 195]. The corresponding list of connected reflexible forms, with $M = 1$, is as follows:

$$A_n = (x^1)^2 - x^1x^2 + (x^2)^2 - \ldots + (x^{n-1})^2 - x^{n-1}x^n + (x^n)^2 \qquad (n = 1, 2, \ldots),$$
$$C_n = 2(x^1)^2 - 2x^1x^2 + 2(x^2)^2 - \ldots + 2(x^{n-1})^2 - 2x^{n-1}x^n + (x^n)^2 \qquad (n = 2, 3, \ldots),$$
$$B_n = (x^1)^2 - x^1x^2 + (x^2)^2 - \ldots + (x^{n-1})^2 - 2x^{n-1}x^n + 2(x^n)^2 \qquad (n = 3, 4, \ldots),$$
$$D_n = (x^1)^2 - x^1x^2 + (x^2)^2 - \ldots + (x^{n-1})^2 - x^{n-2}x^n + (x^n)^2 \qquad (n = 4, 5, \ldots),$$
$$E_n = (x^1)^2 - x^1x^2 + (x^2)^2 - \ldots + (x^{n-1})^2 - x^{n-3}x^n + (x^n)^2 \qquad (n = 6, 7, 8),$$
$$F_4 = (x^1)^2 - x^1x^2 + (x^2)^2 - 2x^2x^3 + 2(x^3)^2 - 2x^3x^4 + 2(x^4)^2,$$
$$G_2 = (x^1)^2 - 3x^1x^2 + 3(x^2)^2.$$

It is often convenient to denote these forms by graphs whose nodes and branches represent the square terms and product terms, as follows:



The values of a_{ii}, other than 1, are marked under the nodes; $2a_{ij}$ can then be deduced from 5.4. The rings that have been drawn round one or two nodes

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are to be ignored for the moment, but will be found useful in §7. The letters A, B, etc. have been rearranged [cf. **17**, p. 297] to agree with the notation of Cartan [**9**, pp. 218-225].

If we are interested in finding the simplest representative of each different class of reflexible forms, we can eliminate all the coefficients 2 and 3 by observing that B_n is equivalent to D_n, F_4 to D_4, G_2 to A_2, and C_n to the disconnected form $\sum(x^i)^2$. In fact,

$$\begin{aligned}
C_n &= (x^1)^2 + (x^1 - x^2)^2 + (x^2 - x^3)^2 + \ldots + (x^{n-1} - x^n)^2,\\
B_n &= (x^1)^2 - x^1x^2 + (x^2)^2 - \ldots + (x^{n-2})^2 - x^{n-2}(x^{n-1} - x^n)\\
&\qquad\qquad + (x^{n-1} - x^n)^2 - x^{n-2}x^n + (x^n)^2,\\
F_4 &= (x^1)^2 - x^1(x^2 - x^3) + (x^2 - x^3)^2 - x^1(x^3 - x^4)\\
&\qquad\qquad + (x^3 - x^4)^2 - x^1x^4 + (x^4)^2,\\
G_2 &= (x^1 - x^2)^2 - (x^1 - x^2)x^2 + (x^2)^2.
\end{aligned}$$

Hence

5.6 *Any connected reflexible form is equivalent to one having every* $a_{ii} = 1$.

This means that the lattice Λ is generated by minimal vectors. By 5.4, each of the remaining coefficients $2a_{ij}$ is either 0 or -1. (By reversing the signs of alternate x's we could arrange to have $+1$ instead of -1, but the negative sign is really preferable, as we shall see in the proof of 5.7.)

We shall call the forms A_n, B_n, D_n, E_n, F_4 and G_2 *properly connected* because they remain connected after the above reduction. But C_n, being equivalent to $\sum(x^i)^2$, is not properly connected, and therefore (by 5.1) not perfect. We proceed to prove that

5.7 *Every properly connected reflexible form is perfect.*

Since equivalent forms are perfect or imperfect together, we may restrict consideration to the forms A_n, D_n, E_n, whose graphs are trees without any marks 2 or 3. The minimal vectors $\mathbf{m} = \sum m^i\mathbf{t}_i$ include the n basic vectors $\mathbf{t}_i$ themselves (since $a_{ii} = 1$) and also any "connected series"

$$\mathbf{t}_i + \mathbf{t}_j + \mathbf{t}_k + \ldots + \mathbf{t}_l,$$

where $2a_{ij} = 2a_{jk} = \ldots = -1$; for when the only non-vanishing x's are $x^i = x^j = x^k = \ldots = x^l = 1$, the form becomes

$$\begin{aligned}
(x^i)^2 - x^ix^j + (x^j)^2 - x^jx^k + (x^k)^2 - \ldots + (x^l)^2&\\
= 1 - 1 + 1 - 1 + 1 - \ldots + 1 = 1.&
\end{aligned}$$

As a test for perfection (cf. 2.1), consider the s equations

$$\sum\sum m^im^jb_{ij} = 0,$$

one for each minimal vector $\mathbf{m}$. Setting $\mathbf{m} = \mathbf{t}_i$ (so that $m^j = \delta^i_j$), we obtain

$$b_{ii} = 0.$$

Setting $\mathbf{m} = \mathbf{t}_i + \mathbf{t}_j$ (where $2a_{ij} = -1$) we obtain

$$b_{ij} = 0.$$

Similarly, a connected series $\mathbf{m} = \mathbf{t}_i + \mathbf{t}_j + \mathbf{t}_k$ yields

$$b_{ik} = 0;$$

and the number of terms in the series can be gradually increased. Since every pair of $\mathbf{t}$'s make the beginning and end of a connected series, all the b's must vanish. Therefore the form is perfect.

Combining 5.7 with 5.2, we deduce from 2.9,

5.8 *Every properly connected reflexible form is extreme.*

It is easily proved by induction that the extreme forms $2A_n$, $2D_n$, $2E_n$ have respective determinants $n + 1$, 4, $9 - n$. This explains why E_n is definite only when $n \leqslant 8$. The three types overlap as follows:

$$\begin{aligned} D_3 &= A_3 \quad \text{(with } x^1 \text{ and } x^2 \text{ interchanged)},\\ E_4 &= A_4 \quad \text{(with } x^1 \text{ and } x^3 \text{ interchanged)},\\ E_5 &= D_5 \quad \text{(with the order } x^1x^2x^3x^4 \text{ reversed)}. \end{aligned}$$

We might even go one stage farther back and say

$$E_3 = (x^1)^2 - x^1x^2 + (x^2)^2 + (x^3)^2 = A_2 + A_1.$$

O'Connor and Pall's integral positive forms of least determinant for $n \leqslant 8$ [**33**, p. 329] are (apart from trivial changes of sign)

$$A_1, A_2, A_3, B_4, B_5$$

and three equivalent to E_n ($n = 6, 7, 8$), namely

$$\begin{aligned} f_n = {} & x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + \ldots + x_{n-3}^2 + x_{n-3}(x_{n-2} - x_n)\\ & + (x_{n-2} - x_n)^2 + (x_{n-2} - x_n)(x_{n-1} + 2x_n) + (x_{n-1} + 2x_n)^2\\ & + x_{n-3}x_n + x_n^2. \end{aligned}$$

6. The minimal vectors. In the present section we shall see how the minimal vectors can be computed for each of the forms represented by trees on page 405. We shall also see how a certain one of the minimal vectors provides formulae for the number s and for the order of the group $\mathbf{S}$. We again use $\mathbf{m}$ to denote a typical minimal vector $\sum m^i \mathbf{t}_i = \sum m_i \mathbf{t}^i$, and we use $\mathbf{z}$ to denote the particular minimal vector whose contravariant components z^i are as large as possible. Thus, if the given form is

$$\sum\sum a_{ij}x^ix^j = f(x^1, \ldots, x^n),$$

we have

$$f(m^1, \ldots, m^n) = f(z^1, \ldots, z^n) = M = 1.$$

The components $m^1, \ldots, m^n$ are conveniently associated with the nodes of the tree. (We may write them *above* the nodes, as a_{ii} has already been written below some of them.) Some of these contravariant components may vanish, but *the nodes that represent non-vanishing m^i's must be the nodes of a connected part of the tree.* For otherwise each separate set of such nodes would represent a positive definite form; and the number $f(m^1, \ldots, m^n)$, being a sum of several positive integers, could not be equal to 1.

The expressions $f(m^1, \ldots, m^n)$ and $f(|m^1|, \ldots, |m^n|)$ differ only in those terms (if any) for which $m^i m^j < 0$ while $a_{ij} < 0$. Since these are positive terms of the former and negative terms of the latter,

$$f(|m^1|, \ldots, |m^n|) \leqslant f(m^1, \ldots, m^n) = 1.$$

But since 1 is the minimum value, $f(|m^1|, \ldots, |m^n|) \geqslant 1$. Hence

$$f(|m^1|, \ldots, |m^n|) = f(m^1, \ldots, m^n),$$

and in fact there are no such terms: $m^i m^j \geqslant 0$ whenever $a_{ij} < 0$. Thus, in the connected part of the tree where non-vanishing m^i's occur, any two adjacent m^i's have the same sign; consequently *all non-vanishing m^i's have the same sign.* Since

$$f(-m^1, \ldots, -m^n) = f(m^1, \ldots, m^n),$$

we may restrict attention to positive values and suppose that every

$$m^i \geqslant 0.$$

For instance, the values of (m^1, m^2) for $G_2 = (x^1)^2 - 3x^1x^2 + 3(x^2)^2$ are $(1, 0)$, $(1, 1)$, $(2, 1)$ and their opposites; i.e., the "positive" minimal vectors in this case are

$$\mathbf{t}_1, \quad \mathbf{t}_1 + \mathbf{t}_2, \quad 2\mathbf{t}_1 + \mathbf{t}_2.$$

Given the contravariant components m^i of a minimal vector $\mathbf{m}$, we can easily compute the covariant components

$$m_k = \sum a_{ki} m^i.$$

For, if the kth node of the tree is joined to the ith, jth, etc., we see from 5.4 that

6.1 $$2m_k = \begin{cases} a_{kk}(2m^k - m^i - m^j - \ldots) & \text{if } a_{kk} > 1, \\ 2m^k - a_{ii}m^i - a_{jj}m^j - \ldots & \text{if } a_{kk} = 1. \end{cases}$$

Thus every $2m_k$ is an integer, in fact, a multiple of a_{kk}.

We proceed to prove that the only possible values for the integer $2m_k$ are 0 and $\pm a_{kk}$, except that when $a_{kk} = 1$ we may have $\mathbf{m} = \mathbf{t}_k$, in which case $m_k = m^k = 1$.

The increment of $f(m^1, \ldots, m^n)$ when one m^k is replaced by $m^k \pm 1$ is

$$(\mathbf{m} \pm \mathbf{t}_k) \cdot (\mathbf{m} \pm \mathbf{t}_k) - \mathbf{m} \cdot \mathbf{m} = \mathbf{t}_k \cdot \mathbf{t}_k \pm 2\mathbf{m} \cdot \mathbf{t}_k = a_{kk} \pm 2m_k.$$

Since $M = 1$, the increment cannot be negative, save in the case when $\mathbf{m} = \mathbf{t}_k$

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is reduced to the zero vector. Hence, except in this trivial case, we have $|2m_k| \leqslant a_{kk}$. Since $2m_k$ is a multiple of a_{kk}, it must be either 0 or $\pm a_{kk}$. But if $\mathbf{m} = \mathbf{t}_k$, so that every m^i vanishes except $m^k = 1$, we have $2m_k = 2$ (by 6.1 with $a_{kk} = 1$).

Moreover, since $\sum m^i m_i = \mathbf{m} \cdot \mathbf{m} = 1$ and $m^i \geqslant 0$, at least one of the integers $2m_i$ must be positive (namely $2m_k = a_{kk}$ for some k, unless $\mathbf{m}$ is basic).

Since the increment of $\mathbf{m} \cdot \mathbf{m}$ is $a_{kk} \pm 2m_k$ when $\mathbf{m}$ is replaced by $\mathbf{m} \pm \mathbf{t}_k$, a given minimal vector yields a new one when we make this change for a value of k such that $2m_k = \mp a_{kk}$. Given any minimal vector that is not basic, we can carry out this procedure with the lower sign, subtracting 1 from m^k for a value of k that makes $2m_k = a_{kk}$, thereby changing $2m_k$ from a_{kk} to $-a_{kk}$ (by 6.1). If the new $\mathbf{m}$ is still not basic we can repeat the procedure; but since the sum $\sum m^i$ is diminished, this can only be done a finite number of times, and eventually we must be left with $\mathbf{m} = \mathbf{t}_i$. (In this final stage, $2m_i = 2$ but, by 6.1 again, every other $2m_k \leqslant 0$.) It follows, by reversing the process (and using the upper sign), that the whole family of "positive" minimal vectors can be built up by starting with those basic vectors $\mathbf{t}_i$ for which $a_{ii} = 1$ and successively adding some $\mathbf{t}_k$ (i.e., increasing m^k by 1) whenever $2m_k < 0$. This will continue until $2m_k \geqslant 0$ for every k.

For instance, in the case of G_2, where $a_{11} = 1$ and $a_{22} = 3$, we have

$$2m_1 = 2m^1 - 3m^2, \qquad 2m_2 = 3(2m^2 - m^1),$$

and the minimal vectors can be written down successively as in the following table:

(m^1, m^2)	$(2m_1, 2m_2)$
(1, 0)	(2, −3)
(1, 1)	(−1, 3)
(2, 1)	(1, 0)

In the case of B_3, where $a_{11} = a_{22} = 1$ and $a_{33} = 2$, we have

$$2m_1 = 2m^1 - m^2, \quad 2m_2 = 2m^2 - m^1 - 2m^3, \quad 2m_3 = 2(2m^3 - m^2),$$

and the first two basic vectors yield other minimal vectors as follows:

(m^1, m^2, m^3)	$(2m_1, 2m_2, 2m_3)$
(1, 0, 0) (0, 1, 0)	(2, −1, 0) (−1, 2, −2)
(1, 1, 0) (0, 1, 1)	(1, 1, −2) (−1, 0, 2)
(1, 1, 1)	(1, −1, 2)
(1, 2, 1)	(0, 1, 0)

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As a third example consider A_3, where $a_{11} = a_{22} = a_{33} = 1$ so that

$$2m_1 = 2m^1 - m^2, \quad 2m_2 = 2m^2 - m^1 - m^3, \quad 2m_3 = 2m^3 - m^2$$

and all the basic vectors are minimal:

(m^1, m^2, m^3)	$(2m_1, 2m_2, 2m_3)$
(1, 0, 0) (0, 1, 0) (0, 0, 1)	(2, −1, 0) (−1, 2, −1) (0, −1, 2)
(1, 1, 0) (0, 1, 1)	(1, 1, −1) (−1, 1, 1)
(1, 1, 1)	(1, 0, 1)

The final vector $\mathbf{m}$ (where $\sum m^k$ is maximum and every $2m_k$ is 0 or a_{kk}) is of such special importance that we shall use a different letter and call it $\mathbf{z}$. Since $f(1, 1, \ldots, 1) = 1$, every $z^k \geqslant 1$. Moreover, since

$$\sum 2z_k z^k = 2\mathbf{z}\cdot\mathbf{z} = 2,$$

only one or two of the $2z_k$'s can take the value a_{kk}: all the rest must be zero. This means that the vector $2\mathbf{z}$ is

6.2 $$a_{kk}\mathbf{t}^k \quad \text{or} \quad \mathbf{t}^k + \mathbf{t}^l$$

for one or two particular numbers k, l (with $a_{kk} = a_{ll} = 1$ in the latter case). (Thus $2\mathbf{z} = \mathbf{t}^1$ for G_2, $\mathbf{t}^2$ for B_3, and $\mathbf{t}^1 + \mathbf{t}^3$ for A_3. A complete list can be read off from the ringed nodes on page 405.)

Geometrically, the operation of adding 1 to m^k (when $2m_k = -a_{kk}$) is the reflection R_k that reverses $\mathbf{t}_k$; for, by 5.3, this reflection changes $\mathbf{m}$ into

$$\mathbf{m} - 2a_{kk}^{-1}m_k\mathbf{t}_k = \mathbf{m} + \mathbf{t}_k.$$

Thus every minimal vector can be derived from a basic vector $\mathbf{t}_i$ (with $a_{ii} = 1$) by applying some operation of the group S generated by $R_1, R_2, \ldots, R_n$. It follows that every minimal vector is reversed by a reflection belonging to S (viz, by one conjugate to R_i). In particular, S contains a reflection that reverses $\mathbf{z}$. Here the reflecting hyperplane is perpendicular to $\mathbf{z}$ through O. Let R_0 denote the reflection in the parallel hyperplane through the midpoint of the segment defining $\mathbf{z}$. This hyperplane cuts off from the angular fundamental region for S a simplex all of whose dihedral angles are submultiples of π. For, the corresponding unit vector

6.3 $$\mathbf{t}_0 = -\mathbf{z}$$

makes with the basic vectors $\mathbf{t}_k$ $(k = 1, \ldots, n)$ the inner products

$$\mathbf{t}_0 \cdot \mathbf{t}_k = -z_k = 0 \quad \text{or} \quad -\tfrac{1}{2}a_{kk};$$

and since $a_{00} = 1$, the cosine of the corresponding dihedral angle is

$$-\mathbf{t}_0\cdot\mathbf{t}_k/a_{kk}^{\frac{1}{2}} = 0 \quad \text{or} \quad \tfrac{1}{2}a_{kk}^{\frac{1}{2}},$$

where $a_{kk} = 1$ or 2 or 3. Since the infinite group **G** can be derived from **S** by adjoining R_0, this simplex serves as a fundamental region. The $n + 1$ vectors $\mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_n$, perpendicular to the bounding hyperplanes of the simplex, yield the semidefinite form

$$\begin{aligned}(x^0\mathbf{t}_0 + x^1\mathbf{t}_1 + \ldots + x^n\mathbf{t}_n)^2 &= (x^0\mathbf{t}_0)^2 - 2x^0(x^1z_1 + \ldots + x^nz_n) \\ &\qquad + (x^1\mathbf{t}_1 + \ldots + x^n\mathbf{t}_n)^2 \\ &= (x^0)^2 - x^0 \sum 2z_kx^k + f(x^1, \ldots, x^n),\end{aligned}$$

which is never negative but vanishes when $x^k = z^k$ and $x^0 = 1$. It is therefore natural to define

6.4 $$z^0 = 1,$$

so that $$z^0\mathbf{t}_0 + z^1\mathbf{t}_1 + \ldots + z^n\mathbf{t}_n = \mathbf{t}_0 + \mathbf{z} = 0 \qquad [17, \text{p. } 183\ (10{\cdot}72)].$$

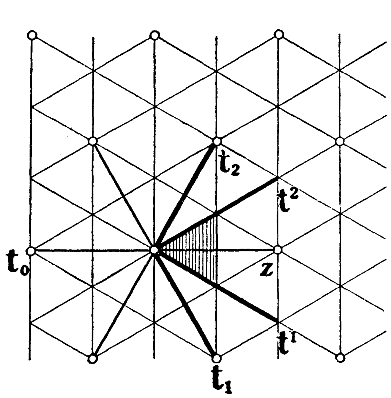

FIG. 1

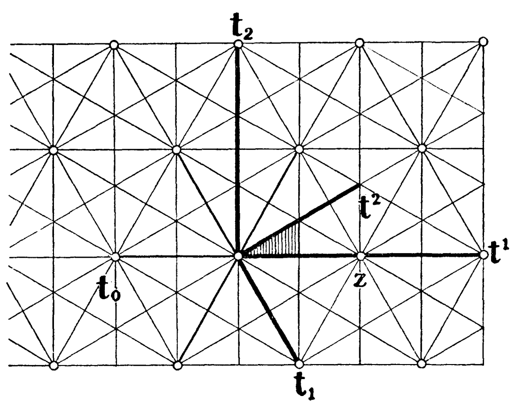

FIG. 2

Figs. 1 and 2 illustrate the two equivalent forms A_2 and G_2, whose "positive" minimal vectors are respectively

$$\mathbf{t}_1,\ \mathbf{t}_2,\ \mathbf{t}_1 + \mathbf{t}_2 = \mathbf{z} = \tfrac{1}{2}(\mathbf{t}^1 + \mathbf{t}^2)$$

and $$\mathbf{t}_1,\ \mathbf{t}_1 + \mathbf{t}_2,\ 2\mathbf{t}_1 + \mathbf{t}_2 = \mathbf{z} = \tfrac{1}{2}\mathbf{t}^1.$$

Returning to the general discussion, we see that the semidefinite form is represented by a graph which is derived from the tree by adding an $(n + 1)$th node, joined to the one or two nodes for which $2z_k \neq 0$. The actual cases, in the same order as in §5, are exhibited on page 412.

In each case the nodes have been marked with the numbers a_{kk} below and z^k above, whenever these numbers are greater than 1. (The marks a_{kk} are an essential part of the symbol, but the values of z^k can be derived from them, as we shall soon see.) It is to be understood that 5.4 holds whenever the ith

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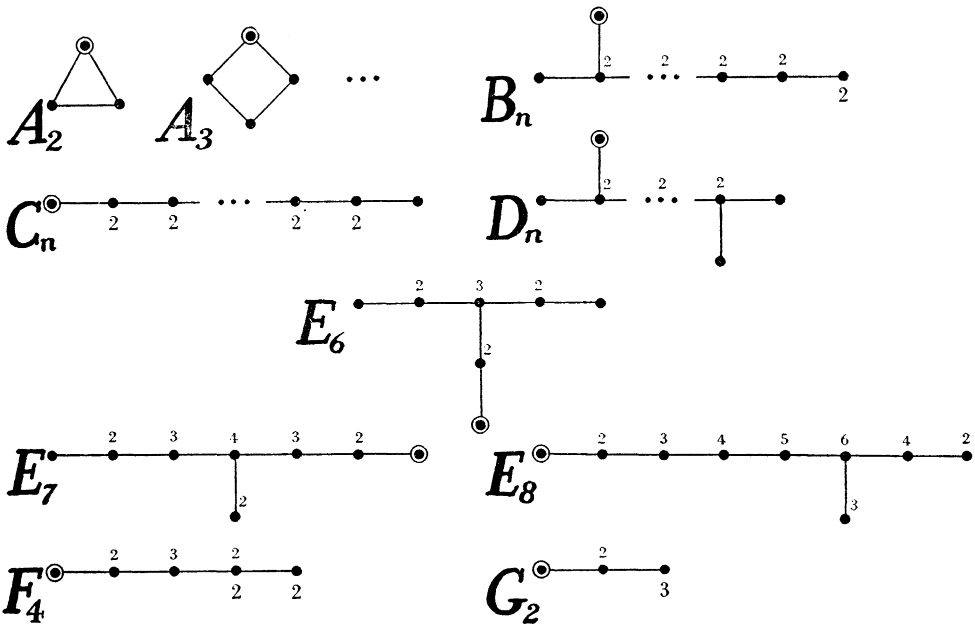

and jth nodes are joined by a branch,[2] even if $i = 0$. The node representing $\mathbf{t}_0$ has been ringed.

Given the semidefinite form $(\sum_0^n x^i \mathbf{t}_i)^2$, we can write down the z^i's, beginning with $z^0 = 1$, by the following simple rule: If the node representing $\mathbf{t}_k$ is joined to those representing $\mathbf{t}_i$, $\mathbf{t}_j$, etc., we have

6.5 $$2z^k = \begin{cases} z^i + \quad z^j + \ldots & \text{if } a_{kk} > 1, \\ a_{ii}z^i + a_{jj}\, z^j + \ldots & \text{if } a_{kk} = 1 \end{cases}$$

This rule follows at once from 6.1 (with $m_k = z_k = 0$) except when z^0 is involved. It remains to be verified when i or k is zero. If $i = 0$, so that the k-node is joined to the 0-node and also to one or more j-nodes, 6.1 gives

$$2z_k = \begin{cases} a_{kk}(2z^k - \sum z^j) & (a_{kk} > 1), \\ 2z^k - \sum a_{jj}z^j & (a_{kk} = 1). \end{cases}$$

But, by 6.2, $2z_k = a_{kk}$. Hence

$$2z^k = \begin{cases} 1 + \sum z^j & (a_{kk} > 1), \\ 1 + \sum a_{jj}z^j & (a_{kk} = 1). \end{cases}$$

This is 6.5 with $i = 0$, $z^0 = 1$ and $a_{00} = 1$. Finally, to verify 6.5 with $k = 0$, we observe that 6.2 gives either

$$\mathbf{z} = \tfrac{1}{2}a_{ii}\mathbf{t}^i \quad \text{or} \quad \mathbf{z} = \tfrac{1}{2}(\mathbf{t}^i + \mathbf{t}^j).$$

[2]The unary form A_1 has been omitted from the above list, because the corresponding semidefinite form $(x^0)^2 - 2x^0x^1 + (x^1)^2$ violates this rule.

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In the former case every covariant component vanishes except $z_i = \frac{1}{2}a_{ii}$, and therefore

$$z^0 = 1 = z_i z^i = \tfrac{1}{2}a_{ii}z^i,$$

as desired. In the latter case $z_i = z_j = \frac{1}{2}$, and therefore

$$z^0 = 1 = z_i z^i + z_j z^j = \tfrac{1}{2}(z^i + z^j) = \tfrac{1}{2}(a_{ii}z^i + a_{jj}z^j).$$

It is interesting to compare these results with [**17**, pp. 176-194], where the vectors $\mathbf{e}_k$ have the same directions as the present vectors $\mathbf{t}_k$, but different magnitudes: $|\mathbf{e}_k|$ is the reciprocal of the distance between two parallel hyperplanes, whereas $|\mathbf{t}_k|$ is twice this distance (so that the translation along $\mathbf{t}_k$ is the product of two parallel reflections). Moreover, we now have a different unit of measurement: not just the distance between closest parallel hyperplanes but twice this distance. Taking these two changes into consideration, we can reconcile the expression 11.93 [**17**, p. 207] with our present notation by defining

$$y^k = a_{kk}z^k.$$

Then the order of the group S is

6.6 $$f y^0 y^1 \ldots y^n n!,$$

where f (called h in [**9**]) is the number of y^k's that are equal to 1 (including $y^0 = z^0$, which is always 1), i.e., f is the number of "special" vertices of the simplex. (The "paramètre dominant" of Borel and De Siebenthal [**3a**, p. 219] is $\sum y^k \varphi_k$.)

The expression 6.6, involving the product of the y's, has an interesting companion involving their sum: The number of minimal vectors is

$$2s = c(y^0 + y^1 + \ldots + y^n),$$

where c is the number of a_{kk}'s that are equal to 1 (*excluding* a_{00}), i.e., c is the number of minimal basic vectors. This empirical formula can be verified in each case from the table on page 414; but no explanation for it has so far been found.

The groups S are listed in the notation of [**14**]. The last two columns will be explained in §§7 and 8.

7. The point-lattices and their vertex figures. In §5 we saw that the translations and reflections associated with the lattice for a reflexible form generate an infinite discrete group G which contains a finite subgroup S generated by reflections in hyperplanes through the original lattice point O, and we found it desirable to restrict consideration to forms for which these hyperplanes are perpendicular to the n basic vectors. In §6 we saw that, in the case of a connected form, the fundamental region for G is a simplex bounded by these n hyperplanes and one more, perpendicular to the vector $\mathbf{z}$ (see 6.3). It follows that the lattice points are the vertices of a honeycomb whose

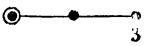
TABLE OF CONNECTED REFLEXIBLE FORMS

Form	S	Order	f	$y^0y^1\cdots y^n$	$2s$	c	$y^0+y^1+\cdots+y^n$	Π	Λ
A_n	$[3^{n-1}]$	$(n+1)!$	$n+1$	1	$n(n+1)$	n	$n+1$	$t_{0,n-1}\alpha_n$	$\alpha_n \mathrm{h}$
C_n	$[3^{n-2}, 4]$	$2^n n!$	2	2^{n-1}	$2n$	1	$2n$	β_n	δ_{n+1}
B_n					$2n(n-1)$	$n-1$	$2n$	$\mathrm{t}_1\beta_n$	$\mathrm{h}\delta_{n+1}$
D_n	$[3^{n-3,1,1}]$	$2^{n-1}n!$	4	2^{n-3}	$2n(n-1)$	n	$2(n-1)$		
E_6	$[3^{2,2,1}]$	$72 \cdot 6!$	3	24	72	6	12	1_{22}	2_{22}
E_7	$[3^{3,2,1}]$	$8 \cdot 9!$	2	288	126	7	18	2_{31}	3_{31}
E_8	$[3^{4,2,1}]$	$192 \cdot 10!$	1	17280	240	8	30	4_{21}	5_{21}
F_4	$[3, 4, 3]$	1152	1	48	24	2	12	$\{3, 4, 3\}$	$\{3, 3, 4, 3\}$
G_2	$[6]$	12	1	6	6	1	6	$\{6\}$	$\{3, 6\}$

graphical symbol (in the notation of [**17**, p. 196]) is essentially one of those listed on page 412. For this purpose the marks z^k (above the nodes) are to be disregarded, and the marks a_{kk} are to be modified in accordance with the transformation 5.5. By 5.4, the cosine of the dihedral angle between the hyperplanes perpendicular to $\mathbf{t}_i$ and $\mathbf{t}_j$ is

$$- a_{ij}(a_{ii}a_{jj})^{-\frac{1}{2}} = \tfrac{1}{2}\max\{(a_{ii}/a_{jj})^{\frac{1}{2}}, (a_{jj}/a_{ii})^{\frac{1}{2}}\}.$$

Since $\cos \pi/4 = \frac{1}{2}2^{\frac{1}{2}}$ and $\cos \pi/6 = \frac{1}{2}3^{\frac{1}{2}}$, the proper way to mark the branches of the graph is as follows: When $a_{ii} = a_{jj}$ we make no mark (and the angle $\pi/3$ is understood); but when a_{ii} and a_{jj} are different the appropriate mark is *twice the quotient of the larger by the smaller;* e.g., the symbol

 , meaning  , becomes  .

The particular cases are given in the final column of the above table. No confusion need be caused by adopting the same symbol Λ for the honeycomb and for the point-lattice of its vertices. (Cf. [**14**, p. 468], where Λ was called Π^+.)

The honeycomb $\alpha_n h$ [**11**, p. 366], whose graphical symbol consists of an $(n + 1)$-gon with one vertex ringed, has been described by Schoute [**37**] as an oblique section of the $(n + 1)$-dimensional cubic honeycomb; e.g.,

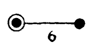
$$\alpha_2 h = \{3, 6\}, \qquad \alpha_3 h = \left\{3, \begin{matrix}3\\4\end{matrix}\right\}$$

in the notation of [**17**, pp. 59, 69, 87-88]. The different cells of $\alpha_n h$ are the regular simplex α_n and all its simple truncations $t_l\alpha_n$.

The n-dimensional cubic honeycomb is denoted (not too happily) by δ_{n+1}. Its alternate vertices form Schoute's "half measure polytope net" $h\delta_{n+1}$ [**38**, p. 90; **17**, pp. 156, 201]; e.g.,

$$h\delta_2 = \delta_2, \quad h\delta_3 = \delta_3, \quad h\delta_4 = \alpha_3 h, \quad h\delta_5 = \{3, 3, 4, 3\}.$$

The penultimate column of the table contains the polytope Π which is the vertex figure of Λ. Its graphical symbol is derived by removing the ringed node and ringing the adjacent node or nodes [**15**, p. 336]. Its $2s$ vertices, being the extremities of the vectors $\pm\mathbf{m}$, are derived from the extremity of $\mathbf{z}$ by applying the operations of $\mathbf{S}$. When $\mathbf{z} = \frac{1}{2}a_{kk}\mathbf{t}^k$, this point lies in the direction of $\mathbf{t}^k$, and the kth node of the tree is ringed; but when $\mathbf{z} = \frac{1}{2}(\mathbf{t}^k + \mathbf{t}^l)$ it lies on the bisector of the angle between $\mathbf{t}^k$ and $\mathbf{t}^l$, so the kth and lth nodes are both ringed. The particular cases are the trees on page 405, with marks transferred from nodes to branches by the above rule; e.g.,

 becomes 

(and the remaining cases can be seen in [**14**, p. 472, the third column of Table I, omitting the last six entries]).

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The "expanded simplex" $t_{0,n-1}\alpha_n$ [**15**, p. 331], sometimes called $e\alpha_n$ [**11**, p. 366], has for its cells the rectangular products $\alpha_{n-k}\times\alpha_{k-1}$ or $[\alpha_{n-k}, \alpha_{k-1}]$ for all values of k from 1 to n; e.g., $\alpha_{n-1}\times\alpha_0$ is just the regular simplex α_{n-1}, and $\alpha_{n-2}\times\alpha_1$ is a right prism based on α_{n-2}. In particular, $t_{0,1}\alpha_2$ is the regular hexagon $\{6\}$, $t_{0,2}\alpha_3$ is the cuboctahedron $\left\{\begin{matrix}3\\4\end{matrix}\right\}$, and $t_{0,3}\alpha_4$ is a four-dimensional polytope whose cells consist of ten tetrahedra and ten triangular prisms (as we may see by analysing the tree in the manner of [**15**, pp. 329, 335]).

The "cross polytope" β_n is the n-dimensional analogue of the square β_2 and octahedron β_3 [**17**, p. 121]. The midpoints of its edges are the vertices of the truncated cross polytope $t_1\beta_n$ or

$$\left\{\begin{matrix}3\\3, \ldots, 3, 4\end{matrix}\right\}$$

[**17**, pp. 147, 200]. In particular, $t_1\beta_4$ is the regular 24-cell $\{3, 4, 3\}$, whose twelve diagonals meet a 3-space of general position in the vertices of three desmic tetrahedra [**27**, p. 170].

For the remaining polytopes and honeycombs (1_{22}, 2_{31}, 4_{21}, and 2_{22}, 3_{31}, 5_{21}), see [**11**, pp. 414, 415] and the analytic treatment in §8.

8. The use of an orthogonal basis. For further discussion of the extreme form A_n (which is absolutely extreme when $n = 2$ or 3) it is convenient to embed the n-space in an $(n + 1)$-space spanned by $n + 1$ mutually orthogonal unit vectors $\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}$. In fact,

$$\begin{aligned}2A_n &= (x^1)^2 + (x^1 - x^2)^2 + (x^2 - x^3)^2 + \ldots + (x^{n-1} - x^n)^2 + (x^n)^2\\ &= \{x^1\mathbf{p}_1 - (x^1 - x^2)\mathbf{p}_2 - (x^2 - x^3)\mathbf{p}_3 - \ldots - (x^{n-1} - x^n)\,\mathbf{p}_n - x^n\mathbf{p}_{n+1}\}^2\\ &= \{x^1(\mathbf{p}_1 - \mathbf{p}_2) + x^2(\mathbf{p}_2 - \mathbf{p}_3) + \ldots + x^n(\mathbf{p}_n - \mathbf{p}_{n+1})\}^2.\end{aligned}$$

Comparing this with $A_n = (x^1\mathbf{t}_1 + x^2\mathbf{t}_2 + \ldots + x^n\mathbf{t}_n)^2$, we see that

$$\mathbf{t}_i = 2^{-\frac{1}{2}}(\mathbf{p}_i - \mathbf{p}_{i+1}) \qquad (i = 1, \ldots, n),$$

whence $\mathbf{z} = \mathbf{t}_1 + \ldots + \mathbf{t}_n = 2^{-\frac{1}{2}}(\mathbf{p}_1 - \mathbf{p}_{n+1})$. (These vectors all lie in the n-space perpendicular to $\mathbf{p}_1 + \ldots + \mathbf{p}_{n+1}$.) The reflection R_i, which reverses $\mathbf{t}_i$, now appears as the transposition $(\mathbf{p}_i\mathbf{p}_{i+1})$; so $\mathbf{S}$ is $[3^{n-1}]$, the symmetric group of degree $n + 1$. Applying these operations to $\mathbf{z}$, and changing the scale to get rid of the multiplier $2^{-\frac{1}{2}}$, we see that Π is the expanded simplex $t_{0,n-1}\alpha_n$ whose $n(n + 1)$ vertices are obtained by permuting

$$(1, 0, \ldots, 0, -1)$$

(with $n - 1$ zeros) in the n-space $\xi^1 + \ldots + \xi^{n+1} = 0$. In other words, $t_{0,n-1}\alpha_n$ is the polytope determined by this equation along with the inequality

$$|\xi^1| + \ldots + |\xi^{n+1}| \leqslant 2.$$

It follows that Λ is the simplicial honeycomb $\alpha_n\mathrm{h}$ whose vertices are all the points having $n + 1$ integral coordinates with sum zero. For instance, the permutations of $(1, 0, -1)$ are the vertices of the hexagon $t_{0,1}\alpha_2$, and the lattice points (ξ^1, ξ^2, ξ^3) with $\xi^1 + \xi^2 + \xi^3 = 0$ form the tessellation of triangles, α_2h.

By choosing a different basis we can obtain an equivalent form; e.g., the basis $\mathbf{p}_i - \mathbf{p}_{n+1}$ $(i = 1, \ldots, n)$ yields

$$\{\sum x^i(\mathbf{p}_i - \mathbf{p}_{n+1})\}^2 = (\sum x^i \mathbf{p}_i - \sum x^i \mathbf{p}_{n+1})^2 = \sum (x^i)^2 + (\sum x^i)^2,$$

which is proportional to the U_n of Korkine and Zolotareff [**31**, p. 367].

When $n = 2$, the basis $\mathbf{p}_1 - \mathbf{p}_2$, $\mathbf{p}_2 - \mathbf{p}_3 - (\mathbf{p}_1 - \mathbf{p}_2)$ yields

$$\begin{aligned}\{x^1(\mathbf{p}_1 - \mathbf{p}_2) + x^2(-\mathbf{p}_1 + 2\mathbf{p}_2 - \mathbf{p}_3)\}^2 \\ &= \{(x^1 - x^2)\mathbf{p}_1 - (x^1 - 2x^2)\mathbf{p}_2 - x^2\mathbf{p}_3\}^2 \\ &= (x^1 - x^2)^2 + (x^1 - 2x^2)^2 + (x^2)^2 \\ &= 2G_2.\end{aligned}$$

On the other hand, the pair of equivalent forms B_n and D_n (which are absolutely extreme when $n = 3, 4$ or 5) are more naturally treated with reference to an n-dimensional orthogonal basis. So likewise is the imperfect form C_n. In fact,

$$\begin{aligned}C_n &= (x^1)^2 + (x^1 - x^2)^2 + \ldots + (x^{n-1} - x^n)^2 \\ &= \{x^1\mathbf{p}_1 - (x^1 - x^2)\mathbf{p}_2 - \ldots - (x^{n-1} - x^n)\mathbf{p}_n\}^2 \\ &= \{x^1(\mathbf{p}_1 - \mathbf{p}_2) + \ldots + x^{n-1}(\mathbf{p}_{n-1} - \mathbf{p}_n) + x^n\mathbf{p}_n\}^2,\end{aligned}$$

whence

$$\mathbf{t}_i = \mathbf{p}_i - \mathbf{p}_{i+1} \quad (i = 1, \ldots, n-1) \quad \text{and } \mathbf{t}_n = \mathbf{p}_n,$$

so that

$$\mathbf{z} = \mathbf{t}_1 + \ldots + \mathbf{t}_n = \mathbf{p}_1.$$

The reflections $\mathrm{R}_1, \ldots, \mathrm{R}_{n-1}$ are again transpositions, but R_n reverses the sign of $\mathbf{p}_n$; so $\mathbf{S}$ is the "hyper-octahedral" group $[3^{n-2}, 4]$, of order $2^n n!$, which permutes the n vectors of the orthogonal basis and changes their signs. Thus the minimal vectors are simply $\pm\mathbf{p}_i$, and Π is the cross polytope β_n whose $2n$ vertices are obtained by permuting

$$(\pm 1, 0, \ldots, 0).$$

In other words, β_n is the polytope determined by

$$|\xi^1| + \ldots + |\xi^n| \leqslant 1.$$

It follows that Λ is the cubic lattice δ_{n+1} whose vertices are all the points having n integral coordinates.

Similarly

$$\begin{aligned}2B_n &= (x^1)^2 + (x^1 - x^2)^2 + \ldots + (x^{n-2} - x^{n-1})^2 + (x^{n-1} - 2x^n)^2 \\ &= \{x^1\mathbf{p}_1 - (x^1 - x^2)\mathbf{p}_2 - \ldots - (x^{n-2} - x^{n-1})\mathbf{p}_{n-1} - (x^{n-1} - 2x^n)\mathbf{p}_n\}^2 \\ &= \{x^1(\mathbf{p}_1 - \mathbf{p}_2) + \ldots + x^{n-1}(\mathbf{p}_{n-1} - \mathbf{p}_n) + 2x^n\mathbf{p}_n\}^2,\end{aligned}$$

whence

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$$\mathbf{t}_i = 2^{-\frac{1}{2}}(\mathbf{p}_i - \mathbf{p}_{i+1}) \quad (i = 1, \ldots, n-1) \quad \text{and } \mathbf{t}_n = 2^{\frac{1}{2}}\mathbf{p}_n,$$

so that

$$\mathbf{z} = \mathbf{t}_1 + 2(\mathbf{t}_2 + \ldots + \mathbf{t}_{n-1}) + \mathbf{t}_n = 2^{-\frac{1}{2}}(\mathbf{p}_1 + \mathbf{p}_2).$$

Since the reflections $R_1, \ldots, R_n$ are the same as before, S is again $[3^{n-2}, 4]$. (Cf. Stiefel [**41**, p. 183].) Thus the minimal vectors are $\pm \mathbf{p}_i \pm \mathbf{p}_j$ $(i \neq j)$, and Π is the truncated cross polytope $t_1\beta_n$ whose $2n(n+1)$ vertices are obtained by permuting

$$(\pm 1, \pm 1, 0, 0, \ldots, 0)$$

(with $n - 2$ zeros). In other words, $t_1\beta_n$ is determined by

$$|\xi^i| \leqslant 1 \quad \text{and} \quad \sum |\xi^i| \leqslant 2.$$

Λ is now the alternated cubic lattice $h\delta_{n+1}$ whose vertices are all the points having n integral coordinates with an even sum [**17**, p. 158]. For instance, the permutations of

$$(\pm 1, \pm 1, 0, 0)$$

are the vertices of the regular 24-cell $t_1\beta_4$, and the lattice points $(\xi^1, \xi^2, \xi^3, \xi^4)$ with $\xi^1 + \xi^2 + \xi^3 + \xi^4 \equiv 0 \pmod 2$ form the regular honeycomb of 16-cells, $h\delta_5$.

The form D_n, equivalent to B_n, is given by

$$\begin{aligned}\{x^1(\mathbf{p}_1 - \mathbf{p}_2) + \ldots + x^{n-1}(\mathbf{p}_{n-1} - \mathbf{p}_n) + x^n(\mathbf{p}_{n-1} + \mathbf{p}_n)\}^2 \\ = (x^1)^2 + (x^1 - x^2)^2 + \ldots + (x^{n-3} - x^{n-2})^2 + (x^{n-2} - x^{n-1} - x^n)^2 \\ + (x^{n-1} - x^n)^2 \\ = 2D_n.\end{aligned}$$

In this case S is the group $[3^{n-1,1,1}]$, of order $2^{n-1}n!$, which permutes the n vectors of the orthogonal basis while changing any even number of signs.

Another possible basis, for a form equivalent to $2D_n$, is

$$\mathbf{p}_n - \mathbf{p}_i \quad (i = 1, \ldots, n-1), \quad \mathbf{p}_n + \mathbf{p}_{n-1},$$

giving

$$\begin{aligned}\{\sum_1^{n-1} x^i(\mathbf{p}_n - \mathbf{p}_i) + x^n(\mathbf{p}_n + \mathbf{p}_{n-1})\}^2 &= \sum_1^{n-2} (x^i)^2 + (x^{n-1} - x^n)^2 + (\sum_1^n x^i)^2 \\ &= \sum_1^n (x^i)^2 + (\sum_1^n x^i)^2 - 2x^{n-1}x^n,\end{aligned}$$

which is proportional to Korkine and Zolotareff's V_n.

Again, we might take

$$\mathbf{p}_i - \mathbf{p}_{i+1} \quad (i = 1, \ldots, n-1), \quad \mathbf{p}_1 + \mathbf{p}_n,$$

which yields

$$\left\{\sum_1^{n-1} x^i(\mathbf{p}_i - \mathbf{p}_{i+1}) + x^n(\mathbf{p}_1 + \mathbf{p}_n)\right\}^2$$
$$= (x^1 + x^n)^2 + (x^1 - x^2)^2 + \ldots + (x^{n-1} - x^n)^2$$
$$= 2(A_n + x^1x^n).$$

When n is odd we can reverse the signs of $x^2, x^4, x^6, \ldots, x^{n-1}$ to obtain the cyclically symmetrical form

$$(x^1)^2 + x^1x^2 + (x^2)^2 + x^2x^3 + \ldots + (x^n)^2 + x^nx^1.$$

So this again is equivalent to D_n. (But such a symmetrical form with n even is obviously semidefinite.)

F_4, equivalent to D_4, is given by

$$\{x^1(\mathbf{p}_1 - \mathbf{p}_2) + x^2(\mathbf{p}_2 - \mathbf{p}_3) + x^3(-\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4) + x^4(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4)\}^2$$
$$= (x^1 - x^3 + x^4)^2 + (x^1 - x^2 + x^3 - x^4)^2 + (x^2 - x^3 - x^4)^2 + (x^3 - x^4)^2$$
$$= 2F_4;$$

and in this case $\mathbf{S}$ is [3, 4, 3], of order 1152 [**17**, p. 149].

When considering A_n we embedded the Euclidean n-space in a Euclidean $(n + 1)$-space. In dealing with B_n, C_n and D_n, no such embedding was necessary. In the case of E_n (which is absolutely extreme when $n = 5, 6, 7$ or 8) we shall find it convenient to embed the Euclidean n-space in a *Minkowskian* $(n + 1)$-space, i.e., to use $n + 1$ "vectors" $\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}$ which are mutually orthogonal with

$$\mathbf{p}_1^2 = 1, \ldots, \mathbf{p}_n^2 = 1, \quad \text{but } \mathbf{p}_{n+1}^2 = -1.$$

In fact,

$$\begin{aligned}
2E_n &= (x^1)^2 + (x^1 - x^2)^2 + (x^2 - x^3)^2 + \ldots + (x^{n-4} - x^{n-3})^2 \\
&\quad + (x^{n-3} - x^{n-2} - x^n)^2 + (x^{n-2} - x^{n-1} - x^n)^2 + (x^{n-1} - x^n)^2 - (x^n)^2 \\
&= \{x^1\mathbf{p}_1 - (x^1 - x^2)\mathbf{p}_2 - (x^2 - x^3)\mathbf{p}_3 - \ldots - (x^{n-4} - x^{n-3})\mathbf{p}_{n-3} \\
&\quad - (x^{n-3} - x^{n-2} - x^n)\mathbf{p}_{n-2} - (x^{n-2} - x^{n-1} - x^n)\mathbf{p}_{n-1} - (x^{n-1} - x^n)\mathbf{p}_n \\
&\qquad + x^n\mathbf{p}_{n+1}\}^2 \\
&= \{x^1(\mathbf{p}_1 - \mathbf{p}_2) + x^2(\mathbf{p}_2 - \mathbf{p}_3) + \ldots + x^{n-3}(\mathbf{p}_{n-3} - \mathbf{p}_{n-2}) \\
&\quad + x^{n-2}(\mathbf{p}_{n-2} - \mathbf{p}_{n-1}) + x^{n-1}(\mathbf{p}_{n-1} - \mathbf{p}_n) \\
&\qquad + x^n(\mathbf{p}_{n-2} + \mathbf{p}_{n-1} + \mathbf{p}_n + \mathbf{p}_{n+1})\}^2,
\end{aligned}$$

whence

$$2^{\frac{1}{2}}\mathbf{t}_i = \mathbf{p}_i - \mathbf{p}_{i+1} \quad (i = 1, \ldots, n-1) \text{ and } 2^{\frac{1}{2}}\mathbf{t}_n = \mathbf{p}_{n-2} + \mathbf{p}_{n-1} + \mathbf{p}_n + \mathbf{p}_{n+1}$$

(all lying in the Euclidean n-space perpendicular to the time-like vector $\mathbf{p}_1 + \ldots + \mathbf{p}_n + 3\mathbf{p}_{n+1}$). Combining these vectors, we see that Λ (suitably magnified) consists of all the points whose Minkowskian coordinates are integers satisfying the equation

$$\xi^1 + \ldots + \xi^n = 3\xi^{n+1} \qquad (n = 4, 5, 6, 7, 8).$$

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We have thus obtained, in a single scheme, coordinates for the vertices of the five honeycombs

$$\alpha_4 h\ (n = 4),\ h\delta_6\ (n = 5),\ 2_{22}\ (n = 6),\ 3_{31}\ (n = 7),\ 5_{21}\ (n = 8).$$

The vertex figures

$$t_{0,3}\alpha_4, \qquad t_1\beta_5, \qquad 1_{22}, \qquad 2_{31}, \qquad 4_{21}$$

can be derived by picking out those points for which

$$(\xi^1)^2 + \ldots + (\xi^n)^2 = (\xi^{n+1})^2 + 2.$$

It follows that the number of minimal vectors, $2s$, is given in these five cases by the formula

$$s = \binom{n}{2} + \binom{n}{3} + \binom{n}{6} + 8\binom{n}{8}.$$

The same vectors were considered, as long ago as 1894, in the thesis of Cartan [**9a**, pp. 142, 143]. His ω_i is our $\mathbf{p}_i - \frac{1}{3}(\mathbf{p}_1 + \ldots + \mathbf{p}_7)$ for E_6, and $\mathbf{p}_i + \frac{1}{3}\mathbf{p}_8$ for E_7.

Du Val [**18**, p. 24] uses symbols R^σ, S^ν, and $S^{\sigma,\nu}$ to denote the hyperplane

$$\xi^1 + \ldots + \xi^n = 3\xi^{n+1} - \sigma,$$

the sphere

$$(\xi^1)^2 + \ldots + (\xi^n)^2 = (\xi^{n+1})^2 - \nu,$$

and their sphere of intersection. In this notation, Λ and Π for E_n are formed by the integer points on R^0 and $S^{0,-2}$. Du Val showed [**18**, pp. 32-34] that the integer points on $S^{1,-1}$, $S^{2,0}$, $S^{3,1}$ are the vertices of q_{21}, 2_{q1}, 1_{q2}, where $q = n - 4$. (Our n is his ϵ.) Thus $S^{0,-2}$ resembles $S^{1,-1}$ when $n = 8$, $S^{2,0}$ when $n = 7$, and $S^{3,1}$ when $n = 6$. In fact, the translation

$$(\xi^1, \ldots, \xi^n, \xi^{n+1}) \rightarrow (\xi^1 + 1, \ldots, \xi^n + 1, \xi^{n+1} + 3)$$

converts $S^{\sigma,\nu}$ into $S^{\sigma+9-n,\ \nu+2\sigma+9-n}$.

A more familiar description of 3_{31} is in terms of the points in Euclidean 8-space whose coordinates are mutually congruent mod 2, with sum zero [**11**, p. 390] (i.e., on a different scale, 8 integers, or 8 halves of odd integers, with sum zero; thus the vertices of 3_{31} belong to two superposed α_7h's). This coordinate scheme may be established quite elegantly by picking out the basis

$$\mathbf{t}_1 = \mathbf{p}_1 - \mathbf{p}_2, \ldots, \mathbf{t}_6 = \mathbf{p}_6 - \mathbf{p}_7,\ \mathbf{t}_7 = \tfrac{1}{2}(-\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4 + \mathbf{p}_5 + \mathbf{p}_6 + \mathbf{p}_7 + \mathbf{p}_8)$$

and observing that

$$\begin{aligned}(x^1\mathbf{t}_1 + \ldots + x^7\mathbf{t}_7)^2 &= (x^1 - \tfrac{1}{2}x^7)^2 + (-x^1 + x^2 - \tfrac{1}{2}x^7)^2 + (-x^2 + x^3 - \tfrac{1}{2}x^7)^2 \\ &+ (-x^3 + x^4 - \tfrac{1}{2}x^7)^2 + (-x^4 + x^5 + \tfrac{1}{2}x^7)^2 + (-x^5 + x^6 + \tfrac{1}{2}x^7)^2 + (-x^6 + \tfrac{1}{2}x^7)^2 + (\tfrac{1}{2}x^7)^2 \\ &= 2\{(x^1)^2 - x^1x^2 + (x^2)^2 - x^2x^3 + (x^3)^2 - x^3x^4 \\ &+ (x^4)^2 - x^4x^5 + (x^5)^2 - x^5x^6 + (x^6)^2 - x^4x^7 + (x^7)^2\} = 2E_7.\end{aligned}$$

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Similarly [**11**, p. 393] the vertices of 5_{21} are the points in Euclidean 9-space whose coordinates are mutually congruent mod 3, with sum zero. (Thus they belong to three superposed α_8h's.) To establish this scheme we pick out the basis

$$\mathbf{t}_1 = \mathbf{p}_1 - \mathbf{p}_2, \ldots, \mathbf{t}_7 = \mathbf{p}_7 - \mathbf{p}_8,$$
$$\mathbf{t}_8 = \tfrac{1}{3}(-\mathbf{p}_0 - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4 - \mathbf{p}_5 + 2\mathbf{p}_6 + 2\mathbf{p}_7 + 2\mathbf{p}_8)$$

and observe that

$$\begin{aligned}(x^1\mathbf{t}_1 + \ldots + x^8\mathbf{t}_8)^2 &= (-\tfrac{1}{3}x^8)^2 + (x^1 - \tfrac{1}{3}x^8)^2 + (-x^1 + x^2 - \tfrac{1}{3}x^8)^2 + (-x^2 + x^3 - \tfrac{1}{3}x^8)^2 \\ &\quad + (-x^3 + x^4 - \tfrac{1}{3}x^8)^2 + (-x^4 + x^5 - \tfrac{1}{3}x^8)^2 + (-x^5 + x^6 + \tfrac{2}{3}x^8)^2 \\ &\quad + (-x^6 + x^7 + \tfrac{2}{3}x^8)^2 + (-x^7 + \tfrac{2}{3}x^8)^2 \\ &= 2E_8.\end{aligned}$$

(In this case the ω_i of Cartan [**9a**, p. 144] is $\mathbf{p}_i - \frac{1}{9}\sum \mathbf{p}$.)

To obtain a comparably elegant system of coordinates for 2_{22}, we have to use complex (or unitary) 3-space, where the distance between two points is equal to the square root of the sum of the norms of the differences of their coordinates. It is well known that the quadratic integers

$$a + b\omega \qquad (\omega = e^{2\pi i/3}),$$

where a and b are rational integers, fall into three classes modulo

$$\lambda = 1 - \omega,$$

typified by 0, 1 and -1 [**26**, p. 187, Theorem 222]. Let us consider the points whose coordinates are three of these quadratic integers, mutually congruent mod λ. As a basis for this lattice we may use the vectors

$$\mathbf{t}_1 = \lambda\omega\mathbf{p}_1,\ \mathbf{t}_2 = \lambda\mathbf{p}_1,\ \mathbf{t}_3 = -\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3,\ \mathbf{t}_4 = \lambda\mathbf{p}_2,\ \mathbf{t}_5 = \lambda\omega\mathbf{p}_2,\ \mathbf{t}_6 = \lambda\mathbf{p}_3$$

[**16**, p. 473]. Then

$$\sum x^i\mathbf{t}_i = (\lambda\omega x^1 + \lambda x^2 - x^3)\,\mathbf{p}_1 + (-x^3 + \lambda x^4 + \lambda\omega x^5)\,\mathbf{p}_2 + (-x^3 + \lambda x^6)\,\mathbf{p}_3,$$

and the norm of this vector is

$$\begin{aligned}&(\lambda\omega x^1 + \lambda x^2 - x^3)(-\lambda\omega x^1 + \bar\lambda x^2 - x^3) \\ &+ (-x^3 + \lambda x^4 + \lambda\omega x^5)(-x^3 + \bar\lambda x^4 - \lambda\omega x^5) + (-x^3 + \lambda x^6)(-x^3 + \bar\lambda x^6) \\ &= 3E_6.\end{aligned}$$

9. Automorphs. We saw in §5 that the group of automorphs of a reflexible form, being the symmetry group of the polytope Π, has a subgroup $\mathbf{S}$ whose typical generator R_k leaves invariant every x^i except x^k, which it changes into

$$x^k - 2x_k/a_{kk}$$

(see 5.3). If the non-vanishing a_{ik}'s $(i \neq k)$ are a_{ik}, a_{jk}, etc. (so that the kth node of the tree is joined to the ith, jth, etc.) this transformed x^k is

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$$\begin{cases} -x^k + \quad x^i + \quad x^j + \dots & \text{if } a_{kk} > 1, \\ -x^k + a_{ii}x^i + a_{jj}x^j + \dots & \text{if } a_{kk} = 1 \end{cases}$$

(by 6.1 with x for m).

For instance, in the case of $G_2 = (x^1)^2 - 3x^1x^2 + 3(x^2)^2$, R_1 and R_2 change (x^1, x^2) into $(-x^1 + 3x^2, x^2)$ and $(x^1, -x^2 + x^1)$, respectively. Again, in the case of the form $A_3 = (x^1)^2 - x^1x^2 + (x^2)^2 - x^2x^3 + (x^3)^2$, the three R's transform (x^1, x^2, x^3) into

$$(-x^1+x^2, x^2, x^3), \qquad (x^1, x^1-x^2+x^3, x^3), \qquad (x^1, x^2, x^2-x^3).$$

In addition to **S**, the group of automorphs contains the symmetry group of the marked tree (p. 405), i.e., the "obvious" automorphs of the form, such as the transposition (x^1x^3) in the case of A_3, or the symmetric group on the three branches of the tree for D_4. In the single case of B_4 there are still other operations, namely the cyclic automorphs of the equivalent form D_4. The whole group of automorphs is most easily obtained as the symmetry group of **Π**. Referring to the table on page 414, we see that this is **S** itself in the following cases:

$$C_n, \quad B_n\ (n \neq 4), \quad E_7, \qquad E_8, \qquad F_4, \qquad G_2.$$

Hence the numbers of automorphs in these cases are the orders of the groups

$$[3^{n-2}, 4], \ [3^{n-2}, 4], \ [3^{3,2,1}], \qquad [3^{4,2,1}], \qquad [3, 4, 3], \qquad [6],$$

namely

$$2^n n!, \qquad 2^n n!, \qquad 8 \cdot 9!, \qquad 192 \cdot 10!, \qquad 1152, \qquad 12.$$

Of course, the forms A_2, B_4 and D_4, D_n $(n > 4)$ have just as many automorphs as the respectively equivalent forms

$$G_2, \qquad F_4, \qquad B_n,$$

namely $\qquad 12, \qquad 1152, \qquad 2^n n!.$

The symmetry groups of the polytopes $t_{0,\, n-1}\alpha_n$ and 1_{22} are derived from the corresponding reflection groups by adding the "central inversion" which reverses the signs of all the x's [**11**, pp. 368, 392]. In the notation of Du Val [**18**, p. 32] the groups are

$$2[3^{n-1}] \quad \text{and} \quad 2[3^{2,2,1}].$$

Hence in these cases the number of automorphs is twice the order of **S**, namely

$$2(n+1)! \text{ for } A_n, \quad 144 \cdot 6! \text{ for } E_6$$

[**8**, pp. 366-368].

10. The enumeration of simple Lie groups. We saw, in §§5 and 6, that the fundamental region for the group **G** corresponding to a connected reflexible form is a Euclidean simplex whose dihedral angles are submultiples of π. We

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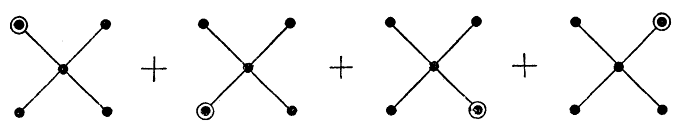
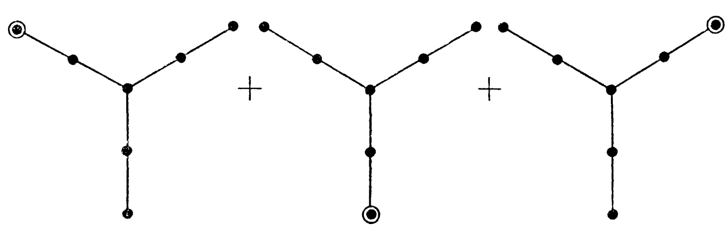
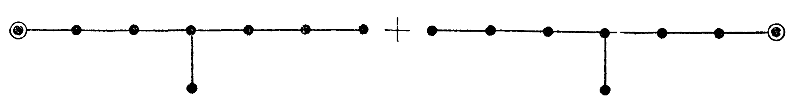
saw also that such a simplex is conveniently denoted by a graph whose nodes represent the $n + 1$ bounding hyperplanes or the respectively opposite vertices, and that a certain number f of the vertices (most naturally described as the "sharpest corners" of the simplex) are *special* in the sense that they lie on the greatest possible number of reflecting hyperplanes of $\mathbf{G}$. (Stiefel [**40**, p. 363] calls them *Knotenpunkte*. His l, m, γ are our n, s, $\mathbf{T}$.) This simplex is the polytope (P) of Cartan [**9**, pp. 216-228]. The discrete group $\mathbf{G}$ (Cartan's $\overline{\mathfrak{G}}_1$) is generated by reflections in the bounding hyperplanes of the simplex, and transforms any *special* vertex O into the point-lattice Λ (his $\overline{R}$) which is conveniently symbolized by ringing the corresponding node of the graph, as in §7 above. If $f = 1$, this is the lattice of all special points. But if $f > 1$, the simplex has f special vertices $O_0, O_1, \ldots, O_{f-1}$, and all the special points form a more complicated point-lattice, Λ^f, consisting of f superposed Λ's [**17**, p. 206]. This lattice Λ^f is naturally symbolized by ringing each of the f special nodes in turn; e.g., for D_4, Λ^4 is



and for E_6, Λ^3 is



and for E_7, Λ^2 is



The points of Λ^f are distributed on s families of parallel hyperplanes. When $a_{ii} = 1$, so that $\mathbf{G}$ is a "trigonal" group [**17**, p. 204], the distance between consecutive planes is the same in all the families, and there is a minimal vector of Λ perpendicular to the hyperplanes of each family. Hence, in the trigonal cases (namely A_n, D_n, E_n), Λ and Λ^f are *reciprocal* lattices. In particular, they are *similar* lattices in each of the cases A_2, D_4, since the lattices

$$\alpha_2\mathrm{h} = \{3, 6\}, \qquad \mathrm{h}\delta_5 = \{3, 3, 4, 3\}$$

are similar to their own reciprocals [**17**, p. 181]. These are the same as the lattices for G_2, F_4, respectively, where Λ and Λ^f coincide for the simple reason that $f = 1$.

For C_n, Λ is the ordinary cubic lattice δ_{n+1}, while Λ^f is the body-centered cubic lattice formed by two dual[3] δ_{n+1}'s [**9**, pp. 229, 230]. For B_n, Λ is the alternated cubic lattice $h\delta_{n+1}$, while Λ^f is the δ_{n+1} formed by two complementary $h\delta_{n+1}$'s. In fact, Λ^f for B_n is the same as Λ for C_n, namely the self-reciprocal lattice δ_{n+1}; Λ for B_n is the same as Λ for D_n, and Λ^f for C_n is the same as Λ^f for D_n. Hence

10.1 Λ^f (*the lattice of special points*) *and* Λ (*the lattice of transforms of one special point*) *are reciprocal lattices except in the cases* B_n *and* C_n $(n > 2)$, *where* Λ^f *for each is reciprocal to* Λ *for the other.*

The translation group T of the lattice Λ is a subgroup of index f in the translation group T^f of Λ^f. The larger group T^f includes translations from O_0 to O_1, O_2, etc. When f is composite, say $f = qr$, T may be a subgroup of index r in an intermediate group T^r which is itself a subgroup of index q in T^f. The f points O_i then fall into q sets of r, such that T^r is transitive on each set. The corresponding lattice Λ^r (which we shall sometimes prefer to write as ${}^r\Lambda$) consists of r superposed Λ's. Similarly Λ^f consists of q superposed Λ^r's. By considering the individual cases, we shall find that such a lattice Λ^r occurs for every divisor of f. Sometimes there are two different lattices for the same value of r; that is why we need the modified symbol ${}^r\Lambda$.

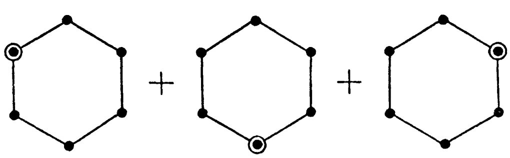
In the case of A_n, the fundamental simplex (whose vertices are *all* special, so that $f = n + 1$) is given in terms of $n + 1$ Cartesian coordinates by

$$\xi^1 \leqslant \xi^2 \leqslant \ldots \leqslant \xi^{n+1} \leqslant \xi^1 + 1, \qquad \xi^1 + \ldots + \xi^{n+1} = 0$$

[**12**, p. 162; cf. **9**, p. 219, where the coordinates are oblique]. Thus the coordinates for its vertex O_i consist of i repetitions of $-1 + i/(n + 1)$ followed by $n + 1 - i$ repetitions of $i/(n + 1)$. The coordinates for the transforms of this point O_i are all congruent to $i/(n + 1)$ mod 1 (with sum zero). Hence the special points on the line O_0O_q are transforms of all the points O_i for which i is a multiple of q. It follows that any divisor r of $n + 1$ yields a point-lattice Λ^r consisting of all the transforms of each of the r points

10.2 $$O_0, O_q, O_{2q}, \ldots, O_{(r-1)q},$$

where $q = (n + 1)/r$; e.g., for A_5, Λ^3 consists of three superposed α_5h's:

10.3 

Since the transforms of O_{jq} have coordinates congruent to j/r mod 1, it is

[3]We speak of *dual* honeycombs, rather than *reciprocal* honeycombs [**17**, p. 182], to avoid confusion with the different concept of reciprocal *lattices*.

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natural to alter the unit of measurement so as to describe Λ^r as consisting of the points whose coordinates are integers mutually congruent mod r, with sum zero.

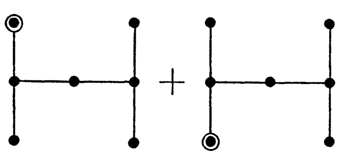
Similarly, the fundamental simplex for D_n, given by

$$\xi^1 \geqslant \xi^2 \geqslant \ldots \geqslant \xi^n, \quad \xi^{n-1} + \xi^n \geqslant 0, \quad \xi^1 + \xi^2 \leqslant 1$$

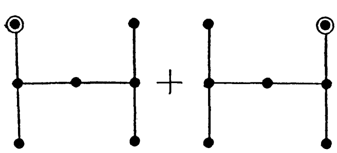
[9, p. 220], has four special vertices:

10.4
$$O_0 = (0, 0, \ldots, 0, 0), \qquad O_1 = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\tfrac{1}{2}),$$
$$O_2 = (1, 0, \ldots, 0, 0), \qquad O_3 = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \ \ \tfrac{1}{2}).$$

The coordinates of the transforms of O_0 or O_2 are integers whose sum is even or odd, respectively. The coordinates of the transforms of O_1 or O_3 are numbers, congruent to $\frac{1}{2}$ mod 1, whose sum differs from $\frac{1}{2}n$ by a number which is odd or even, respectively [9, p. 230]. Hence the transforms of O_0 and O_2 together form a cubic lattice δ_{n+1}, but the transforms of O_0 and O_1 (or of O_0 and O_3) form a lattice only when n is even. It is convenient to distinguish these two cases (e.g.



and



when $n = 6$) by the respective symbols ${}^2\Lambda$ and Λ^2. When $n = 4$, the special vertices of the simplex are completely symmetrical, so the distinction disappears.

Having found the various lattices Λ^r (or ${}^r\Lambda$), we obtain appropriate symbols for the corresponding classes of forms by using, in place of the letter Λ, the "family" symbol A_n or B_n, etc. Thus the complete list is as follows:

10.5
$$\begin{array}{ll}
A_n{}^r & (r \mid n+1;\ n = 1,\ 2,\ 3, \ldots), \\
C_n,\ C_n{}^2 & (n = 2,\ 3,\ 4, \ldots), \\
B_n,\ B_n{}^2 & (n = 3,\ 4,\ 5, \ldots), \\
D_n,\ {}^2D_n,\ D_n{}^4 & (n = 4,\ 5,\ 6, \ldots), \\
D_n{}^2 & (n = 6,\ 8, 10, \ldots), \\
E_6,\ E_6{}^3,\ E_7,\ E_7{}^2,\ E_8,\ F_4,\ G_2. &
\end{array}$$

(We exclude $D_4{}^2$ because it is the same as ${}^2D_4$. This is clear from the graph, which should really be drawn in three dimensions like the structural formula for methane.)

Later on we shall adopt the same symbols for particular forms in these classes. Since one lattice may arise from several different groups, the following equivalences occur among the forms:

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$$
\begin{array}{lccc}
 & A_1 \sim A_1^2, & A_2 \sim A_2^3 \sim G_2, & C_2 \sim C_2^2, \\
 & A_3 \sim B_3, & A_3^2 \sim B_3^2 \sim C_3, & A_3^4 \sim C_3^2, \\
10.6 & & C_4^2 \sim B_4 \sim D_4 \sim D_4^4 \sim F_4 & \\
 & B_n \sim D_n, & C_n \sim B_n^2 \sim {}^2D_n, & C_n^2 \sim D_n^4, \\
 & & A_7^2 \sim E_7, \qquad A_7^4 \sim E_7^2, & \\
 & & A_8^3 \sim D_8^2 \sim E_8. &
\end{array}
$$

(Of course, A_n^r does not mean the rth power of A_n.) Thus the most concise list of the classes is

$$A_n^r\ (r \mid n+1), \quad C_n, \quad C_n^2, \quad D_n, \quad D_n^2\ (n \text{ even}, \geqslant 6), \quad E_6, \quad E_6^3,$$

with the inevitable duplications

$$A_1 \sim A_1^2,\ A_2 \sim A_2^3,\ C_2 \sim C_2^2,\ A_3^2 \sim C_3,\ A_3^4 \sim C_3^2,\ C_4^2 \sim D_4,\ A_8^3 \sim D_8^2.$$

In §§12-14 we shall obtain simple expressions for A_n^r, C_n^2, D_n^2 and E_6^3. We shall find that they are not all perfect. But 4.3 (with group **S**) shows that they are all eutactic.

The application of these geometrical ideas to the theory of Lie groups, developed by Cartan, Witt [**45**] and Hopf [**29**], may be summarized as follows. Every Euclidean simplex whose dihedral angles are submultiples of π is the fundamental region for a group generated by reflections; this discrete group **G** represents a family of locally isomorphic simple Lie groups; and every compact simple Lie group arises in this manner. Stiefel [**40**, p. 374] showed that the individual Lie groups in each family may be distinguished by associating them with the lattices Λ^r (or ${}^r\Lambda$). Thus 10.5 *can be interpreted as a complete list of compact simple Lie groups.*

In particular, A_1 is the group of quaternions of norm 1 [**40**, p. 378]; A_1^2 is the group of rotations of a sphere with a fixed centre, or the group of displacements in the elliptic plane; and G_2 is the group of automorphisms of the algebra of Cayley numbers [**8**, p. 370].

11. Determinants. The determinant of a reflexible form (or, more generally, of any form represented by a tree in the manner of §5) is easily computed by the following rule. Let Δ denote the whole determinant, Δ' the cofactor obtained by deleting a node of degree 1 (say the kth node) and its single branch, and Δ'' the cofactor obtained by deleting also the remaining end of this branch along with any branches occurring there. Then, since the only non-vanishing elements in the kth row or column of Δ are a_{kk} and one a_{ik},

11.1 $$\Delta = a_{kk}\Delta' - a_{ik}^2\Delta''.$$

(The case when $a_{kk} = 2$ and $a_{ik} = -\lambda^{\frac{1}{2}}$ was described by Witt [**45**, p. 302].) It follows almost immediately that the determinants of

$$A_n, \quad C_n, \quad B_n, \quad D_n, \quad E_n, \quad F_4, \quad G_2$$

are

$$\frac{n+1}{2^n}, \quad 1 \quad \frac{4}{2^n}, \quad \frac{4}{2^n}, \quad \frac{9-n}{2^n}, \quad \tfrac{1}{4}, \quad \tfrac{3}{4}.$$

A. J. Coleman has made the interesting observation that these numbers are related to f in each case by the formula

$$\Delta = fa_{11} \ldots a_{nn}/2^n.$$

This means that $f = \det(C_{ij})$, where

$$C_{ij} = 2a_{ij}(a_{ii}\, a_{jj})^{-\frac{1}{2}}$$

so that $C_{ii} = 2$ and any other C_{ij} is twice the cosine of the angle between $\mathbf{t}_i$ and $\mathbf{t}_j$. This enables us to replace the expression 6.6, for the order of $\mathbf{S}$, by

$$2^n \Delta z^0\, z^1 \ldots z^n n!.$$

Of course, $2^n\Delta$ is simply the determinant of the doubled form $\sum\sum 2a_{ij}x^ix^j$, whose minimum is 2 instead of 1; e.g., the determinant of $2A_n$ is $n + 1$.

It is interesting to observe that the determinants for the new classes of forms $A_n{}^r$, etc., can be computed before we have obtained any particular forms in these classes. In fact, Λ^r has r times as many lattice points (in a large region of n-space) as Λ itself; therefore the period parallelotope is r^{-1} times as great. We saw in §3 that the content of the parallelotope is $\Delta^{\frac{1}{2}}$. Hence the determinant for Λ^r is r^{-2} times the determinant for Λ; e.g., for $2A_n{}^r$ it is

11.2 $$(n + 1)r^{-2}.$$

12. The forms $A_n{}^r$. We seek a form whose point-lattice consists of the transforms of the r points 10.2 under the group $\mathbf{G}$ generated by the symmetric group on the $n + 1$ coordinates (which is $\mathbf{S}$) along with the translation $(1, 0, 0, \ldots, 0, -1)$. Since $2A_n$ is such a form when $r = 1$, it is natural to use such a unit that a suitable form in the general case is denoted by $2A_n{}^r$. The precise expression for this form depends on our choice of a basis for the lattice. An obvious but redundant basis is afforded by any basis for Λ, say

$$\mathbf{p}_1 - \mathbf{p}_2,\ \mathbf{p}_2 - \mathbf{p}_3, \ldots, \mathbf{p}_n - \mathbf{p}_{n+1},$$

along with the vector

$$\begin{aligned} O_0O_q &= (-1 + r^{-1})(\mathbf{p}_1 + \ldots + \mathbf{p}_q) + r^{-1}(\mathbf{p}_{q+1} + \ldots + \mathbf{p}_{n+1}) \\ &= -(\mathbf{p}_1 + \ldots + \mathbf{p}_q) + r^{-1}(\mathbf{p}_1 + \ldots + \mathbf{p}_{n+1}). \end{aligned}$$

The desired basis of n vectors is derived by omitting one of these $n + 1$, namely one that can be expressed in terms of the remaining n. Accordingly, we ask whether $\mathbf{p}_n - \mathbf{p}_{n+1}$ can be expressed in terms of

12.1 $$\begin{aligned} &\mathbf{t}_1 = \mathbf{p}_1 - \mathbf{p}_2,\ \mathbf{t}_2 = \mathbf{p}_2 - \mathbf{p}_3, \ldots, \mathbf{t}_{n-1} = \mathbf{p}_{n-1} - \mathbf{p}_n, \\ &\mathbf{t}_n = -(\mathbf{p}_1 + \ldots + \mathbf{p}_q) + r^{-1}\textstyle\sum \mathbf{p} \qquad (qr = n + 1). \end{aligned}$$

In this investigation we assume $r > 1$, since otherwise $\mathbf{t}_n$ would vanish.

We have

12.2
$$\mathbf{p}_i - \mathbf{p}_j = \mathbf{t}_i + \mathbf{t}_{i+1} + \ldots + \mathbf{t}_{j-1} \qquad (i < j \leqslant n)$$

and

$$-r\mathbf{t}_n = (r-1)(\mathbf{p}_1 + \ldots + \mathbf{p}_q) - (\mathbf{p}_{q+1} + \ldots + \mathbf{p}_{qr}) = \sum_{i=1}^{q} \sum_{k=1}^{r-1} (\mathbf{p}_i - \mathbf{p}_{kq+i}).$$

The only term in this double sum that is not of the form $\mathbf{p}_i - \mathbf{p}_j$ with $i < j \leqslant n$ is the final term $\mathbf{p}_q - \mathbf{p}_{qr}$. Using 12.2 for all the preceding terms, we obtain an expression for $\mathbf{p}_q - \mathbf{p}_{qr}$ involving the $\mathbf{t}$'s alone. Finally

$$\mathbf{p}_n - \mathbf{p}_{n+1} = (\mathbf{p}_q - \mathbf{p}_{qr}) - (\mathbf{p}_q - \mathbf{p}_n).$$

Thus the vectors 12.1 do constitute a basis, the general lattice point is given by the vector

$$x^1\mathbf{t}_1 + \ldots + x^n\mathbf{t}_n = \{x^1 - (1-r^{-1})x^n\}\,\mathbf{p}_1 + \sum_{i=2}^{q} \{-x^{i-1} + x^i - (1-r^{-1})x^n\}\,\mathbf{p}_i$$
$$+ \sum_{i=q+1}^{n-1} (-x^{i-1} + x^i + r^{-1}x^n)\,\mathbf{p}_i + (-x^{n-1} + r^{-1}x^n)\,\mathbf{p}_n + r^{-1}x^n\,\mathbf{p}_{n+1},$$

and the desired form $2A_n^{\,r}$ is

$$\begin{aligned}(x^1\mathbf{t}_1 + \ldots + x^n\mathbf{t}_n)^2 &= \{x^1 - (1-r^{-1})x^n\}^2 + \sum_{i=2}^{q} \{-x^{i-1} + x^i - (1-r^{-1})x^n\}^2 \\ &\quad + \sum_{i=q+1}^{n-1} (-x^{i-1} + x^i + r^{-1}x^n)^2 + (-x^{n-1} + r^{-1}x^n)^2 + (r^{-1}x^n)^2 \\ &= 2\{(x^1)^2 - x^1x^2 + (x^2)^2 - \ldots + (x^{n-1})^2 - x^qx^n\} + q(1-r^{-1})(x^n)^2 \\ &= 2\{A_{n-1} - x^qx^n + \tfrac{1}{2}q(1-r^{-1})(x^n)^2\}.\end{aligned}$$

Accordingly we define, as one of the simplest representatives of its class,

12.3
$$A_n^{\,r} = A_{n-1} - x^qx^n + \frac{q}{2}\left(1 - \frac{1}{r}\right)(x^n)^2 \qquad (r > 1,\ qr = n+1).$$

In particular, $A_7^{\,2} = E_7$, and $A_8^{\,3}$ is obviously equivalent to E_8.

By 11.2, the determinants for $2A_n$ and $2A_n^{\,n+1}$ are $n+1$ and $(n+1)^{-1}$. By 10.1, the corresponding point-lattices are reciprocal, which means that either form is equivalent to the reciprocal of the other (see 3.3). More generally,

12.4 *The two forms*
$$2A_n^{\,r} = 2A_{n-1} - 2x^qx^n + q(1 - r^{-1})(x^n)^2$$
and
$$2A_n^{\,q} = 2A_{n-1} - 2x^rx^n + r(1 - q^{-1})(x^n)^2$$
belong to reciprocal classes whenever $qr = n + 1 \quad (q > 1, r > 1)$.

To prove this we use the covariant basis 12.1 and compute the contravariant basis $\mathbf{t}^1, \ldots, \mathbf{t}^n$, given by

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$$\mathbf{t}^i \cdot \mathbf{t}_j = \delta^i_j, \qquad \mathbf{t}^i \cdot \sum \mathbf{p} = 0,$$

where $\sum \mathbf{p}$ means $\mathbf{p}_1 + \ldots + \mathbf{p}_{n+1}$. These relations with $j < n$ yield (for some k)

$$\mathbf{t}^i = (k+1)(\mathbf{p}_1 + \ldots + \mathbf{p}_i) + k(\mathbf{p}_{i+1} + \ldots + \mathbf{p}_n) - (nk + i)\,\mathbf{p}_{n+1} \qquad (i<n).$$

Using $\mathbf{t}^i \cdot \mathbf{t}_n = 0$, we soon find that $k = -iq^{-1}$ or -1 according as $i \leqslant q$ or $i \geqslant q$. Thus

$$\mathbf{t}^i = \begin{cases} \mathbf{p}_1 + \ldots + \mathbf{p}_i - iq^{-1}\sum \mathbf{p} + i(r-1)\,\mathbf{p}_{n+1} & (i \leqslant q), \\ -(\mathbf{p}_{i+1} + \ldots + \mathbf{p}_n) + (n-i)\,\mathbf{p}_{n+1} & (q \leqslant i < n). \end{cases}$$

Also $$\mathbf{t}^n = -q^{-1}\sum \mathbf{p} + r\,\mathbf{p}_{n+1}.$$

These vectors generate the reciprocal lattice, which represents the reciprocal form. To identify this with the lattice generated by $\mathbf{p}_i - \mathbf{p}_j$ and $-(\mathbf{p}_1 + \ldots + \mathbf{p}_r) + q^{-1}\sum \mathbf{p}$, we observe that

$$\begin{aligned} \mathbf{p}_1 - \mathbf{p}_i &= \mathbf{t}^1 + \mathbf{t}^{i-1} - \mathbf{t}^i && (1 < i \leqslant q), \\ \mathbf{p}_1 - \mathbf{p}_{n+1} &= \mathbf{t}^1 - \mathbf{t}^n, \\ \mathbf{p}_{n+1} - \mathbf{p}_i &= \mathbf{t}^{i-1} - \mathbf{t}^i && (q < i < n), \\ \mathbf{p}_{n+1} - \mathbf{p}_n &= \mathbf{t}^{n-1}, \end{aligned}$$

$$\mathbf{p}_1 + \ldots + \mathbf{p}_r - q^{-1}\sum \mathbf{p} = \begin{cases} \mathbf{t}^r - (r-1)\,\mathbf{t}^n & (r \leqslant q), \\ \mathbf{t}^r - (q-1)\,\mathbf{t}^n & (r \geqslant q). \end{cases}$$

Thus 12.4 is proved.

In particular, the reciprocal of $A_7{}^4$ is equivalent to $A_7{}^2 = E_7$. By 10.1, this is equivalent to the reciprocal of $E_7{}^2$. Accordingly, we define

$$E_7{}^2 = A_7{}^4.$$

By 11.1, the determinant of $2A_{n-1} - 2x^q x^n + q(1 - r^{-1})(x^n)^2$ is

$$\Delta = q(1 - r^{-1})\Delta' - \Delta'',$$

where Δ' is the determinant of $2A_{n-1}$ and Δ'' is the determinant of the form derived from $2A_{n-1}$ by setting $x_q = 0$, namely

$$2A_{q-1}(x^1, \ldots, x^{q-1}) + 2A_{n-q-1}(x^{q+1}, \ldots, x^{n-1}).$$

Thus $\Delta' = n$, $\Delta'' = q(n-q)$, and

12.5 $$\Delta = q\left(1 - \frac{q}{n+1}\right)n - q(n-q) = \frac{q^2}{n+1} = \frac{q}{r} = \frac{n+1}{r^2},$$

in agreement with 11.2.

It is interesting to compare this with the value of M, which we compute by considering the lattice points nearest to the origin. The coordinates of such a nearest point are obviously either

$$1, \; -1 \quad \text{and} \quad n-1 \quad \text{zeros}$$

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or q coordinates $\pm(1 - r^{-1})$ and $q(r - 1)$ coordinates $\mp r^{-1}$. Thus $2A_n{}^r$ has

12.6 $$M = \min\{2, q(1 - r^{-1})\}.$$

(Clearly, this minimum is attained when one of the x's is 1 while all the rest vanish.)

The two sets of vectors that we have just been comparing are the transforms of $\pm\mathbf{t}_1$ and $\pm\mathbf{t}_n$ under the symmetric group $\mathbf{S}$. The minimal vectors for $A_n{}^r$ $(n = qr - 1, r > 1)$ consist of one or both of these sets: the former set if $(q - 2)(r - 1) > 2$, the latter if $(q - 2)(r - 1) < 2$, both together if $(q - 2)(r - 1) = 2$. Since q and r must be positive integers, $A_7{}^2$ and $A_8{}^3$ are the only cases where both sets are minimal.

The transforms of $\pm\mathbf{t}_1$ are the $n(n + 1)$ vectors $\mathbf{p}_i - \mathbf{p}_j$ $(i \neq j)$. The number of transforms of $\pm\mathbf{t}_n$ is evidently $\binom{qr}{q}$ or $2\binom{qr}{q}$ according as $r = 2$ or $r > 2$. Thus $s \geqslant \frac{1}{2}n(n + 1)$ in every case except

$$A_n{}^{n+1}\,(s = n + 1),\quad A_3{}^2\,(s = 3)\quad\text{and}\quad A_5{}^2\,(s = 10).$$

These exceptional forms, violating 1.1, cannot be extreme. But we shall find that they are the only failures:

12.7 *Every form $A_n{}^r$ is extreme, except $A_n{}^{n+1}$ $(n > 2)$, $A_3{}^2$, $A_5{}^2$.*

By the remark at the end of §10, what remains to be proved is that every form $A_n{}^r$ satisfying 1.1 is perfect. In the two cases where $(q - 2)(r - 1) = 2$, we know this already from 5.7, since $A_7{}^2 = E_7$ and $A_8{}^3 \sim E_8$.

By 3.5, $A_n{}^r$ is perfect when $(q - 2)(r - 1) > 2$, since its minimal vectors $\mathbf{p}_i - \mathbf{p}_j$ are the same as those of A_n. (This may seem paradoxical. But in saying that a perfect form is uniquely determined by the value of its minimum and all the representations, Voronoï was speaking of the algebraic representations, which depend on the basis $\mathbf{t}_1, \ldots, \mathbf{t}_n$; he did not mean that such a form is uniquely determined by the geometrical arrangement of minimal vectors.)

The remaining possibility is $(q - 2)(r - 1) < 2$. Since we are assuming $r > 1$ and excluding the exceptional forms $A_n{}^{n+1}$ $(q = 1)$, $A_3{}^2$ $(q = 2)$, $A_5{}^2$ $(q = 3)$, this inequality reduces to

$$q = 2,\ r > 2,$$

in which case $A_n{}^r$ is obviously equivalent to

$$D_n - r^{-1}(x^n)^2 \qquad (n = 2r - 1).$$

To test this for perfection, we investigate the possibility of a quadric cone

$$\sum_1^{2r}\sum_1^{2r} b_{ij}\,\xi^i\xi^j = 0$$

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containing all the vectors derived from $(r-1)(\mathbf{p}_1+\mathbf{p}_2)-(\mathbf{p}_3+\ldots+\mathbf{p}_{2r})$ by permuting the $2r$ $\mathbf{p}$'s. Direct substitution yields

$$(r-1)^2(b_{11}+2b_{12}+b_{22})-2(r-1)\sum_3^{2r}(b_{1j}+b_{2j})+\sum_3^{2r}\sum_3^{2r}b_{ij}=0.$$

In terms of $e_{ij}=2b_{ij}-b_{ii}-b_{jj}$, this becomes

12.8 $$(r-1)^2e_{12}-(r-1)\sum_3^{2r}(e_{1j}+e_{2j})+\sum_{2<i<j}\sum e_{ij}=0.$$

Interchanging subscripts 2 and 3, subtracting, and dividing by r, we obtain

12.9 $$(r-1)(e_{12}-e_{13})-\sum_4^{2r}(e_{2j}-e_{3j})=0.$$

Since $e_{ij}=e_{ji}$ and $e_{jj}=0$, this implies

$$r(e_{12}-e_{13})=\sum_1^{2r}(e_{2j}-e_{3j}).$$

Since there is nothing special about the subscript 1, we deduce that $e_{i2}-e_{i3}$ is the same for all values of i (other than 2 or 3), say

$$e_{i2}-e_{i3}=d.$$

By 12.9, $(r-1)d=(2r-3)d$, whence $(r-2)d=0$. Since $r>2$, $d=0$. Thus $e_{i2}=e_{i3}$, and since there is nothing special about 2 and 3, we deduce that e_{ij} has the same value for all $i\neq j$, say

$$e_{ij}=c.$$

By 12.8, $(r-1)^2c-4(r-1)^2c+\binom{2r-2}{2}c=0$, whence $r(r-1)c=0$, $c=0$, $2b_{ij}=b_{ii}+b_{jj}$, and

$$\sum\sum b_{ij}\xi^i\xi^j=\tfrac{1}{2}\sum\sum(b_{ii}+b_{jj})\xi^i\xi^j=\sum\sum b_{jj}\xi^i\xi^j=\sum\xi^i\,.\,\sum b_{jj}\xi^j.$$

Thus the only quadric cone containing all the minimal vectors is the degenerate cone that consists of the n-space $\sum\xi^i=0$ (which contains the whole lattice) and an arbitrary second n-space. Hence the form is perfect, by 3.5. But we have already seen that it is eutactic. Hence it is extreme, and 12.7 is proved.

By 12.4 with $q=r$, the form

$$2A_{n-1}-2x^rx^n+(r-1)(x^n)^2 \qquad (n=r^2-1)$$

is equivalent to its own reciprocal. When $r=2$ this is the imperfect form C_3. When $r=3$ it is equivalent to $2E_8$. For any odd value of r we can halve it to obtain the extreme form

$$A_{n-1}-x^rx^n+\tfrac{1}{2}(r-1)(x^n)^2 \qquad (n=r^2-1)$$

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which is remarkable as having the least possible determinant for a positive definite $(r^2 - 1)$-ary form with integral coefficients. O'Connor and Pall [**33**, p. 329] found an imperfect form of the same determinant ($\Delta = 2^{-n}$) consisting of the sum of $n/8$ forms E_8 in separate sets of variables.

The coefficient of $(x^n)^2$ in $A_n{}^r$ is again integral when $q = 2r$. Then the form is

$$A_{n-1} - x^{2r}x^n + (r-1)(x^n)^2 \qquad (n = 2r^2 - 1),$$

with $M = 1$ and $\Delta = 2^{1-n}$. When r is even, this is an integral $(2r^2 - 1)$-ary form of least possible determinant: e.g., when $r = 2$ it is E_7.

By 12.5 and 12.6, the form $A_n{}^r$ with $n = 2^r - 1$ has

$$\frac{M^n}{\Delta} = (r-1)^n\left(\frac{2}{r}\right)^{n-1}.$$

In particular, the quinary form

$$A_5{}^3 = (x^1)^2 - x^1x^2 + (x^2)^2 - x^2x^3 + (x^3)^2 - x^3x^4 + (x^4)^2 - x^2x^5 + \tfrac{2}{3}(x^5)^2,$$

whose lattice is 10.3, has

$$\frac{M^5}{\Delta} = 2^5\left(\frac{2}{3}\right)^4 = \frac{512}{81}.$$

But the extreme quinary forms are all known, viz, $A_5\, D_5$ (or B_5) and an extra one which Korkine and Zolotareff named Z [**32**, pp. 243, 247]. Hence Z must be equivalent to $A_5{}^3$, and we can verify this directly by deriving Z from the basis

$$-\mathbf{c}_{15},\ \mathbf{c}_{25},\ \mathbf{c}_{35},\ \mathbf{c}_{45},\ \mathbf{c}_{16},$$

where

$$\mathbf{c}_{ij} = -(\mathbf{p}_i + \mathbf{p}_j) + \tfrac{1}{3}\textstyle\sum \mathbf{p}.$$

Another interesting special case is the septenary form $A_7{}^4 = E_7{}^2$, whose 28 pairs of minimal vectors $\pm\mathbf{c}_{ij}$ correspond to the 28 bitangents of the general plane quartic curve [**11**, p. 406].

13. The forms $C_n{}^2$ and $D_n{}^2$. The n-dimensional body-centred cubic lattice, representing $C_n{}^2$, has the obvious basis

$$\mathbf{t}_1 = \mathbf{p}_1,\ \ldots,\ \mathbf{t}_{n-1} = \mathbf{p}_{n-1},\ \mathbf{t}_n = -\tfrac{1}{2}(\mathbf{p}_1 + \ldots + \mathbf{p}_n),$$

in terms of which $\mathbf{p}_n = -(\mathbf{t}_1 + \ldots + \mathbf{t}_{n-1} + 2\mathbf{t}_n)$. Accordingly we define

$$C_n^2 = (x^1\mathbf{t}_1 + \ldots + x^{n-1}\mathbf{t}_{n-1} + x^n\mathbf{t}_n)^2 = \left\{(x^1 - \tfrac{1}{2}x^n)\mathbf{p}_1 + \ldots + (x^{n-1} - \tfrac{1}{2}x^n)\mathbf{p}_{n-1} - \tfrac{1}{2}x^n\mathbf{p}_n\right\}^2$$
$$= \sum_1^n (x^i - \tfrac{1}{2}x^n)^2.$$

Since $\Delta = 1$ for C_n, the principle at the end of §11 yields $\Delta = \frac{1}{4}$ for $C_n{}^2$. Clearly, a minimal vector is either $\pm\mathbf{p}_i$ or $\frac{1}{2}(\pm\mathbf{p}_1 \pm \mathbf{p}_2 \pm \ldots \pm \mathbf{p}_n)$. Thus

$$M = \min(1, \tfrac{1}{4}n).$$

Since there are $2n$ vectors $\pm\, \mathbf{p}_i$, and 2^n of the other type,

$$s = 2^{n-1} \quad \text{or} \quad n + 2^{n-1} \quad \text{or} \quad n$$

according as $\quad n < 4 \quad$ or $\quad n = 4 \quad$ or $\quad n > 4$.

By 1.1, C_n^2 is imperfect whenever $n \neq 4$. Hence

13.1 *The only extreme form C_n^2 is C_4^2, which is obviously equivalent to D_4.*

The ternary form

$$\phi' = x^2 + \tfrac{1}{2}(x + 2y + z)^2 + z^2 = \tfrac{1}{2}\{(x - z)^2 + (x + 2y + z)^2 + (x + z)^2\}$$

of §1 is equivalent to $2C_3^2$ by the transformation

$$x^1 = x,\ x^2 = -\,y,\ x^3 = x + z, \text{ or } x = x^1,\ y = -\,x^2,\ z = x^3 - x^1.$$

Its reciprocal form, ϕ or A_3, is equivalent to D_3. More generally (see 10.1). the reciprocal of $2C_n^2$ is equivalent to B_n, and therefore also to D_n. In other words, the reciprocal of C_n^2 (which has $\Delta = \frac{1}{4}$) is equivalent to $2D_n$ (which has $\Delta = 4$).

In the case of C_n^2, S is the "hyper-octahedral" group $[3^{n-2}, 4]$, of order $2^n n!$, generated by the permutations and sign changes of the $\mathbf{p}$'s. This is the whole group of automorphs except when $n = 4$.

The form D_n^2 is more interesting. Here S is the group $[3^{n-3,1,1}]$, of order $2^{n-1}n!$, generated by reflections in the hyperplanes $\xi^i \pm \xi^j = 0$, and G contains also the reflections in $\xi^i \pm \xi^j = 1$. The point-lattice consists of the transforms of O_0 and O_3, in the notation of 10.4. Since these points are just as densely distributed as the transforms of O_0 and O_2, which are the points of the ordinary cubic lattice, we still have $\Delta = 1$.

A minimal vector is either $\pm\,\mathbf{p}_i \pm \mathbf{p}_j$, or $\frac{1}{2}(\pm\,\mathbf{p}_1 \pm \mathbf{p}_2 \pm \ldots \pm \mathbf{p}_n)$ with an even number of minus signs; thus

$$M = \min(2, \tfrac{1}{4}n).$$

A convenient basis is

$$\mathbf{t}_1 = \mathbf{p}_1 - \mathbf{p}_2, \ldots, \mathbf{t}_{n-2} = \mathbf{p}_{n-2} - \mathbf{p}_{n-1},\ \mathbf{t}_{n-1} = \mathbf{p}_{n-2} + \mathbf{p}_{n-1},\ \mathbf{t}_n = -\tfrac{1}{2}(\mathbf{p}_1 + \ldots + \mathbf{p}_n),$$

in terms of which

$$\mathbf{p}_{n-1} + \mathbf{p}_n = -\{\mathbf{t}_1 + 2\mathbf{t}_2 + \ldots + (n-3)\mathbf{t}_{n-3} + (\tfrac{1}{2}n - 1)(\mathbf{t}_{n-2} + \mathbf{t}_{n-1}) + 2\mathbf{t}_n\}.$$

(Notice that the lattice includes every vector $2\mathbf{p}_i$, but not $\mathbf{p}_i$ itself.) Thus the form is

$$\begin{aligned}(\sum_1^n x^i \mathbf{t}_i)^2 &= \{(x^1 - \tfrac{1}{2}x^n)\mathbf{p}_1 + \sum_2^{n-3}(-x^{i-1} + x^i - \tfrac{1}{2}x^n)\mathbf{p}_i \\ &\quad + (-x^{n-3} + x^{n-2} + x^{n-1} - \tfrac{1}{2}x^n)\mathbf{p}_{n-2} + (-x^{n-2} + x^{n-1} - \tfrac{1}{2}x^n)\mathbf{p}_{n-1} - \tfrac{1}{2}x^n\mathbf{p}_n\}^2 \\ &= (x^1 - \tfrac{1}{2}x^n)^2 + \sum_2^{n-3}(-x^{i-1} + x^i - \tfrac{1}{2}x^n)^2 \\ &\quad + (-x^{n-3} + x^{n-2} + x^{n-1} - \tfrac{1}{2}x^n)^2 + (-x^{n-2} + x^{n-1} - \tfrac{1}{2}x^n)^2 + (\tfrac{1}{2}x^n)^2\end{aligned}$$

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$$= 2\{(x^1)^2 - x^1x^2 + (x^2)^2 - x^2x^3 + \ldots - x^{n-3}x^{n-2} + (x^{n-2})^2 - x^{n-3}x^{n-1} + (x^{n-1})^2 - x^{n-1}x^n\} + \tfrac{1}{4}n(x^n)^2$$
$$= 2D_{n-1} - 2x^{n-1}x^n + \tfrac{1}{4}n(x^n)^2.$$

Halving this for closer analogy with the other forms, we define

$$D_n^2 = D_{n-1} - x^{n-1}x^n + \tfrac{1}{8}n(x^n)^2.$$

Since there are $2n(n-1)$ vectors $\pm \mathbf{p}_i \pm \mathbf{p}_j$, and 2^{n-1} of the other type,

$$s = 2^{n-2} \quad \text{or} \quad n(n-1) + 2^{n-2} \quad \text{or} \quad n(n-1)$$

according as

$$n < 8 \quad \text{or} \quad n = 8 \quad \text{or} \quad n > 8.$$

Thus the form is imperfect when $n = 2$ or 4 or 6. But it is perfect when $n \geqslant 8$, since the vectors $\pm \mathbf{p}_i \pm \mathbf{p}_j$ are likewise minimal for D_n, as we saw in §8. Hence

13.2 *The forms D_n^2 with $n = 8, 10, 12, \ldots$ are extreme.*

We easily find by inspection the contravariant basis

$$\mathbf{t}^1 = \mathbf{p}_1 - \mathbf{p}_n,\ \mathbf{t}^2 = \mathbf{p}_1 + \mathbf{p}_2 - 2\mathbf{p}_n,\ \ldots,\ \mathbf{t}^{n-3} = \mathbf{p}_1 + \ldots + \mathbf{p}_{n-3} - (n-3)\mathbf{p}_n,$$
$$\mathbf{t}^{n-2} = \mathbf{t}^{n-1} - \mathbf{p}_{n-1} + \mathbf{p}_n,\ \mathbf{t}^{n-1} = \tfrac{1}{2}(\mathbf{p}_1 + \ldots + \mathbf{p}_n) - \tfrac{1}{2}n\mathbf{p}_n,\ \mathbf{t}^n = -2\mathbf{p}_n.$$

When $\frac{1}{2}n$ is even, these contravariant $\mathbf{t}$'s generate the same lattice as the covariant $\mathbf{t}$'s. But when $\frac{1}{2}n$ is odd they generate the image of that lattice by reflection in one of the hyperplanes $\xi^i = 0$; for then we can give $\mathbf{p}_n$ an even coefficient by writing $\mathbf{t}^{n-1} = \frac{1}{2}(\mathbf{p}_1 + \ldots + \mathbf{p}_{n-1} - \mathbf{p}_n) - (\frac{1}{2}n - 1)\mathbf{p}_n$, which shows that the reciprocal lattice consists of the transforms of O_0 and O_1, instead of O_0 and O_3. In other words, the reciprocal lattice either coincides with the original or is its reflected image. Hence

13.3 *The form $2D_n^2$ is equivalent to its own reciprocal.*

Clearly $\mathbf{S}$, of order $2^{n-1}n!$, is the whole group of automorphs except when $n = 4$ or 8. But $2D_4^2$ is obviously equivalent to C_4, and therefore also to $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$; and D_8^2 is obviously equivalent to E_8. Incidentally D_n^2 (with $\Delta = 2^{-n}$) remains an integral form of least possible determinant whenever n is divisible by 8. (Since $D_8^2 \sim A_8^3$, we might expect D_{24}^2 to be equivalent to A_{24}^5; but this is not so, as $s = 552$ for the former and 300 for the latter.)

To prove that D_n^2 is equivalent to the W_n of Korkine and Zolotareff [**31**, p. 367] we can use the basis

$$\mathbf{p}_1 + \mathbf{p}_2,\ \mathbf{p}_1 - \mathbf{p}_2,\ \mathbf{p}_1 + \mathbf{p}_3,\ \mathbf{p}_1 + \mathbf{p}_4,\ \ldots,\ \mathbf{p}_1 + \mathbf{p}_{n-1},\ \tfrac{1}{2}(\mathbf{p}_1 + \ldots + \mathbf{p}_n),$$

which yields the form

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$$\{x^1(\mathrm{p}_1+\mathrm{p}_2)+x^2(\mathrm{p}_1-\mathrm{p}_2)+x^3(\mathrm{p}_1+\mathrm{p}_3)+\ldots+x^{n-1}(\mathrm{p}_1+\mathrm{p}_{n-1})+\tfrac{1}{2}x^n(\mathrm{p}_1+\ldots+\mathrm{p}_n)\}^2$$
$$= (x^1+x^2+\ldots+x^{n-1}+\tfrac{1}{2}x^n)^2+(x^1-x^2+\tfrac{1}{2}x^n)^2$$
$$+(x^3+\tfrac{1}{2}x^n)^2+\ldots+(x^{n-1}+\tfrac{1}{2}x^n)^2+(\tfrac{1}{2}x^n)^2$$
$$= 2\{\sum_1^n (x^i)^2+\sum_{j<k} x^jx^k - x^1x^2 - x^2x^n + \frac{n-8}{8}(x^n)^2\}.$$

14. The form E_6^3. Beniamino Segre [**39**, p. 3, §4] denotes the twenty-seven lines on the general cubic surface by the symbols

$$0jk, \quad k0j, \quad jk0,$$

where j and k take the values 1, 2, 3, independently. Typical relations of incidence are as follows: 011 intersects the ten lines

$$022, \quad 023, \quad 032, \quad 033, \quad k01, \quad j10$$

(each having just one coordinate in common with 011) and is skew to the remaining sixteen lines.

The corresponding vertices of the six-dimensional polytope 2_{21} [**16**, p. 469] are

14.1 $$(0, \omega^j, -\omega^k), \quad (-\omega^k, 0, \omega^j), \quad (\omega^j, -\omega^k, 0),$$

where $\omega = e^{2\pi i/3}$. Here the values of j and k are most conveniently taken to be 0, 1, 2, but can just as well be 1, 2, 3, making the agreement complete. Corresponding to the relations of incidence, we have the fact that two vertices

$$(\xi^1, \xi^2, \xi^3) \quad \text{and} \quad (\eta^1, \eta^2, \eta^3)$$

belong to a diagonal or to an edge according as the real part of

$$\xi^1\bar{\eta}^1+\xi^2\bar{\eta}^2+\xi^3\bar{\eta}^3$$

is equal to -1 or $\frac{1}{2}$.

If we interpret ω as a primitive root of the field $GF(2^2)$, the minus signs in 14.1 can be omitted, and we have Frame's notation for the twenty-seven lines [**21**, p. 660]. This notation is exactly the same as Segre's, except that Frame uses the symbols 1, ω, $\bar{\omega}$ where Segre uses 1, 2, 3.

Returning to the interpretation of ω as an ordinary complex number, we observe that the twenty-seven vectors 14.1 (in complex 3-space) generate the lattice of points (ξ^1, ξ^2, ξ^3) whose coordinates are integers of the Eisenstein field $R(\omega)$ satisfying

$$\xi^1+\xi^2+\xi^3 \equiv 0 \pmod{\lambda},$$

where $\lambda = 1-\omega$. This lattice, whose vertex figure consists of the two "semi-reciprocal" 2_{21}'s

$$(0, \pm\omega^j, \mp\omega^k), \quad (\mp\omega^k, 0, \pm\omega^j), \quad (\pm\omega^j, \mp\omega^k, 0),$$

is easily identified with the lattice representing E_6^3 (see page 423). In fact, one of the three superposed 2_{22}'s is given by

$$\xi^1 \equiv \xi^2 \equiv \xi^3 \pmod{\lambda},$$

as we saw at the end of §8. The other two are derived from this by adding, in turn, the vectors $(0, 1, -1)$ and $(0, -1, 1)$.

As a basis for the whole lattice we may take the six vectors

$$\mathbf{t}_1 = (\lambda\omega, 0, 0),\ \mathbf{t}_2 = (\lambda, 0, 0),\ \mathbf{t}_3 = (-1, -1, -1),$$
$$\mathbf{t}_4 = (0, \lambda, 0),\quad \mathbf{t}_5 = (0, \omega, -\omega),\quad \mathbf{t}_6 = (0, 0, \lambda\omega).$$

For, using $x^1 \ldots x^6$ as an abbreviation for $x^1\mathbf{t}_1 + \ldots + x^6\mathbf{t}_6$, we find

$$\begin{array}{lll}
000010 = (0, \omega, -\omega), & 000011 = (0, \omega, -\bar{\omega}), & 012221 = (\bar{\omega}, 0, -1),\\
000110 = (0, 1, -\omega), & 000111 = (0, 1, -\bar{\omega}), & 112221 = (\omega, 0, -1),\\
001110 = (-1, 0, \bar{\omega}), & 001111 = (-1, 0, \omega), & 122221 = (1, 0, -1),\\
011110 = (-\omega, 0, \bar{\omega}), & 011111 = (-\omega, 0, \omega), & 012342 = (\bar{\omega}, -\bar{\omega}, 0),\\
111110 = (-\bar{\omega}, 0, \bar{\omega}), & 111111 = (-\bar{\omega}, 0, \omega), & 112342 = (\omega, -\bar{\omega}, 0),\\
 & & 122342 = (1, -\bar{\omega}, 0),\\
000121 = (0, -\bar{\omega}, 1), & & 123342 = (0, \omega, -1),\\
001121 = (-1, \omega, 0), & 001221 = (-1, 1, 0), & 123442 = (0, 1, -1),\\
011121 = (-\omega, \omega, 0), & 011221 = (-\omega, 1, 0), & 123452 = (0, -\bar{\omega}, \bar{\omega}),\\
111121 = (-\bar{\omega}, \omega, 0), & 111221 = (-\bar{\omega}, 1, 0), & 123453 = (0, -\bar{\omega}, \omega).
\end{array}$$

The general vector

$$x^1x^2x^3x^4x^5x^6 = (\lambda\omega x^1 + \lambda x^2 - x^3,\ - x^3 + \lambda x^4 + \omega x^5,\ - x^3 - \omega x^5 + \lambda\omega x^6)$$

has norm

$$\begin{aligned}
&(\lambda\omega x^1 + \lambda x^2 - x^3)\,(-\lambda\omega x^1 + \bar{\lambda} x^2 - x^3)\\
&+(- x^3 + \lambda x^4 + \omega x^5)\,(- x^3 + \bar{\lambda} x^4 + \bar{\omega} x^5)\\
&+(- x^3 - \omega x^5 + \lambda\omega x^6)\,(- x^3 - \bar{\omega} x^5 - \lambda\omega x^6)\\
= {}& 3(x^1)^2 - 3x^1x^2 + 3(x^2)^2 - 3x^2x^3 + 3(x^3)^2 - 3x^3x^4 + 3(x^4)^2 - 3x^4x^5\\
&+ 2(x^5)^2 - 3x^5x^6 + 3(x^6)^2\\
= {}& 3A_6 - (x^5)^2.
\end{aligned}$$

We therefore define

$$E_6{}^3 = A_6 - \tfrac{1}{3}(x^5)^2.$$

Alternatively we may use a Euclidean hyperplane of Minkowskian 7-space. As we remarked in §8, Du Val [**18**, p. 33] obtained the vertices of 2_{q1} as integer points on $S^{2,0}$. In particular, the vertices of 2_{21} are those integer points which satisfy the two equations

$$\xi^1 + \ldots + \xi^6 = 3\xi^7 - 2, \quad (\xi^1)^2 + \ldots + (\xi^6)^2 = (\xi^7)^2,$$

namely the twenty-seven points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_6, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_6, \mathbf{c}_{12}, \mathbf{c}_{13}, \ldots, \mathbf{c}_{56}$, where

$$\mathbf{a}_1 = (1, 0, 0, 0, 0, 0; 1),\ \mathbf{c}_{12} = (0, 0, 1, 1, 1, 1; 2),\ \mathbf{b}_1 = (2, 1, 1, 1, 1, 1; 3),$$

and so on. Shifting the origin to the centre, we obtain, in the Euclidean 6-space $\xi^1 + \ldots + \xi^6 = 3\xi^7$,

$$\mathbf{a}_1 = (\tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{2}{3}; -1), \quad \mathbf{c}_{12} = (-\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}; 0),$$
$$\mathbf{b}_1 = (\tfrac{4}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}; 1),$$

etc. Identifying these points with the corresponding vectors, we record the useful combinations

$$\mathbf{a}_1 - \mathbf{a}_2 = \mathbf{b}_1 - \mathbf{b}_2 = (1, -1, 0, 0, 0, 0; 0), \qquad \text{etc.},$$
$$\mathbf{b}_1 - \mathbf{a}_1 = \mathbf{b}_2 - \mathbf{a}_2 = \ldots = (1, 1, 1, 1, 1, 1; 2),$$
$$\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 + \mathbf{a}_6 = (-3, -3, -3, -3, -3, -3; -6) = -3(\mathbf{b}_1 - \mathbf{a}_1).$$

Since $\mathbf{a}_i + \mathbf{b}_j + \mathbf{c}_{ij} = 0$, the lattice generated by the $\mathbf{a}$'s, $\mathbf{b}$'s and $\mathbf{c}$'s is actually generated by the $\mathbf{a}$'s and $\mathbf{b}$'s alone (though not by the $\mathbf{a}$'s alone, since we would need the non-integral combination $\frac{1}{3}\sum \mathbf{a}_i$; see Burnside [7, p. 487]). In fact, a convenient basis is

14.2
$$\mathbf{t}_1 = \mathbf{a}_1 - \mathbf{a}_2,\ \mathbf{t}_2 = \mathbf{a}_2 - \mathbf{a}_3,\ \mathbf{t}_3 = \mathbf{a}_3 - \mathbf{a}_4,$$
$$\mathbf{t}_4 = \mathbf{a}_4 - \mathbf{a}_5,\ \mathbf{t}_5 = \mathbf{a}_5, \qquad \mathbf{t}_6 = \mathbf{b}_1 - \mathbf{a}_1,$$

in terms of which

$$\begin{aligned} \mathbf{a}_i &= \mathbf{t}_i + \mathbf{t}_{i+1} + \ldots + \mathbf{t}_5 \qquad (i \leqslant 5), \\ \mathbf{a}_6 &= \mathbf{a}_1 + \ldots + \mathbf{a}_6 - (\mathbf{a}_1 + \ldots + \mathbf{a}_5) \\ &= -3\mathbf{t}_6 - (\mathbf{t}_1 + 2\mathbf{t}_2 + 3\mathbf{t}_3 + 4\mathbf{t}_4 + 5\mathbf{t}_5) \\ &= -(\mathbf{t}_1 + 2\mathbf{t}_2 + 3\mathbf{t}_3 + 4\mathbf{t}_4 + 5\mathbf{t}_5 + 3\mathbf{t}_6) \end{aligned}$$

and, of course,

$$\mathbf{b}_i = \mathbf{a}_i + \mathbf{t}_6 \qquad (i \leqslant 6).$$

Expanding 14.2, we have

$$\mathbf{t}_1 = (1, -1, 0, 0, 0, 0; 0), \quad \mathbf{t}_2 = (0, 1, -1, 0, 0, 0; 0),$$
$$\mathbf{t}_3 = (0, 0, 1, -1, 0, 0; 0),$$
$$\mathbf{t}_4 = (0, 0, 0, 1, -1, 0; 0), \quad \mathbf{t}_5 = (-\tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{1}{3}, -\tfrac{2}{3}; -1),$$
$$\mathbf{t}_6 = (1, 1, 1, 1, 1, 1; 2),$$

whence

$$\begin{aligned} (\textstyle\sum x^i\mathbf{t}_i)^2 &= (x^1 - \tfrac{2}{3}x^5 + x^6)^2 + (-x^1 + x^2 - \tfrac{2}{3}x^5 + x^6)^2 \\ &\quad + (-x^2 + x^3 - \tfrac{2}{3}x^5 + x^6)^2 + (-x^3 + x^4 - \tfrac{2}{3}x^5 + x^6)^2 \\ &\quad + (-x^4 + \tfrac{1}{3}x^5 + x^6)^2 + (-\tfrac{2}{3}x^5 + x^6)^2 - (-x^5 + 2x^6)^2 \\ &= 2A_6 - \tfrac{2}{3}(x^5)^2 = 2E_6^3. \end{aligned}$$

This senary form is eutactic, since all the forms 10.5 are eutactic (by 4.3). To test it for perfection, we ask whether its minimal vectors can lie on a cone

$$\sum_1^7 \sum_1^7 b_{ij}\xi^i\xi^j = 0, \ \sum_1^6 \xi^i = 3\xi^7.$$

Now, the fifteen vectors $-\mathbf{c}_{ij} = \mathbf{p}_i + \mathbf{p}_j - \frac{1}{3}(\mathbf{p}_1 + \ldots + \mathbf{p}_6)$ all lie in the 5-space $\xi^1 + \ldots + \xi^6 = 0$, $\xi^7 = 0$; and we saw on page 431 that any cone containing them must degenerate into this 5-space and another. Thus it

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only remains to be seen whether the twelve vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ all lie in one 5-space. They certainly do not, since the corresponding points form two simplexes in parallel 5-spaces. Hence

14.3 *The senary form* $E_6^3 = A_6 - \frac{1}{3}(x^5)^2$ *is extreme.*

The determinant of this form is easily found from 11.1 by computing, in turn, the determinants of $2A_1$, $2A_2$, $2A_3$, $2A_4$, $2A_4 - 2x^4x^5 + \frac{4}{3}(x^5)^2$ and $2A_4 - 2x^4x^5 + \frac{4}{3}(x^5)^2 - 2x^5x^6 + 2(x^6)^2 = 2E_6^3$, namely

$$2, \quad 2 \cdot 2 - 1 = 3, \quad 2 \cdot 3 - 2 = 4, \quad 2 \cdot 4 - 3 = 5, \quad \tfrac{4}{3} \cdot 5 - 4 = \tfrac{8}{3}$$

and $2 \cdot \frac{8}{3} - 5 = \frac{1}{3}$. Thus $\Delta = \frac{1}{3}$. Since $M = \frac{4}{3}$,

$$\frac{M^6}{\Delta} = \frac{2^{12}}{3^5} = \frac{4096}{243}.$$

The two forms D_6 and E_6^3 provide the surprising spectacle of two ways of packing equal spheres in six dimensions, the number of spheres touching any one sphere being 60 or 54, respectively, although the latter is the denser packing (since $2^4 < 2^{12}/3^5$).

Instead of 14.2, we could have taken as basis

$$\mathbf{t}_1 = \mathbf{c}_{56} - \mathbf{a}_1, \quad \mathbf{t}_2 = \mathbf{a}_1 - \mathbf{a}_2, \quad \mathbf{t}_3 = \mathbf{a}_2 - \mathbf{a}_3, \quad \mathbf{t}_4 = \mathbf{a}_3 - \mathbf{a}_4, \quad \mathbf{t}_5 = -\mathbf{c}_{45}, \quad \mathbf{t}_6 = \mathbf{a}_6 - \mathbf{a}_5,$$

obtaining the same expression again for $(\sum x^i\, \mathbf{t}_i)^2$. Since $\mathbf{a}_i - \mathbf{a}_j = \mathbf{c}_{jk} - \mathbf{c}_{ik}$, the five vectors $\mathbf{t}_2, \ldots, \mathbf{t}_6$ (without $\mathbf{t}_1$) generate the fifteen $\mathbf{c}$'s and are thus a basis for the lattice representing A_5^3 or $D_5 - \frac{1}{3}(x^5)^2$ (which we found to be equivalent to the Z of Korkine and Zolotareff). Leaving out $\mathbf{t}_1$ means setting $x^1 = 0$. Hence, after another trivial change of notation,

14.4 *The extreme quinary form* A_5^3 *is equivalent to* $A_5 - \frac{1}{3}(x^4)^2$.

This equivalence can be verified directly by comparing the basis

$$\mathbf{p}_1 - \mathbf{p}_2, \quad \mathbf{p}_2 - \mathbf{p}_3, \quad \mathbf{p}_3 - \mathbf{p}_4, \quad \mathbf{p}_4 + \mathbf{p}_5 - \tfrac{1}{3}\textstyle\sum \mathbf{p}, \quad \mathbf{p}_6 - \mathbf{p}_5$$

for $A_5 - \frac{1}{3}(x^4)^2$ with the basis

$$\mathbf{p}_1 - \mathbf{p}_2, \quad \mathbf{p}_2 - \mathbf{p}_3, \quad \mathbf{p}_3 - \mathbf{p}_4, \quad \mathbf{p}_4 - \mathbf{p}_5, \quad \mathbf{p}_4 + \mathbf{p}_5 - \tfrac{1}{3}\textstyle\sum \mathbf{p}$$

for $D_5 - \frac{1}{3}(x^5)^2$.

15. Conclusion. The forms that we have been discussing are all derived from groups generated by reflections in the manner explained in §§5 and 10. The principal results are epitomized in the table (page 439) of n-ary forms up to $n = 11$, which covers all classes of extreme forms for $n \leqslant 6$, possibly also for $n = 7$. The actual expressions are given on pages 394 and 405.

For an extreme form, M^n/Δ is (locally) maximum, i.e., Δ/M^n is minimum. The table records the more convenient number $2^n\Delta/M^n$. This attains its

TABLE OF THE SIMPLEST EXTREME FORMS

Form	Korkine and Zolotareff's symbol	Group of automorphs	Order	s	M	$2^n\Delta$	$\left(\frac{2}{M}\right)^n\Delta$
A_1	U_1	[ ]	2	1	1	2	2
$A_2 \sim G_2$	U_2	[6]	12	3	1	3	3
$A_3 \sim B_3 \sim D_3$	U_3	[3, 4]	48	6	1	4	4
A_4	U_4	$2[3^3]$	240	10	1	5	5
$B_4 \sim D_4 \sim F_4$	V_4	[3, 4, 3]	1152	12	1	4	4
A_5	U_5	$2[3^4]$	$2 \cdot 6!$	15	1	6	6
A_5^3	Z	$2[3^4]$	$2 \cdot 6!$	15	$\frac{2}{3}$	$\frac{2}{3}$	$3^4/2^4$
$B_5 \sim D_5$	V_5	$[3^3, 4]$	$2^5 5!$	20	1	4	4
A_6	U_6	$2[3^5]$	$2 \cdot 7!$	21	1	7	7
$B_6 \sim D_6$	V_6	$[3^4, 4]$	$2^6 6!$	30	1	4	4
E_6^3		$2[3^{2,2,1}]$	$144 \cdot 6!$	27	$\frac{2}{3}$	$\frac{1}{3}$	$3^5/2^6$
E_6	X	$2[3^{2,2,1}]$	$144 \cdot 6!$	36	1	3	3
A_7	U_7	$2[3^6]$	$2 \cdot 8!$	28	1	8	8
$B_7 \sim D_7$	V_7	$[3^5, 4]$	$2^7 7!$	42	1	4	4
$A_7^4 = E_7^2$		$[3^{3,2,1}]$	$8 \cdot 9!$	28	$\frac{3}{4}$	$\frac{1}{2}$	$2^{13}/3^7$
$A_7^2 = E_7$	Y	$[3^{3,2,1}]$	$8 \cdot 9!$	63	1	2	2
A_8	U_8	$2[3^7]$	$2 \cdot 9!$	36	1	9	9
$B_8 \sim B_8$	V_8	$[3^6, 4]$	$2^8 8!$	56	1	4	4
$A_8^3 \sim D_8^2 \sim E_8$	W_8	$[3^{4,2,1}]$	$192 \cdot 10!$	120	1	1	1
A_9	U_9	$2[3^8]$	$2 \cdot 10!$	45	1	10	10
$B_9 \sim D_9$	V_9	$[3^7, 4]$	$2^9 9!$	72	1	4	4
A_9^5		$2[3^8]$	$2 \cdot 10!$	45	$\frac{4}{5}$	$\frac{2}{5}$	$5^8/2^{17}$
A_9^2		$2[3^8]$	$2 \cdot 10!$	45	1	$\frac{5}{2}$	$\frac{5}{2}$
A_{10}	U_{10}	$2[3^9]$	$2 \cdot 11!$	55	1	11	11
$B_{10} \sim D_{10}$	V_{10}	$[3^8, 4]$	$2^{10} 10!$	90	1	4	4
D_{10}^2	W_{10}	$[3^{7,1,1}]$	$2^9 10!$	90	1	1	1
A_{11}	U_{11}	$2[3^{10}]$	$2 \cdot 12!$	66	1	12	12
$B_{11} \sim D_{11}$	V_{11}	$[3^9, 4]$	$2^{11} 11!$	110	1	4	4
A_{11}^2		$2[3^{10}]$	$2 \cdot 12!$	66	1	3	3
A_{11}^6		$2[3^{10}]$	$2 \cdot 12!$	66	$\frac{5}{6}$	$\frac{1}{3}$	$2^{11}3^{10}/5^{11}$
A_{11}^3		$2[3^{10}]$	$2 \cdot 12!$	66	1	$\frac{4}{3}$	$\frac{4}{3}$
A_{11}^4		$2[3^{10}]$	$2 \cdot 12!$	66	1	$\frac{3}{4}$	$\frac{3}{4}$

smallest possible value (for each n) when the form is absolutely extreme (viz, $A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8$). The absolutely extreme forms for $n > 8$ are not listed, because they are not related to groups generated by reflections; in fact, they are essentially more complicated: their groups of automorphs are not transitive on their minimal vectors.

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