Proceedings of the Edinburgh Mathematical Society (2024) **67**, 1148–1166
doi:10.1017/S0013091524000580

TEICHMÜLLER DISPLACEMENT THEOREM ON GROMOV HYPERBOLIC SPACES

QINGSHAN ZHOU[1], SAMINATHAN PONNUSAMY[2] AND QIANGHUA LUO[1]

[1] *School of Mathematics, Foshan University, Foshan, Guangdong Province, People's Republic of China*
[2] *Department of Mathematics, Indian Institute of Technology Madras, Chennai, Tamilnadu, India*

Corresponding author: Qianghua Luoemail: luo.qh@fosu.edu.cn

(Received 28 September 2023)

Abstract Given a Gromov hyperbolic domain $G \subsetneq \mathbb{R}^n$ with uniformly perfect Gromov boundary, Zhou and Rasila recently proved that for all quasiconformal homeomorphisms $\psi\colon G \to G$ with identity value on the Gromov boundary, the quasihyperbolic displacement $k_G(x, \psi(x))$ for all $x \in G$ is bounded above. In this paper, we generalize this result and establish Teichmüller displacement theorem for quasi-isometries of Gromov hyperbolic spaces in a quantitative way. As applications, we obtain its connections to bilipschitz extensions of certain Gromov hyperbolic spaces.

Keywords: Gromov hyperbolic space; Teichmüller displacement theorem; quasi-isometry; roughly starlike; uniformly perfect

2010 Mathematics subject classification: Primary 30L10; Secondary 30L05

1. Introduction and main results

1.1. Background

Let $G \subsetneq \mathbb{R}^n$ $(n \geq 2)$ be a domain, where the closure $\overline{G}$ and the boundary ∂G of G are taken in the topology of the Riemann sphere $\overline{R}^n = \mathbb{R}^n \cup \{\infty\}$. We define

$$\begin{aligned}\mathcal{T}_H(G) := \big\{\psi : \overline{G} \to \overline{G} \;\big|\; &\psi \text{ is a homeomorphism such that} \\ &\text{the restriction } \ \psi|_G \text{ is } H\text{-}QC \text{ and } \psi|_{\partial G} = \mathrm{id}_{\partial G}\big\},\end{aligned}$$

where the abbreviation H-QC is used for H-quasiconformal and $\mathrm{id}_{\partial G}$ denotes the identity map on ∂G.

Originally, Teichmüller displacement problem is to determine how far a given point $x \in G$ can be mapped under a map $\psi \in \mathcal{T}_H(G)$. For the domain $G = \mathbb{R}^2 \setminus \{(0,0), (1,0)\}$, it was shown by Teichmüller [19] that the displacement with respect to the hyperbolic distance h_G of G satisfies inequality

$$h_G(x, \psi(x)) \leq \log H \text{ for all } x \in G.$$

From then onwards, many researchers considered the same problem in several different settings and applied Teichmüller type results in the study of quasiconformal homogeneity of domains, see [11, 12, 14, 23, 30] and the references therein. The recent monograph by Hariri et al. [8] provides a chapter which systematically introduces many results and background information about Teichmüller displacement problem.

For example, Manojlović and Vuorinen [14] investigated spatial quasiconformal homeomorphisms of the unit balls onto itself with identity boundary value and obtained an analogue of Teichmüller's result. In [23], Vuorinen and Zhang studied the Teichmüller displacement problem with respect to the quasihyperbolic metric on uniform and convex domains with uniformly perfect boundaries. It was proved by Bonfert–Taylor et al. [1] that all quasiconformal homeomorphisms of hyperbolic manifolds onto itself with identity boundary value are uniformly close to isometries.

Inspired by these investigations, Zhou and Rasila [30] recently studied the Teichmüller displacement problem from the point of view of Gromov hyperbolic geometry [4, 5]. On the one hand, it was shown in [30, Theorem 1.1] that both the displacements with respect to the distance ratio metric j_G and its modification $\widetilde{j}_G$ are bounded above for all quasiconformal homeomorphisms $\psi \in \mathcal{T}_H(G)$ provided ∂G is uniformly perfect. For the Gromov hyperbolicity of these two metrics, we refer to [7, 28].

On the other hand, the authors [30] investigated Teichmüller displacement problem for the class of Gromov hyperbolic domains which was introduced by Bonk et al. [2]. Recall that $G \subsetneq \mathbb{R}^n$ is said to be a Gromov hyperbolic domain if the domain G equipped with its quasihyperbolic metric k_G is δ-hyperbolic for some $\delta \geq 0$. For a given Gromov hyperbolic domain $G \subsetneq \mathbb{R}^n$ with uniformly perfect Gromov boundary $\partial_\infty G$, it was proved in [30, Theorem 1.2] that for all quasiconformal mappings $\psi\colon G \to G$ with $\psi|_{\partial_\infty G} = \mathrm{id}_{\partial_\infty G}$, the quasihyperbolic displacement $k_G(x, \psi(x))$ is bounded above for all $x \in G$.

It follows from [2, Proposition 2.8] that (G, k_G) is a proper geodesic metric space. By [6, Theorem 3], we know that each quasiconformal homeomorphism $\psi\colon G \to G$ is a quasi-isometry with respect to the quasihyperbolic metric. Motivated by this study, we consider here the Teichmüller displacement problem on Gromov hyperbolic spaces when quasiconformal maps are replaced by quasi-isometries.

1.2. Main results

Throughout this paper, we assume that (X, d) is a proper geodesic Gromov hyperbolic space with $X^* = X \cup \partial_\infty X$ its Gromov closure, and that $f\colon X \to X$ is a (λ, μ)-quasi-isometry. The displacement of $x \in X$ under f is denoted by $d(x, f(x))$. The number $\sup\{d(x, f(x)) | x \in X\}$ is called the *displacement* of f on X. It is not difficult to see from [3, Proposition 6.3] that f has a natural bijective extension from X to $\partial_\infty X$, denoted by $f|_{\partial_\infty X}$. This means that the image of any Gromov sequence under f is also Gromov. Set

https://doi.org/10.1017/S0013091524000580 Published online by Cambridge University Press

$$\mathcal{T}_{\lambda,\mu}(X^*) = \big\{ f : X^* \to X^* \;\big|\; f|_X \text{ is a } (\lambda,\mu)\text{-quasi-isometry},\ f|_{\partial_\infty X} = \mathrm{id}_{\partial_\infty X} \big\}.$$

Our main result is the following:

Theorem 1.1. *Let $\delta, K, \mu \geq 0$ and $C, \lambda \geq 1$. If (X, d) is a proper geodesic space that is δ-hyperbolic and K-roughly starlike with respect to $\xi \in \partial_\infty X$, and $\partial_\infty X$ is a C-uniformly perfect set, then there is a number $\Lambda = \Lambda(\delta, K, C, \lambda, \mu)$ such that $d(x, f(x)) \leq \Lambda$ for every $f \in \mathcal{T}_{\lambda,\mu}(X^*)$ and for all $x \in X$.*

Remark 1.2. We say that $\partial_\infty X$ is a uniformly perfect set if it is C-uniformly perfect with respect to a certain visual metric. This makes sense because $\partial_\infty X$ equipped with any two visual metrics are quasimöbius to each other by [5, Corollary 5.2.9], and uniform perfectness is preserved under quasimöbius maps due to [24, Lemma C]. Also, we note that the uniform perfectness for $\partial_\infty X$ cannot be removed, see [30, Remark 1.1]. All connected metric spaces are uniformly perfect. For more background and applications of uniformly perfect sets in geometric function theory and analysis on metric spaces, we refer to [5, 9, 15, 17, 18, 24].

Remark 1.3. Note that Theorem 1.1 is a generalization of [30, Theorem 1.2]. The strategy for proving Theorem 1.1 is different from that of [30, Theorem 1.2], where they applied the bounded uniformization of Gromov hyperbolic spaces due to Bonk et al. [2]. In this paper, our main tool is the unbounded uniformization theory, developed recently in [29].

For the definition of rough starlikeness, we refer the reader to § 2.5. We remark that if X is roughly starlike with respect to $\xi \in \partial_\infty X$, then $\partial_\infty X$ contains at least two points. The class of Gromov hyperbolic spaces that are roughly starlike is very large. For example, it includes metric trees, Gromov hyperbolic domains in $\mathbb{R}^n$ or annular quasiconvex spaces [10, 22], Gromov hyperbolic manifolds [25], negatively curved solvable Lie groups [16, 26] and hyperbolic fillings [3, 5]. Hence, Theorem 1.1 is valid for these Gromov hyperbolic spaces.

The notion of rough starlikeness with respect to a distinguished point within the space was introduced by Bonk et al. [2]. This concept is equivalent to the visual property defined by Bonk and Schramm [3], where they demonstrated that Gromov hyperbolic spaces with locally bounded geometry can be quasi-isometrically embedded into the classical hyperbolic spaces $\mathbb{H}^n$. This property has served as an important tool in [10, 27, 30].

In this paper, we establish the following relationships between these two concepts in a quantitative way.

Theorem 1.4. *Let X be a proper geodesic δ-hyperbolic space, where $\partial_\infty X$ contains at least two points. Then the following conditions are equivalent:*

(1) *X is K_1-roughly starlike with respect to $\xi \in \partial_\infty X$,*
(2) *X is K_2-roughly starlike with respect to each point of X,*

(3) *X is K_3-roughly starlike with respect to $w \in X$ and*

$$diam(\partial_\infty X, d_{w,\varepsilon}) \geq \tau_0 > 0,$$

where $d_{w,\varepsilon}$ is a visual metric on $\partial_\infty X$ with parameter ε and base point w.

The constants K_i, for $i = 1, 2, 3$, depend only on each other, τ_0, and δ.

In view of the above considerations, we establish the Teichmüller displacement theorem on Gromov hyperbolic spaces that is roughly starlike with respect to an interior point. Employing Theorem 1.4, we obtain the following consequence of Theorem 1.1.

Corollary 1.5. *Let $\delta, K, \mu \geq 0$, $C, \lambda \geq 1$ and $\vartheta > 0$. Suppose (X, d) is a proper geodesic δ-hyperbolic space and K-roughly starlike with respect to $w \in X$. If $\partial_\infty X$ is a C-uniformly perfect set with $\vartheta = diam(\partial_\infty X, d_{w,\varepsilon}) > 0$, then there is a number $\Lambda_1 = \Lambda_1(\delta, K, C, \lambda, \mu, \vartheta)$ such that $d(x, f(x)) \leq \Lambda_1$ for all $x \in X$ and for every $f \in \mathcal{T}_{\lambda,\mu}(X^*)$.*

This paper is organized as follows. In § 2, we focus on Gromov hyperbolic geometry and properties of quasi-isometries, and then we prove Theorem 1.4. The proof of Theorem 1.1 is given in § 3. In § 4, we provide two examples and some applications of our main results.

2. Gromov hyperbolic spaces and quasi-isometric maps

2.1. Metric geometry

Let (Z, d) be a metric space. The open ball and the closed ball of radius r centred at $x \in Z$ are denoted by $B(x, r)$ and $\overline{B}(x, r)$, respectively. The space Z is called *proper* if its closed balls are compact. We use $\mathrm{diam}(W)$ to denote the diameter of a set $W \subset Z$. For $C \geq 1$, a metric space Z is called *C-uniformly perfect*, if for each $x \in Z$ and every $r > 0$, $B(x, r) \setminus B(x, r/C) \neq \emptyset$ provided $Z \setminus B(x, r) \neq \emptyset$.

A *geodesic arc* α between x and y in Z is a map $\alpha \colon I = [0, l] \to Z$ from an interval I to Z such that $\alpha(0) = x$, $\alpha(l) = y$, and $d(\alpha(t), \alpha(t')) = |t - t'|$ for all $t, t' \in I$. If $I = [0, \infty)$, then α is called a *geodesic ray*. If $I = \mathbb{R}$, then α is called a *geodesic line*. The space Z is said to be *geodesic* if every pair of points can be connected with a geodesic arc. Let $[x, y]$ denote the geodesic between x and y in Z.

2.2. Maps

The identity map of a set W is denoted by id_W. Let $f\colon (Z, d) \to (Z', d')$ be a map (not necessarily continuous) between metric spaces Z and Z', and let $\lambda \geq 1$ and $\mu \geq 0$ be constants. We say that f is a *(λ, μ)-quasi-isometric map* if for all $x, y \in Z$,

$$\lambda^{-1} d(x, y) - \mu \leq d'(f(x), f(y)) \leq \lambda d(x, y) + \mu.$$

If in addition, every point $y \in Z'$ has the distance at most μ from the set $f(Z)$, then f is called a *(λ, μ)-quasi-isometry*. Moreover, if f is a homeomorphism and $\mu = 0$, then it

https://doi.org/10.1017/S0013091524000580 Published online by Cambridge University Press

is called a λ-*bilipschitz map*. A curve $\gamma\colon I \to Z$ is called a (λ, μ)-*quasigeodesic* if γ is a (λ, μ)-quasi-isometric map.

2.3. Gromov hyperbolicity

Let (X, d) be a metric space. Fix a base point $w \in X$. For $x, y \in X$, we define

$$(x|y)_w = \frac{1}{2}\big(d(x, w) + d(y, w) - d(x, y)\big).$$

This number is called the *Gromov product* of x and y with respect to w. We say that X is *Gromov hyperbolic*, if there is a constant $\delta \geq 0$ such that

$$(x|y)_w \geq \min\{(x|z)_w, (z|y)_w\} - \delta \text{ for all } x, y, z, w \in X.$$

In this paper, we assume that Gromov hyperbolic spaces are unbounded.

Suppose X is a Gromov hyperbolic space. A sequence $\{x_i\}$ in X is called a *Gromov sequence* if $(x_i|x_j)_w \to \infty$ as i, $j \to \infty$. Two such sequences $\{x_i\}$ and $\{y_j\}$ are said to be *equivalent* if $(x_i|y_i)_w \to \infty$ as $i \to \infty$. The *Gromov boundary* $\partial_\infty X$ of X is defined to be the set of all equivalence classes of Gromov sequences, and $X^* = X \cup \partial_\infty X$ is called the *Gromov closure* of X. If (X, d) is proper geodesic, then the Gromov boundary is also equivalent to the *geodesic boundary*, which is defined as the set of equivalence classes of geodesic rays, where two geodesic rays are equivalent if they have finite Hausdorff distance.

Lemma 2.1. *([4, Chapter III.H. Lemmas* 3.1 *and* 3.2*]) Suppose that X is a proper geodesic space that is δ-hyperbolic. Then for each $x \in X$ and $\xi \in \partial_\infty X$, there exists a geodesic ray $\gamma\colon [0, \infty) \to X$ with $\gamma(0) = x$ and $\gamma(\infty) = \xi$. Similarly, for each pair of distinct points $\xi, \eta \in \partial_\infty X$, there exists a geodesic line $\gamma\colon \mathbb{R} \to X$ with $\gamma(-\infty) = \xi$ and $\gamma(\infty) = \eta$.*

For all $x \in X$ and $\xi \in \partial_\infty X$, the Gromov product $(x|\xi)_w$ of x and ξ is defined by $(x|\xi)_w = \inf\{\liminf_{i\to\infty}(x|y_i)_w \mid \{y_i\} \in \xi\}$. For all ξ, $\zeta \in \partial_\infty X$, the Gromov product $(\xi|\zeta)_w$ of ξ and ζ is defined by $(\xi|\zeta)_w = \inf\{\liminf_{i\to\infty}(x_i|y_i)_w \mid \{x_i\} \in \xi \text{ and } \{y_i\} \in \zeta\}$.

Next, we recall the following results about the Gromov product.

Lemma 2.2. *([21, Lemma* 5.11*]) Let X be a δ-hyperbolic space with $o, z \in X$, and let $\xi, \xi' \in \partial_\infty X$. Then for any sequences $\{y_i\} \in \xi$, $\{y_i'\} \in \xi'$, we have*

(1) $(z|\xi)_o \leq \liminf\limits_{i\to\infty}(z|y_i)_o \leq \limsup\limits_{i\to\infty}(z|y_i)_o \leq (z|\xi)_o + \delta$;
(2) $(\xi|\xi')_o \leq \liminf\limits_{i\to\infty}(y_i|y_i')_o \leq \limsup\limits_{i\to\infty}(y_i|y_i')_o \leq (\xi|\xi')_o + 2\delta$.

Let (X, d) be a δ-hyperbolic space and $w \in X$ be given. For $0 < \varepsilon < \min\{1, 1/(5\delta)\}$, we define

$$\rho_{w,\varepsilon}(\xi, \zeta) = \mathrm{e}^{-\varepsilon(\xi|\zeta)_w}$$

for all ξ and ζ in the Gromov boundary $\partial_\infty X$ of X with the convention that $\mathrm{e}^{-\infty} = 0$.

We now define

$$d_{w,\varepsilon}(\xi,\zeta) := \inf\left\{\sum_{i=1}^{n} \rho_{w,\varepsilon}(\xi_{i-1},\xi_i) \;\middle|\; n \geq 1, \xi = \xi_0, \xi_1, \ldots, \xi_n = \zeta \in \partial_\infty X\right\}.$$

Then $(\partial_\infty X, d_{w,\varepsilon})$ is a metric space with

$$\frac{1}{2}\rho_{w,\varepsilon} \leq d_{w,\varepsilon} \leq \rho_{w,\varepsilon}, \tag{2.3}$$

and we call $d_{w,\varepsilon}$ the *visual metric* of $\partial_\infty X$ with base point $w \in X$ and parameter ε.

2.4. Busemann functions

Let (X, d) be a Gromov δ-hyperbolic space with $o \in X$ and $\xi \in \partial_\infty X$. Let $\mathcal{B}(\xi)$ be the class of Busemann functions based at ξ; see [5, Section 3.1] for more background information. Let $b \in \mathcal{B}(\xi)$ be a Busemann function. For all $x \in X$,

$$b(x) = b_{\xi,o}(x) = b_\xi(x, o) = (\xi|o)_x - (\xi|x)_o.$$

We define the Gromov product of $x, y \in X$ with respect to the Busemann function $b \in \mathcal{B}(\xi)$ as

$$(x|y)_b = \frac{1}{2}(b(x) + b(y) - d(x, y)).$$

Similarly, for $x \in X$ and $\zeta \in \partial_\infty X \setminus \{\xi\}$, the Gromov product $(x|\zeta)_b$ of x and ζ based at b is defined by $(x|\zeta)_b = \inf\{\liminf_{i\to\infty}(x|z_i)_b \mid \{z_i\} \in \zeta\}$. For points ξ_1 and ξ_2 belonging to $\partial_\infty X \setminus \{\xi\}$, we define their Gromov product based at b by $(\xi_1|\xi_2)_b = \inf\{\liminf_{i\to\infty}(x_i|y_i)_b \mid \{x_i\} \in \xi_1, \{y_i\} \in \xi_2\}$.

For $\varepsilon > 0$ with $\mathrm{e}^{22\varepsilon\delta} \leq 2$, we define

$$\rho_{b,\varepsilon}(\xi_1,\xi_2) = \mathrm{e}^{-\varepsilon(\xi_1|\xi_2)_b} \qquad \text{for all } \xi_1, \xi_2 \in \partial_\infty X \setminus \{\xi\}.$$

Then for $i = 1, 2, 3$ with $\xi_i \in \partial_\infty X \setminus \{\xi\}$, we have

$$\rho_{b,\varepsilon}(\xi_1,\xi_2) \leq \mathrm{e}^{22\varepsilon\delta} \max\{\rho_{b,\varepsilon}(\xi_1,\xi_3), \rho_{b,\varepsilon}(\xi_3,\xi_2)\}.$$

We now define

$$d_{b,\varepsilon}(\omega,\zeta) := \inf\left\{\sum_{i=1}^{m} \rho_{b,\varepsilon}(\zeta_{i-1},\zeta_i) \mid m \geq 1, \zeta = \zeta_0, \zeta_1, \ldots, \zeta_m = \omega \in \partial_\infty X \setminus \{\xi\}\right\}.$$

By [5, Lemma 3.3.3], it follows that $(\partial_\infty X \setminus \{\xi\}, d_{b,\varepsilon})$ is a metric space such that $\rho_{b,\varepsilon}/2 \leq d_{b,\varepsilon} \leq \rho_{b,\varepsilon}$. We call $d_{b,\varepsilon}$ a *Hamenstädt metric* on the punctured space $\partial_\infty X \setminus \{\xi\}$ based at ξ with parameter ε.

https://doi.org/10.1017/S0013091524000580 Published online by Cambridge University Press

2.5. Rough starlikeness

We first recall the definition of rough starlikeness of Gromov hyperbolic spaces. See [2, 10, 22, 27, 30] for more information and backgrounds on this topic.

Let X be a proper geodesic δ-hyperbolic space, $w \in X$, $\xi \in \partial_\infty X$ and $K \geq 0$.

Definition 2.4. *We say that X is K-roughly starlike with respect to ξ if for each $x \in X$, there is a point $\zeta \in \partial_\infty X$ and a geodesic line $\gamma = [\xi, \zeta]$ connecting ξ and ζ such that $dist(x, \gamma) \leq K$.*

Definition 2.5. *We say that X is K-roughly starlike with respect to w if for each $x \in X$, there is a point $\zeta \in \partial_\infty X$ and a geodesic ray $\gamma = [w, \zeta]$ emanating from w to ζ such that $dist(x, \gamma) \leq K$.*

Next, we show that the rough starlikeness of Gromov hyperbolic spaces is preserved under quasi-isometries. Although this result is well-known, we have failed to find a reference containing its proof. For completeness, we give a proof here.

Lemma 2.6. *Let $\delta, K, \mu \geq 0$, $\lambda \geq 1$ and let $f\colon (X, d) \to (X', d')$ be a (λ, μ)-quasi-isometry between proper geodesic δ-hyperbolic spaces. We have the following:*

(1) *If X is K-roughly starlike with respect to $\xi \in \partial_\infty X$, then X' is K'-roughly starlike with respect to a point $\xi' \in \partial_\infty X'$, where $K' = K'(\delta, K, \mu, \lambda)$;*
(2) *If X is K-roughly starlike with respect to $w \in X$, then X' is K'-roughly starlike with respect to a point $w' \in X'$, where $K' = K'(\delta, K, \mu, \lambda)$.*

Proof. We only prove (1), because the proof of (2) is similar. It follows from [3, Proposition 6.3] that f induces a bijective map $f\colon \partial_\infty X \to \partial_\infty X'$. Let $\xi' = f(\xi)$. Then we check that X' is K'-roughly starlike with respect to the point $\xi' \in \partial_\infty X'$, where $K' = K'(\delta, K, \mu, \lambda)$.

On the one hand, for a given $x' \in X'$, there is an $x \in X$ such that

$$d'(f(x), x') \leq \mu. \tag{2.7}$$

As X is K-roughly starlike with respect to $\xi \in \partial_\infty X$, there is a $\zeta \in \partial_\infty X$ and a geodesic line $\gamma = [\xi, \zeta]$ joining ξ and ζ such that

$$\mathrm{dist}(x, \gamma) \leq K. \tag{2.8}$$

On the other hand, because $f\colon X \to X'$ is a (λ, μ)-quasi-isometry, we see that $f(\gamma)$ is a (λ, μ)-quasigeodesic line with endpoints $\xi' = f(\xi)$ and $\zeta' = f(\zeta)$. Furthermore, because X' is a proper geodesic δ-hyperbolic space, by [10, Lemma 3.5], it follows that there is a

https://doi.org/10.1017/S0013091524000580 Published online by Cambridge University Press

geodesic line $\alpha = [\xi', \zeta']$ joining ξ' and ζ' such that the Hausdorff distance satisfies the inequality

$$d'_{\mathcal{H}}(\alpha, f(\gamma)) \leq M \tag{2.9}$$

for some constant $M = M(\lambda, \mu, \delta)$. Therefore, we obtain from Equations (2.7), (2.8) and (2.4) that

$$\mathrm{dist}(x', \alpha) \leq \mu + \lambda K + \mu + M =: K'.$$

The lemma follows. □

Finally, we are ready to supply the proof of Theorem 1.4 which connects the above two notions of rough starlikeness.

2.6. Proof of Theorem 1.4

Suppose that X is a proper geodesic δ-hyperbolic space, and $\partial_\infty X$ contains at least two points.

(1) $\Rightarrow$ (2) : Let $w \in X$. For each $x \in X$, there is a point $\xi_x \in \partial_\infty X$ and a geodesic line $[\xi, \xi_x]$ connecting ξ and ξ_x such that

$$\mathrm{dist}(x, [\xi, \xi_x]) \leq K_1, \tag{2.10}$$

because X is K_1-roughly starlike with respect to ξ. By Lemma 2.1, it follows that there are two geodesic rays $[w, \xi]$ and $[w, \xi_x]$ joining w to ξ and ξ_x, respectively. Considering the extended geodesic triangle $\Delta = [w, \xi_x] \cup [\xi_x, \xi] \cup [\xi, w]$ and then applying [21, Theorem 6.24], we see that there is a positive integer N such that

$$\mathrm{dist}(y, [w, \xi_x] \cup [\xi, w]) \leq N\delta \text{ for all } y \in [\xi, \xi_x].$$

This inequality, together with Equation (2.10), shows that

$$\mathrm{dist}(x, [w, \xi_x] \cup [\xi, w]) \leq K_1 + N\delta =: K_2,$$

as desired.

(2) $\Rightarrow$ (3) : Because $\partial_\infty X$ contains at least two points, we may choose two distinct points ξ and ζ from $\partial_\infty X$. By Lemma 2.1, it follows that there is a geodesic line $[\xi, \zeta]$ connecting ξ to ζ. Now, fix a point $w \in X$ in the line $[\xi, \zeta]$. By Lemma 2.2, we find that $(\xi|\zeta)_w \leq 2\delta$. Therefore, by Equation (2.3), we obtain that

$$\mathrm{diam}(\partial_\infty X, d_{w,\varepsilon}) \geq d_{w,\varepsilon}(\xi, \zeta) \geq \frac{1}{2}\mathrm{e}^{-\varepsilon(\xi|\zeta)_w} \geq \frac{1}{2}\mathrm{e}^{-2\varepsilon\delta} =: \tau_0,$$

as required.

https://doi.org/10.1017/S0013091524000580 Published online by Cambridge University Press

(3) $\Rightarrow$ (1) : Because $\mathrm{diam}(\partial_\infty X, d_{w,\varepsilon}) \geq \tau_0 > 0$, we see from Equation (2.3) that there are two points ξ and ζ in $\partial_\infty X$ such that

$$\tau_0 \leq d_{w,\varepsilon}(\xi,\zeta) \leq \mathrm{e}^{-\varepsilon(\xi|\zeta)w},$$

which implies that

$$(\xi|\zeta)_w \leq \frac{1}{\varepsilon}\log\frac{1}{\tau_0}.$$

Again, by Lemma 2.1, there is a geodesic line $[\xi,\zeta]$ connecting ξ to ζ. Moreover, by the extended standard estimate (cf. [21, 6.20]), it follows that there is a positive integer $N_1 \geq 0$ such that

$$\mathrm{dist}(w,[\xi,\zeta]) \leq (\xi|\zeta)_w + N_1\delta \leq \frac{1}{\varepsilon}\log\frac{1}{\tau_0} + N_1\delta =: C_2.$$

Thus there is a point $w_0 \in [\xi,\zeta]$ such that

$$d(w,w_0) \leq C_2.$$

Now, we check that X is K_1-roughly starlike with respect to ξ with a constant $K_1 \geq 0$ depending only on δ that will be decided below.

Fix $x \in X$. Because X is K_3-roughly starlike with respect to w, there is a point $\xi_x \in \partial_\infty X$ and a geodesic ray $[w,\xi_x]$ connecting w and ξ_x such that

$$\mathrm{dist}(x,[w,\xi_x]) \leq K_3. \tag{2.11}$$

Lemma 2.1 ensures that there is a geodesic ray $[w_0,\xi_x]$ joining w_0 to ξ_x. Because $d(w,w_0) \leq C_2$, by the Closeness Lemma (cf. [21, 6.9]), we have the following Hausdorff distance:

$$d_{\mathcal{H}}([w_0,\xi_x],[w,\xi_x]) \leq C_2 + N_2\delta, \tag{2.12}$$

for some positive integer N_2.

Pick a geodesic line $[\xi,\xi_x]$ connecting ξ to ξ_x and consider the extended geodesic triangle $\Delta = [w_0,\xi_x] \cup [\xi_x,\xi] \cup [\xi,w_0]$. Now it follows from [21, Theorem 6.24] that there is a positive integer N_3 such that for all $z \in [w_0,\xi_x]$,

$$\mathrm{dist}(z,[w_0,\xi] \cup [\xi,\xi_x]) \leq N_3\delta. \tag{2.13}$$

Hence we obtain from Equations (2.11), (2.12) and (2.13) that

$$\mathrm{dist}(x,[w_0,\xi] \cup [\xi,\xi_x]) \leq K_3 + C_2 + N_2\delta + N_3\delta =: K_1.$$

This implies that

$$\mathrm{dist}(x, [\zeta, \xi] \cup [\xi, \xi_x]) \leq K_1,$$

completing the proof.

3. Teichmüller displacement theorem

In this section, we study Teichmüller displacement problem on Gromov hyperbolic spaces in a quantitative way and prove Theorem 1.1. For the proof of Theorem 1.1, we use the unbounded uniformization procedure that was developed recently in [29]. We begin with some definitions.

3.1. Quasihyperbolic metric and uniform spaces

Let (Ω, d) be a metric space. The metric completion and metric boundary of Ω are denoted by $\overline{\Omega}$ and $\partial\Omega = \overline{\Omega} \setminus \Omega$, respectively. The space Ω is incomplete if $\partial\Omega \neq \emptyset$. For $z \in \Omega$, the distance between z and $\partial\Omega$ is denoted by $d(z) = \mathrm{dist}(z, \partial\Omega)$.

In this subsection, we assume that (Ω, d) is an incomplete, locally compact and rectifiably connected metric space, and that the identity map $(\Omega, d) \to (\Omega, \ell)$ is continuous, where ℓ is the length metric of Ω induced by d. See [2, Appendix] for more discussions.

Definition 3.1. *As in [2], the quasihyperbolic metric k in (Ω, d) is defined by*

$$k(x, y) = \inf_{\alpha} \int_{\alpha} \frac{ds}{d(z)},$$

where the infimum is taken over all rectifiable curves α in Ω connecting x and y and ds denotes the arc length element with respect to the metric d.

It follows from [2, Proposition 2.8] that (Ω, k) is a proper geodesic space. Next we recall the definition of uniform spaces from the work of Bonk et al. [2] and use this to establish their bounded uniformization theory of Gromov hyperbolic spaces. For more backgrounds, we refer to [10, 23, 27] and the references therein.

Definition 3.2. *Let $A \geq 1$. The space (Ω, d) is called A-uniform if each pair of points x and y in Ω can be connected with a rectifiable arc α in Ω satisfying:*

(1) $\ell(\alpha) \leq A\, d(x, y)$, *and*
(2) $\min\{\ell(\alpha[x, z]), \ell(\alpha[z, y])\} \leq A\, d(z)$ *for all* $z \in \alpha$,

where $\ell(\alpha)$ is the length of α and $\alpha[x, z]$ is the part of α between x and z.

3.2. Unbounded uniformization of Gromov hyperbolic spaces

In this subsection, we assume that (X, d) is a proper geodesic space that is δ-hyperbolic, and $\partial_\infty X$ contains at least two points. Let $o \in X$, $\xi \in \partial_\infty X$ and $b = b_{\xi,o} \colon X \to \mathbb{R}$ a Busemann function based at ξ.

https://doi.org/10.1017/S0013091524000580 Published online by Cambridge University Press

Following the notation of [29], consider the family of conformal deformations of X induced by the densities

$$\rho_\varepsilon(x) = \mathrm{e}^{-\varepsilon b(x)} \text{ for } \varepsilon > 0.$$

The resulting spaces are denoted by $X_\varepsilon = (X, d_\varepsilon)$. One observes that d_ε is a metric on X defined by

$$d_\varepsilon(x, y) = \inf_\alpha \int_\alpha \rho_\varepsilon \, ds, \tag{3.3}$$

where the infimum is taken over all rectifiable curves α in (X, d) joining the points x and y. The metric completion and the boundary of X_ε are denoted by $\overline{X_\varepsilon}$ and $\partial_\varepsilon X := \partial X_\varepsilon = \overline{X_\varepsilon} \setminus X_\varepsilon$, respectively. Let k_ε be the quasihyperbolic metric of (X, d_ε).

Now, we recall certain auxiliary results from [29] for later use.

Lemma 3.4. *([29, Theorem 1.2]) For all $0 < \varepsilon \le \varepsilon_0(\delta)$, the conformal deformations $X_\varepsilon = (X, d_\varepsilon)$ of X are unbounded A-uniform spaces with a constant $A = A(\delta)$.*

Lemma 3.5. *([29, Lemma 5.1]) There is a constant $A_1 = A_1(\delta) \ge 1$ such that*

$$\frac{1}{A_1} d_\varepsilon(x, y) \le \frac{1}{\varepsilon} \mathrm{e}^{-\varepsilon (x|y)_b} \big(\min\{1, \varepsilon d(x, y)\} \big) \le A_1 d_\varepsilon(x, y),$$

for all $x, y \in X$.

Lemma 3.6. *([29, Lemma 5.5]) There is a natural bijective map $\phi\colon \partial_\infty X \to \partial_\varepsilon X \cup \{\infty\}$ with $\phi(\xi) = \infty$.*

Lemma 3.7. *([29, Lemma 5.24]) If X is K-roughly starlike with respect to $\xi \in \partial_\infty X$, then for $0 < \varepsilon \le \varepsilon_0(\delta)$, the identity map $(X, d) \to (X_\varepsilon, k_\varepsilon)$ is M-bilipschitz with $M = M(\delta, K, \varepsilon)$.*

3.3. Proof of Theorem 1.1

Let $\delta, K, \mu \ge 0$ and $C, \lambda \ge 1$. Suppose that (X, d) is a proper geodesic space that is δ-hyperbolic and K-roughly starlike with respect to $\xi \in \partial_\infty X$. Let $\partial_\infty X$ be a C-uniformly perfect set. Recall that

$$\mathcal{T}_{\lambda,\mu}(X^*) = \big\{ f : X^* \to X^* \;\big|\; f|_X \text{ is a } (\lambda, \mu)\text{-quasi-isometry and } \; f|_{\partial_\infty X} = \mathrm{id}_{\partial_\infty X} \big\}.$$

The issue is to find a constant Λ such that

$$d(x, f(x)) \le \Lambda$$

for each $f \in \mathcal{T}_{\lambda,\mu}(X^*)$ and for all $x \in X$.

Let $b = b_{\xi,o}\colon X \to \mathbb{R}$ be a Busemann function based at ξ with $o \in X$. Fix a constant $\varepsilon = \varepsilon(\delta, K)$. Let $X_\varepsilon := (X, d_\varepsilon)$ be the uniformization of (X, d) induced by the conformal deformation as in Equation (3.3).

One observes from Lemma 3.4 that X_ε is unbounded and A-uniform with $A = A(\delta)$. According to Lemma 3.5, we know that

$$\phi\colon (\partial_\infty X \setminus \{\xi\}, d_{b,\varepsilon}) \to (\partial_\varepsilon X, d_\varepsilon)$$

is actually a bilipschitz map, where $d_{b,\varepsilon}$ is a Hamenstädt metric based at ξ with parameter ε. Because (X, d) is K-roughly starlike with respect to $\xi \in \partial_\infty X$, it follows from Lemma 3.7 that the identity map

$$\varphi\colon (X, d) \to (X_\varepsilon, k_\varepsilon)$$

is M-bilipschitz with $M = M(\delta, K, \varepsilon)$.

Thanks to [24, Theorem C], we see that the uniform perfectness is preserved under quasimöbius maps; for the definition of quasimöbius maps, see [20]. It follows from [5, Theorem 5.2.17] that $\partial_\infty X$ equipped with any two visual metrics or Hamenstädt metrics are quasimöbius equivalent to each other with the control function depending only on δ. As $\partial_\infty X$ is C-uniformly perfect with respect to a certain visual metric, one thus finds that $\partial_\varepsilon X$ is C_0-uniformly perfect with $C_0 = C_0(C, \delta)$.

Fix $f \in \mathcal{T}_{\lambda,\mu}(X^*)$. We observe that f induces a map $g\colon \overline{X_\varepsilon} \cup \{\infty\} \to \overline{X_\varepsilon} \cup \{\infty\}$ with

$$g|_X := \varphi \circ f|_X \circ \varphi^{-1}\colon (X_\varepsilon, k_\varepsilon) \to (X_\varepsilon, k_\varepsilon)$$

and

$$g|_{\partial_\varepsilon X} := \phi \circ f|_{\partial_\infty X} \circ \phi^{-1}, \qquad g(\infty) = \infty.$$

Next, we show that the following three items:

(1) $g|_{\partial_\varepsilon X} = \mathrm{id}_{\partial_\varepsilon X}$;
(2) The continuous extension of g from X_ε to the one-point extended boundary $\partial_\varepsilon X \cup \{\infty\}$ is exactly $g|_{\partial_\varepsilon X \cup \{\infty\}}$;
(3) There is a homeomorphism $\eta_0\colon [0, \infty) \to [0, \infty)$ such that

$$\frac{d_\varepsilon(g(x), g(a))}{d_\varepsilon(g(y), g(a))} \leq \eta_0\left(\frac{d_\varepsilon(x, a)}{d_\varepsilon(y, a)}\right), \tag{3.8}$$

for all three distinct points $x, y \in \overline{X_\varepsilon}$ and $a \in \partial_\varepsilon X$.

Because $f \in \mathcal{T}_{\lambda,\mu}(X^*)$, we have $f|_{\partial_\infty X} = \mathrm{id}_{\partial_\infty X}$ and this proves (1).

To prove (2), for each sequence $\{x_n\}$ which is d_ε-convergent to $a \in \partial_\varepsilon X \cup \{\infty\}$, we check that the sequence $\{g(x_n)\} = \{f(x_n)\}$ is d_ε-convergent to $g(a) = a$. By Lemma 3.6, one observes that $\{x_n\}$ is a Gromov sequence of X such that $\{x_n\} \in \phi^{-1}(a) \in \partial_\infty X$. As $f\colon X \to X$ is a quasi-isometry which has a continuous extension to $\partial_\infty X$ such that

https://doi.org/10.1017/S0013091524000580 Published online by Cambridge University Press

$f|_{\partial_\infty X} = \mathrm{id}_{\partial_\infty X}$, we see from [3, Proposition 6.3] that $\{f(x_n)\}$ is also a Gromov sequence of X satisfying

$$\{f(x_n)\} \in f|_{\partial_\infty X} \circ \phi^{-1}(a) = \phi^{-1}(a) \in \partial_\infty X.$$

Then Lemma 3.6 guarantees that $\{f(x_n)\}$ is d_ε-convergent to $\phi[\phi^{-1}(a)] = a = g(a) \in \partial_\varepsilon X \cup \{\infty\}$, as desired.

It remains to show the last item (3). For any given three distinct points $x, y \in \overline{X_\varepsilon}$ and $a \in \partial_\varepsilon X$, we let

$$d_\varepsilon(x,a) = t\, d_\varepsilon(y,a) \quad \text{and} \quad d_\varepsilon(g(x),g(a)) = T\, d_\varepsilon(g(y),g(a)).$$

Choose sequences $\{x_n\}$, $\{y_n\}$ and $\{a_n\}$ in X so that they are d_ε-convergent to x, y and a, respectively. From the statement (2), it follows that $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(a_n)\}$ are d_ε-convergent to $g(x)$, $g(y)$ and $g(a)$, respectively. Moreover, by Lemma 3.6, we have $\{a_n\} \in \phi^{-1}(a) \in \partial_\infty X$ and $\phi^{-1}(x) \neq \phi^{-1}(a) \neq \phi^{-1}(y)$, because $x \neq a \neq y$. Without loss of generality, we may assume that for all n,

$$\min\Big\{d(x_n,a_n),\ d(y_n,a_n),\ d(f(x_n),f(a_n)),\ d(f(y_n),f(a_n))\Big\} \geq 1.$$

By Lemma 3.5, there is a constant $A_1 = A_1(\delta) \geq 1$ such that

$$\begin{aligned} t_n = \frac{d_\varepsilon(x_n,a_n)}{d_\varepsilon(y_n,a_n)} &\geq \frac{1}{A_1^2}\mathrm{e}^{\varepsilon(y_n|a_n)_b - \varepsilon(x_n|a_n)_b}\,\frac{\min\{1,[\varepsilon d(x_n,a_n)]\}}{\min\{1,[\varepsilon d(y_n,a_n)]\}} \\ &= \frac{1}{A_1^2}\mathrm{e}^{\varepsilon(y_n|a_n)_b - \varepsilon(x_n|a_n)_b}. \end{aligned}$$

This ensures that

$$(y_n|a_n)_b - (x_n|a_n)_b \leq A_2 + \frac{1}{\varepsilon}\log t_n \tag{3.9}$$

with $A_2 = 2(\log A_1)/\varepsilon$.

By a similar argument as above, we see from Lemma 3.5 that

$$T_n = \frac{d_\varepsilon(g(x_n),g(a_n))}{d_\varepsilon(g(y_n),g(a_n))} = \frac{d_\varepsilon(f(x_n),f(a_n))}{d_\varepsilon(f(y_n),f(a_n))} \tag{3.10}$$

$$\leq A_1^2 e^{\varepsilon(f(y_n)|f(a_n))_b - \varepsilon(f(x_n)|f(a_n))_b}.$$

On the other hand, we note that $b \in \mathcal{B}(\xi)$ and $f(\xi) = \xi$. As $f\colon (X,d) \to (X,d)$ is a (λ,μ)-quasi-isometry, it follows from [29, Lemma 3.7] that there exists a control function

https://doi.org/10.1017/S0013091524000580 Published online by Cambridge University Press

$\theta\colon \mathbb{R} \to \mathbb{R}$ depending only on λ, μ and δ with

$$\theta(t) = \max\{\lambda_1 t,\ t/\lambda_1\} + \mu_1$$

such that

$$(f(y_n)|f(a_n))_b - (f(x_n)|f(a_n))_b \leq \theta\big((y_n|a_n)_b - (x_n|a_n)_b\big).$$

This, together with Equations (3.10) and (3.9), implies that

$$T_n \leq A_1^2 \mathrm{e}^{\varepsilon\theta\left(\frac{1}{\varepsilon}\log t_n\right)}. \tag{3.11}$$

Again by Lemma 3.6, we see that $T_n \to T$ and $t_n \to t$ as $n \to \infty$. Therefore, we obtain Equation (3.8) from Equation (3.11) by letting $n \to \infty$. This ensures (3).

We continue the proof of this theorem. Fix $x \in X$, and choose a point $x_0 \in \partial_\varepsilon X$ such that $d_\varepsilon(x, x_0) = d_\varepsilon(x) = \mathrm{dist}_\varepsilon(x, \partial_\varepsilon X)$. We wish to obtain an upper bound for the quasihyperbolic distance k_ε between x and $g(x) = f(x)$. To this end, we first show the following:

Claim. There is a constant $M_1 \geq 1$ such that

$$\frac{d_\varepsilon(x)}{M_1} \leq d_\varepsilon(g(x)) \leq d_\varepsilon(g(x), g(x_0)) \leq M_1 d_\varepsilon(x).$$

We first check that $d_\varepsilon(g(x), g(x_0)) \leq M_0 d_\varepsilon(x)$ for some $M_0 \geq 1$. As $\partial_\varepsilon X$ is unbounded, it is clear that $\partial_\varepsilon X \setminus B_\varepsilon(x_0, d_\varepsilon(x)) \neq \emptyset$. Thus there is a point $x_1 \in \partial_\varepsilon X$ such that

$$\frac{d_\varepsilon(x)}{C_0} \leq d_\varepsilon(x_0, x_1) \leq d_\varepsilon(x), \tag{3.12}$$

because $\partial_\varepsilon X$ is C_0-uniformly perfect. Note that $g|_{\partial_\varepsilon X} = \mathrm{id}_{\partial_\varepsilon X}$. Now, by Equations (3.8) and (3.12), we obtain

$$\begin{aligned} d_\varepsilon(g(x), g(x_0)) &\leq \eta_0\left(\frac{d_\varepsilon(x, x_0)}{d_\varepsilon(x_1, x_0)}\right) d_\varepsilon(g(x_1), g(x_0)) \\ &\leq \eta_0(C_0) d_\varepsilon(x_1, x_0) \\ &\leq \eta_0(C_0) d_\varepsilon(x) = M_0 d_\varepsilon(x). \end{aligned}$$

For the other direction, by an elementary computation, we see from Equation (3.8) that for any three distinct points $x, y \in \overline{X_\varepsilon}$ and $a \in \partial_\varepsilon X$,

$$\frac{d_\varepsilon(x, a)}{d_\varepsilon(y, a)} \leq \eta_1\left(\frac{d_\varepsilon(g(x), g(a))}{d_\varepsilon(g(y), g(a))}\right), \tag{3.13}$$

where $\eta_1 = [\eta_0^{-1}(t^{-1})]^{-1}$ for all $t > 0$.

https://doi.org/10.1017/S0013091524000580 Published online by Cambridge University Press

Thus by Equation (3.13), a similar argument as above guarantees that

$$d_\varepsilon(x) \le M_1 d_\varepsilon(g(x))$$

for some constant $M_1\,(\ge M_0)$ depending only on η_0 and C_0. Therefore, the claim is proved.

Finally, we are ready to complete the proof of Theorem 1.1. Because $g|_{\partial_\varepsilon X} = \mathrm{id}_{\partial_\varepsilon X}$, we see from the claim that

$$\begin{aligned} d_\varepsilon(g(x), x) &\le d_\varepsilon(x_0, x) + d_\varepsilon(g(x), g(x_0)) \\ &\le d_\varepsilon(x) + M_1 d_\varepsilon(x) \\ &\le M_1(M_1+1)\min\{d_\varepsilon(x), d_\varepsilon(g(x))\}. \end{aligned} \tag{3.14}$$

Moreover, because (X, d_ε) is A-uniform, it follows from Equation (3.14) and [2, Lemma 2.13] that

$$\begin{aligned} k_\varepsilon(x, g(x)) &\le 4A^2 \log\left(1 + \frac{d_\varepsilon(g(x), x)}{\min\{d_\varepsilon(x), d_\varepsilon(g(x))\}}\right) \\ &\le 4A^2 \log\big(1 + M_1(M_1+1)\big). \end{aligned}$$

Because the identity map $\varphi\colon (X, d) \to (X_\varepsilon, k_\varepsilon)$ is M-bilipschitz, we obtain

$$d(x, f(x)) = d(x, g(x)) \le M k_\varepsilon(x, g(x)) \le 4MA^2 \log[1 + M_1(M_1+1)] =: \Lambda$$

finishing the proof.

4. Examples and applications

4.1. Examples

While studying the Teichmüller displacement problem on Gromov hyperbolic spaces X that is roughly starlike with respect to an interior point $w \in X$, one observes from Corollary 1.5 that the upper bound for the displacement depends on the diameter of $(\partial_\infty X, d_{w,\varepsilon})$. In the following, we provide two examples to explain this phenomenon.

Example 4.1. Let $\mathbb{H}^2$ be the Poincaré hyperbolic disk with the original point $o \in \mathbb{H}^2$. For a given integer $m \ge 1$, we attach $\mathbb{H}^2$ at the point o with a line segment $I_m = \{o \times [0, m]\}$. We define the space $Y = \mathbb{H}^2 \sqcup I_m$ equipped with the induced length metric d. Then we have the following:

(1) Clearly, (Y, d) is a Gromov hyperbolic metric space that is 0-roughly starlike with respect to $w = (o, m)$. The Gromov boundary $\partial_\infty Y$ is the same as $\partial_\infty \mathbb{H}^2 = \mathbb{S}^1$, which is connected and therefore uniformly perfect.
(2) One easily finds that the diameter of $(\partial_\infty Y, d_{w,\varepsilon})$ is comparable with $e^{-m\varepsilon}$.
(3) We define a mapping $f\colon Y \to Y$ such that $f|_{\mathbb{H}^2} = \mathrm{id}_{\mathbb{H}^2}$ and $f|_{I_m}$ is a linear function with $f(u) = (o, m/4)$, where $u = (o, m/2) \in Y$. It is not hard to see that f is

a $(4,0)$-quasi-isometry which induces a boundary map $f|_{\partial_\infty Y} = \mathrm{id}_{\partial_\infty Y}$. However, $d(u, f(u)) = m/4$.

The second example tells us that the Teichmüller displacement theorem for quasiconformal mappings, namely [23, Theorem 1.9] and [30, Theorem 1.2], is not valid for domains G in the Riemann spheres, where the displacement depends also on the diameter of ∂G with respect to the spherical metric.

Example 4.2. Let $\overline{R}^2 = \mathbb{R}^2 \cup \{\infty\}$ be the Riemann 2-sphere, and $|\cdot|$ the Euclidean metric on $\mathbb{R}^2$. Let $0 < \epsilon \le 1/4$, $\overline{B}(0,\epsilon) = \{z \in \mathbb{R}^2 |\ |z| \le \epsilon\}$, and $D_\epsilon = \overline{R}^2 \setminus \overline{B}(0,\epsilon)$. Define $g\colon D_\epsilon \to D_\epsilon$ with $g(\infty) = \infty$ and

$$g(z) = \frac{1}{\epsilon}|z|z \text{ for all } z \in D_\epsilon \setminus \{\infty\}.$$

Let σ be the spherical metric on $\overline{R}^2$ defined as in [8, (3.6)]. Then we have the following:

(1) (D_ϵ, σ) is a bounded locally compact uniform metric space with

$$\mathrm{diam}_\sigma(\partial D_\epsilon) = \frac{2\epsilon}{1+\epsilon^2}.$$

(2) Let $k_{D_\epsilon}(\sigma)$ be the quasihyperbolic metric of (D_ϵ, σ), see Definition 3.1. It follows from [2, Theorem 3.6] that $(D_\epsilon, k_{D_\epsilon}(\sigma))$ is a proper geodesic Gromov hyperbolic space that is roughly starlike with respect to a point $w \in D_\epsilon$. Moreover, there is a natural quasisymmetric homeomorphism between the metric boundary $(\partial D_\epsilon, \sigma)$ and the Gromov boundary $\partial_\infty D_\epsilon$ of $(D_\epsilon, k_{D_\epsilon}(\sigma))$ endowed with a visual metric.
(3) The mapping $g\colon (D_\epsilon, \sigma) \to (D_\epsilon, \sigma)$ is quasiconformal and has a continuous extension to ∂D_ϵ with $g|_{\partial D_\epsilon} = \mathrm{id}_{\partial D_\epsilon}$. Hence $g\colon (D_\epsilon, k_{D_\epsilon}(\sigma)) \to (D_\epsilon, k_{D_\epsilon}(\sigma))$ is a quasi-isometry and has a continuous extension to $\partial_\infty D_\epsilon$ which is the identity map on $\partial_\infty D_\epsilon$.
(4) Clearly, ∂D_ϵ is connected and so is $\partial_\infty D_\epsilon$. In particular, $\partial_\infty D_\epsilon$ is uniformly perfect. Therefore, we know from Corollary 1.5 that the displacement $k_{D_\epsilon}(\sigma)(z, g(z))$ is bounded above for all $z \in D_\epsilon$. Note that the upper bound depends on ϵ.
(5) As $\epsilon \to 0$, one finds that $\mathrm{diam}_\sigma(\partial D_\epsilon) \to 0$ and

$$k_{D_\epsilon}(\sigma)\big((\sqrt{\epsilon}, 0), g((\sqrt{\epsilon}, 0))\big) \ge \log\left(1 + \frac{\sqrt{1+\epsilon^2}}{\sqrt{2}\epsilon}\right) \to \infty.$$

4.2. Applications

This subsection focuses on some applications of Theorem 1.1. Let $\delta, K \ge 0$, $C \ge 1$ and $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. Suppose that (X, d) and (X', d') are proper geodesic δ-hyperbolic spaces, and $\partial_\infty X$ is C-uniformly perfect which contains at least two points, and $F\colon \partial_\infty X \to \partial_\infty X'$ is η-quasisymmetric with respect to visual metrics. For the definition and properties of quasisymmetric maps, we refer to [9, 20, 24].

https://doi.org/10.1017/S0013091524000580 Published online by Cambridge University Press

It is known that if the Gromov boundaries of two roughly starlike hyperbolic geodesic spaces are powerly quasisymmetrically equivalent, then they are quasi-isometrically equivalent (cf. [3, 5]). As a consequence of Theorem 1.1, we thus obtain that any such two quasi-isometries are bounded above up to a finite distance.

Corollary 4.3. *Suppose that X and X' are K-roughly starlike with respect to points on Gromov boundaries, respectively. Then there is a number $\Lambda_2 = \Lambda_2(K, \delta, C, \eta, \lambda, \mu)$ such that, for (λ, μ)-quasi-isometries $f_1, f_2 \colon X \to X'$ induced by F with $f_1|_{\partial_\infty X} = f_2|_{\partial_\infty X} = F$,*

$$d'(f_1(x), f_2(x)) \leq \Lambda_2 \text{ for all } x \in X.$$

Proof. By [9, Exercise 11.2 and Theorem 11.3], we observe that $\partial_\infty X'$ is C'-uniformly perfect and the inverse of F, $F^{-1} \colon \partial_\infty X' \to \partial_\infty X$, is a powerly η_1-quasisymmetric map with C' and η_1 depending only on C and η. Note that the visual property and rough starlikeness of a proper geodesic hyperbolic space are equivalent. It follows from [5, Corollary 7.2.3] that there is a (λ_1, μ_1)-quasi-isometry $g \colon X' \to X$ whose natural extension $g|_{\partial_\infty X'} = F^{-1}$, where λ_1 and μ_1 depend only on δ, K and η_1. As the composition of quasi-isometries is also a quasi-isometry, we immediately find that

$$g \circ f_1,\ g \circ f_2 \in \mathcal{T}_{\lambda', \mu'}(X^*),$$

for some positive constants λ' and μ' which depend only on $\lambda, \lambda_1, \mu, \mu_1$ and δ.

Now by Theorem 1.1, we see that there is a constant $\Lambda = \Lambda(\lambda', \mu', C', \delta, K)$ such that for all $x \in X$,

$$d(g \circ f_1(x), x) \leq \Lambda \quad \text{and} \quad d(g \circ f_2(x), x) \leq \Lambda.$$

As $g \colon X' \to X$ is a (λ_1, μ_1)-quasi-isometry, the above two inequalities ensure that

$$d'(f_1(x), f_2(x)) \leq \lambda_1 d(g \circ f_1(x), g \circ f_2(x)) + \mu_1 \leq 2\lambda_1 \Lambda + \mu_1 =: \Lambda_2$$

and the proof of the corollary is complete. □

Performing a similar argument as in the proof of Corollary 4.3, we obtain the following result as a consequence of Corollary 1.5.

Corollary 4.4. *Suppose that X and X' are K-roughly starlike with respect to $w \in X$ and $w' \in X'$ respectively, and $\vartheta = diam(\partial_\infty X, d_{w,\varepsilon}) > 0$. Then there is a number $\Lambda_3 = \Lambda_3(K, \delta, C, \eta, \lambda, \mu, \vartheta)$ such that, for (λ, μ)-quasi-isometries $f_1, f_2 \colon X \to X'$ induced by F with $f_1|_{\partial_\infty X} = f_2|_{\partial_\infty X} = F$,*

$$d'(f_1(x), f_2(x)) \leq \Lambda_3 \text{ for all } x \in X.$$

4.3. Concluding remarks

Now, we consider the connection between Theorem 1.1 and the following question proposed by Xie:

Question 4.5. ([25, Question 7.1]) Let Y_1 and Y_2 be two Hadamard n-manifolds (whose sectional curvatures are bounded from below) with $n \neq 4$, and $g\colon Y_1 \to Y_2$ a quasi-isometry. Is g always a finite distance from a bilipschitz homeomorphism?

Because the boundary of a Hadamard manifold is homeomorphic to a sphere, it is not hard to see from Corollary 4.3 that whenever one finds a bilipschitz map $\widetilde{g}\colon Y_1 \to Y_2$ with $\widetilde{g}|_{\partial_\infty Y_1} = g|_{\partial_\infty Y_1}$, then the answer to Question 4.5 is positive.

Finally, we remark that Theorem 1.1 is useful in understanding the arguments in [13, 16] concerning the bilipschitz extension of mappings from Gromov boundaries to the interiors of certain Gromov hyperbolic spaces. Indeed, we may obtain [16, Corollaries 1.2 and 1.4] by combining [16, Theorem 1.1] and the earlier mentioned results, particularly Corollary 4.3. Note that [13, Lemma 3.23] is a quantitative consequence of Theorem 1.1.

Funding Statement. Qingshan Zhou was partly supported by NNSF of China (No. 11901090), by Guangdong Basic and Applied Basic Research Foundation (No. 2022A1515012441), by Department of Education of Guangdong Province, China (No. 2021KTSCX116). Qianghua Luo was supported by Guangdong Basic and Applied Basic Research Foundation (No. 2023A1515110902).

References

(1) P. Bonfert–Taylor, R. D. Canary, G. Martin and E. Taylor, Quasiconformal homogeneity of hyperbolic manifolds, *Math. Ann.* **331**(2) (2005), 281–295.

(2) M. Bonk, J. Heinonen and P. Koskela, Uniformizing Gromov hyperbolic spaces, *Astérisque* **270** (2001), viii+99.

(3) M. Bonk and O. Schramm, Embeddings of Gromov hyperbolic spaces, *Geom. Funct. Anal.* **10**(2) (2000), 266–306.

(4) M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature, Grundlehren der Mathematischen Wissenschaften*, Volume 319, pp. xxii+643 (Springer-Verlag, Berlin, 1999).

(5) S. Buyalo and V. Schroeder, *Elements of asymptotic geometry* (EMS Monographs in Mathematics: Zürich, xii+200 pp. 2007).

(6) F. W. Gehring and B. G. Osgood, Uniform domains and the quasi-hyperbolic metric, *J. Anal. Math.* **36** (1979), 50–74.

(7) P. Hästö, Gromov hyperbolicity of the j_G and $\widetilde{j}_G$ metrics, *Proc. Amer. Math. Soc.* **134**(4) (2006), 1137–1142.

(8) P. Hariri, R. Klén and M. Vuorinen, *Conformally invariant metrics and quasiconformal mappings*. Springer Monographs in Mathematics, pp. xix+502. (Springer, Cham, 2020).

(9) J. Heinonen, *Lectures on Analysis on Metric Spaces* (Springer-Verlag, Berlin-Heidelberg-New York, 2001).

(10) D. Herron, N. Shanmugalingam and X. Xie, Uniformity from Gromov hyperbolicity, *Illinois J. Math.* **52**(4) (2008), 1065–1109.

(11) R. Klén, V. Todorčević and M. Vuorinen, Teichmüller's problem in space, *J. Math. Anal. Appl.* **455**(2) (2017), 1297–1316.

(12) Y. Li, M. Vuorinen and X. Wang, Quasiconformal maps with bilipschitz or identity boundary values in Banach spaces, *Ann. Acad. Sci. Fenn. Math.* **39**(2) (2014), 905–917.

(13) A. Lukyanenko, Bi-Lipschitz extension from boundaries of certain hyperbolic spaces, *Geom. Dedicata* **164** (2013), 47–71.

(14) V. Manojlović and M. Vuorinen, On quasiconformal maps with identity boundary values, *Trans. Amer. Math. Soc.* **363**(4) (2011), 2467–2479.

(15) O. Rainio, T. Sugawa and M. Vuorinen, Uniformly perfect sets, Hausdorff dimension, and conformal capacity, *J. Geom. Anal.* **34**(6) (2024), 154, 10.1007/s12220-024-01599-5.

(16) N. Shanmugalingam and X. Xie, A rigidity property of some negatively curved solvable Lie groups, *Comment. Math. Helv.* **87**(4) (2012), 805–823.

(17) T. Sugawa, Uniformly perfect sets–analytic and geometric aspects, *Sugaku* **53**(4) (2001), 387–402.

(18) T. Sugawa, M. Vuorinen and T. Zhang, Conformally invariant complete metrics, *Math. Proc. Cambridge Philos. Soc.* **174**(2) (2023), 273–300.

(19) O. Teichmüller, Ein Verschiebungssatz der quasikonformen Abbildung, *Deutsche Math*, **7** (1944), 336–343.

(20) J. Väisälä, Quasi-Möbius maps, *J. Anal. Math.* **44** (1984/85), 218–234.

(21) J. Väisälä, Gromov hyperbolic spaces, *Expo. Math.* **23**(3) (2005), 187–231.

(22) J. Väisälä, Hyperbolic and uniform domains in Banach spaces, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **30**(2) (2005), 261–302.

(23) M. Vuorinen and X. Zhang, Distortion of quasiconformal mappings with identity boundary values, *J. Lond. Math. Soc.* **90**(3) (2014), 637–653.

(24) X. Wang and Q. Zhou, Quasimöbius maps, weakly quasimöbius maps and uniform perfectness in quasi-metric spaces, *Ann. Acad. Sci. Fenn. Ser. AI Math.* **42**(1) (2017), 257–284.

(25) X. Xie, Quasiisometries between negatively curved Hadamard manifolds, *J. Lond. Math. Soc.* **79**(1) (2009), 15–32.

(26) X. Xie, Quasisymmetric maps on the boundary of a negatively curved solvable Lie group, *Math. Ann.* **353**(3) (2012), 727–746.

(27) Q. Zhou, Y. Li and A. Rasila, Gromov hyperbolicity, John spaces, and quasihyperbolic geodesics, *J. Geom. Anal.* **32**(9) (2022), 15.

(28) Q. Zhou, S. Ponnusamy and T. Guan, Gromov hyperbolicity of the $\widetilde{j}_G$ metric and boundary correspondence, *Proc. Amer. Math. Soc.* **150**(7) (2022), 2839–2847.

(29) Q. Zhou, S. Ponnusamy and A. Rasila, Busemann functions and uniformization of Gromov hyperbolic spaces, https://arxiv.org/pdf/2008.01399v2.pdf.

(30) Q. Zhou and A. Rasila, Teichmüller's problem for Gromov hyperbolic domains, *Israel J. Math.* **252**(1) (2022), 399–427.