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Słociński–Wold decompositions for row isometries

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Abstract. Słociński gave sufficient conditions for commuting isometries to have a nice Wold-like decomposition. In this note, we provide analogous results for row isometries satisfying certain commutation relations. Other than known results for doubly commuting row isometries, we provide sufficient conditions for a Wold decomposition based on the Lebesgue decomposition of the row isometries.

1 Introduction

Let V be an isometry acting on a Hilbert space H. A well-known result, discovered independently by von Neumann (1929) and Wold (1938), tells us that H decomposes uniquely into V-reducing subspaces $H = H_u \oplus H_s$ where $V|_{H_u}$ is a unitary and V_{H_s} is a unilateral shift. We will follow the convention of calling this result the *Wold decomposition of V*. Over the decades, there have been generalizations of this result, decomposing isometric representations of semigroups into their unitary and nonunitary parts. Suciu's work in [20] is an early example of such results.

The work at hand is largely inspired by the Wold-like decomposition given Słociński [19]. Let V_1 and V_2 be commuting isometries on a Hilbert space H. We say that V_1 and V_2 have a *Słociński–Wold* decomposition if H decomposes as $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$, where each space H_i reduces both V_1 and V_2; $V_1|_{H_1}, V_1|_{H_2}, V_2|_{H_1}, V_2|_{H_3}$ are unitaries; and $V_1|_{H_3}, V_1|_{H_4}, V_2|_{H_2}, V_2|_{H_4}$ are unilateral shifts. Słociński gives sufficient conditions for a pair commuting isometries to have a Słociński–Wold decomposition. Most notable, or at least the most noted, of these results is that a pair of *doubly* commuting isometries V_1 and V_2 has a Słociński–Wold decomposition (where doubly commuting means that $V_1V_2 = V_2V_1$ and $V_1^*V_2 = V_2V_1^*$). Generalizations of this result for n doubly commuting isometries have been given [8]. Słociński also gives sufficient conditions for the existence of a Słociński–Wold decomposition based on the structure of the individual unitary parts of the isometries. Recall that a unitary U can decomposed as $U_{\text{abs}} \oplus U_{\text{sing}}$ where U_{abs} has absolutely continuous spectral measure and U_{sing} has singular spectral measure (both with respect to Lebesgue measure). Słociński gives two results [19, Theorems 4 and 5], showing the existence of a Słociński–Wold decomposition in the absence of absolutely continuous unitary parts.

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Let $S = [S_1, \dots, S_m]$ be a row isometry on a Hilbert space H. That is, $S: H^{(m)} \to H$ is an isometric map. Equivalently, $S = [S_1, \dots, S_m]$ is a row isometry if $S_1, \dots, S_n$ are isometries with pairwise orthogonal ranges. Popescu [14] shows that there is a Wold decomposition for S. That is, H can be decomposed into S-reducing subspaces $H = H_u \oplus H_s$ where $S|_{H_u}$ is a row unitary, and $S|_{H_s}$ is an n-shift. Beyond row isometries, Muhly and Solel [13] give a Wold decomposition for isometric representations of C*-correspondences, decomposing an isometric representation into unitary and induced parts.

Let $S = [S_1, \dots, S_m]$ and $T = [T_1, \dots, T_n]$ be two row isometries on a Hilbert space H. We say that S and T θ-commute if there is a permutation $\theta \in S_{m\times n}$ such that for $1 \le i \le m$ and $1 \le i \le n$, $S_i T_j = T_{j'} S_{i'}$ when $\theta(i, j) = (i', j')$. A pair of θ-commuting row isometries determines an isometric representation of a 2-graph with a single vertex. Thus, a pair of θ-commuting row isometries is an isometric representation of a product system of two finite-dimensional C*-correspondences (see, e.g., [6, Section 4]). Skalski and Zacharias [18] generalized Słociński's Wold decomposition for doubly commuting isometries to isometric representations of product systems of C*-correspondences which satisfy a doubly commuting condition. Thus, Skalski and Zacharias's result gives a Słociński–Wold decomposition for θ-commuting row isometries.

In this note, we will give sufficient conditions for two θ-commuting row isometries to have a Słociński–Wold decomposition mirroring the three theorems proved by Słociński for commuting isometries. Theorems 3–5 of [19] are generalized in Theorems 3.4, 3.8, and 3.10, respectively. In [19, Theorems 4 and 5], Słociński uses the Lebesgue decomposition of a unitary. For row unitaries, we use the Lebesgue decomposition due to Kennedy [10]. This states that any row unitary decomposes into an absolutely continuous row unitary, a singular row unitary, and a third part called a dilation-type row unitary. For a single unitary U, the statements "U has no absolutely continuous part" and "U is singular" are equivalent; for row unitaries, the existence of dilation-type parts means that the latter is a stronger statement than the former. In this note, for a row unitary, the statement "U is singular" will play the role that "U has no absolutely continuous part" played in [19].

2 Row isometries and their structure

A *row isometry* on a Hilbert space H is an isometric map S from $H^{(n)}$ to H. An operator $S: H^{(n)} \to S$ is a row isometry if and only if $S = [S_1, \dots, S_m]$ where $S_1, \dots, S_m$ are isometries on H with pairwise orthogonal ranges. Equivalently, the $S_1, \dots, S_m$ are isometries satisfying

$$\sum_{i=1}^{m} S_i S_i^* \le I_H.$$

A row isometry $S = [S_1, \dots, S_m]$ is a *row unitary* if S is a unitary map. Equivalently, S is a row unitary if

$$\sum_{i=1}^{m} S_i S_i^* = I_H.$$

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Let $S = [S_1, \dots, S_m]$ be a row operator on a Hilbert space H, and let $M \subseteq H$ be a subspace. The subspace M is *S-invariant* if $S_i H \subseteq H$ for each $1 \le i \le m$; M is *S^*-invariant* if $S_i^* H \subseteq H$ for each $1 \le i \le m$; and M is *S-reducing* if M is both S-invariant and S^*-invariant.

Denote by $\mathbb{F}_m^+$ the unital free semigroup on n generators $\{1, \dots, m\}$. For $w = w_1 \dots w_k \in \mathbb{F}_n^+$, denote by S_w the isometry

$$S_{w_1} S_{w_2} \dots S_{w_k}.$$

Here, $S_\varnothing$ will denote I_H.

Example 2.1 Let $H = \ell^2(\mathbb{F}_m^+)$ with orthonormal basis $\{\xi_w : w \in \mathbb{F}_m^+\}$. For $i \in \{1, \dots, m\}$, define the operator L_i by

$$L_i \xi_w = \xi_{iw}.$$

Then $L = [L_1, \dots, L_m]$ is a row isometry on H.

Definition 2.1 Let $S = [S_1, \dots, S_m]$ be a row isometry. Let L be the row isometry described in Example 2.1. We call S an *m-shift of multiplicity α* if S is unitarily equivalent to an ampliation of L by α. That is, $[S_1, \dots, S_m] \simeq [L_1^{(\alpha)}, \dots, L_m^{(\alpha)}]$.

Note that when $m = 1$, an m-shift is a unilateral shift. Thus, the following result, due to Popescu [14], is a generalization of the Wold decomposition of a single isometry.

Theorem 2.2 (Cf. [14, Theorem 1.2]) *Let $S = [S_1, \dots, S_m]$ be a row isometry on H. Then H decomposes into two S-reducing subspaces*

$$H = H_u \oplus H_s,$$

such that $S|_{H_u}$ is a row unitary and $S|_{H_s}$ is an m-shift.

Furthermore,

$$H_u = \bigcap_{k \ge 0} \bigoplus_{|w|=k} S_w H,$$

and

$$H_s = \bigoplus_{w \in \mathbb{F}_n^+} S_w M,$$

where $M = \bigcap_{i=1}^n \ker(S_i^*)$.

Definition 2.2 When S is a row isometry on a Hilbert space H, the decomposition $H = H_s \oplus H_u$ described in Theorem 2.2 is called the *Wold decomposition* of S.

2.1 The Lebesgue–Wold decomposition

Just as a unitary can be decomposed into its singular and absolutely continuous parts, a row unitary can be decomposed further. We will briefly summarize these results now, drawing largely from [2, 10].

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Let $L = [L_1, \ldots, L_m]$ be the m-shift described in Example 2.1. Denote by A_m and $\mathcal{L}_m$ the following two algebras:

$$A_m := \overline{\mathrm{Alg}\{I, L_1, \ldots, L_m\}}^{\|\cdot\|},$$
$$\mathcal{L}_m := \overline{\mathrm{Alg}\{I, L_1, \ldots, L_m\}}^{\mathrm{WOT}}.$$

The algebra A_m is called the *noncommutative disk algebra*, and the algebra $\mathcal{L}_m$ is called the *noncommutative analytic Toeplitz algebra*.

Let $S = [S_1, \ldots, S_m]$ be a row isometry on a Hilbert space H. The *free semigroup algebra* generated by S is the algebra

$$\mathcal{S} := \overline{\mathrm{Alg}\{I, S_1, \ldots, S_m\}}^{\mathrm{WOT}}.$$

Popescu [16] observed that the unital, norm-closed algebra generated by $S_1, \ldots, S_m$ is completely isometrically isomorphic to the noncommutative disk algebra A_m. The free semigroup algebra $\mathcal{S}$, however, can be very different from $\mathcal{L}_m$.

Definition 2.3 Let $S = [S_1, \ldots, S_m]$ be a row isometry on a Hilbert space H with $m \geq 2$.

(i) There is a completely isometric isomorphism

$$\Phi \colon A_m \to \overline{\mathrm{Alg}\{I, S_1, \ldots, S_m\}}^{\|\cdot\|},$$

such that $\Phi(L_i) = S_i$ for $1 \leq i \leq m$. The row isometry S is *absolutely continuous* if Φ extends to a weak-* continuous representation of $\mathcal{L}_m$.

(ii) The row isometry S is *singular* if S has no absolutely continuous restriction to an invariant subspace.

(iii) The row isometry S is of *dilation type* if it has no singular and no absolutely continuous summands.

Remark 2.3 (i) Absolute continuity for row isometries was introduced by Davidson, Li, and Pitts [3]. We refer the reader to [3, Section 2] or [10, Section 2] for details on why Definition 2.3 (i) generalizes the notion of a unitary with absolutely continuous spectral measure.

(ii) By [10, Theorem 5.1], a row isometry $S = [S_1, \ldots, S_m]$, with $m \geq 2$, is singular if and only if the free semigroup algebra $\mathcal{S}$ generated by S is a von Neumann algebra. Read [17] gave the first example of a self-adjoint free semigroup algebra, by showing that $B(H)$ is a free semigroup algebra (see also [1]).

(iii) The name "dilation type" is justified in [10, Proposition 6.2]. If S is a row isometry of dilation type on H, then there is a minimal subspace $V \subseteq H$ such that V is invariant for each S_i^*, $1 \leq i \leq m$, and the restriction of S to $V^\perp$ is an m-shift. In which case, S is the minimal isometric dilation of the compression of S to V. In particular, if $K = (V + \sum_{i=1}^m S_i V) \ominus V$, then $H = V \oplus \bigoplus_{w \in \mathbb{F}_m^+} S_w K$.

We can now describe the Lebesgue–Wold decomposition of a row isometry, due to Kennedy [10].

Theorem 2.4 (Cf. [10, Theorem 6.5]) *If S is a row isometry on H, then H decomposes into four spaces which reduce S:*

$$H = H_{\text{abs}} \oplus H_{\text{sing}} \oplus H_{\text{dil}} \oplus H_s,$$

where $H_{\text{abs}} \oplus H_{\text{sing}} \oplus H_{\text{dil}}$ and H_s are the unitary and m-shift parts of the Wold decomposition, respectively. Furthermore, we have the following properties:

(i) $S|_{H_{\text{abs}}}$ *is absolutely continuous.*
(ii) $S|_{H_{\text{sing}}}$ *is singular.*
(iii) $S|_{H_{\text{dil}}}$ *is of dilation type.*

This decomposition is unique.

Kennedy [10, Theorem 4.16] gives another characterization of absolute continuity. Let $S = [S_1, \ldots, S_m]$ be a row isometry with $m \geq 2$, and let $\mathcal{S}$ be the free semigroup algebra generated by S. Then S is absolutely continuous if and only if $\mathcal{S}$ is isomorphic to $\mathcal{L}_m$. This characterization answered a question asked in [3].

The property of $\mathcal{S}$ being isomorphic to $\mathcal{L}_m$ plays an important role in the work of Davidson, Katsoulis, and Pitts [2] in describing the structure of free semigroup algebras. We summarize the results which will be relevant to us now. Note that what we are calling "absolutely continuous" was called "*type L*" in [2]. The equivalence of the terms is due to the aforementioned work of Kennedy [10].

Theorem 2.5 (Cf. [2, Theorem 2.6]) *Let $S = [S_1, \ldots, S_m]$ be a row isometry on a Hilbert space H with $m \geq 2$. Let $\mathcal{S}$ be the free semigroup algebra generated by S. There is a largest projection P in $\mathcal{S}$ such that $P\mathcal{S}P$ is self-adjoint. Furthermore, the following are satisfied:*

(i) *PH is S^*-invariant.*
(ii) *The restriction of S to $P^\perp H$ is an absolutely continuous row isometry.*

Definition 2.4 Let S be a row isometry, and let P be the projection described in Theorem 2.5. Then P is called *structure projection* for S.

Let $S = [S_1, \ldots, S_m]$ be a row isometry on H, with $H = H_{\text{abs}} \oplus H_{\text{sing}} \oplus H_{\text{dil}} \oplus H_s$ being the Lebesgue–Wold decomposition. Furthermore, write $H_{\text{dil}} = V \oplus \bigoplus_{w \in \mathbb{F}_m^+} S_w K$, as described in Remark 2.3(iii). It follows from Theorems 2.4 and 2.5 that

$$PH = H_{\text{sing}} \oplus V.$$

3 Słociński–Wold decompositions for θ-commuting row isometries

Definition 3.1 Let $A = [A_1, \ldots, A_m]$ and $B = [B_1, \ldots, B_n]$ be two row operators on a Hilbert space H, and let $\theta \in S_{m \times n}$ be a permutation. We say that A and B *θ-commute* if

$$A_i B_j = B_{j'} A_{i'}$$

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when $\theta(i,j) = (i',j')$. When θ is the identity permutation, we will say that A and B *commute*.

If A and B are θ-commuting row operators which further satisfy

$$B_j^* A_i = \sum_{\theta(k,j)=(i,j_k)} A_k B_{j_k}^* \text{ and}$$
$$A_i^* B_j = \sum_{\theta(i,k)=(i_k,j)} B_k A_{i_k}^*,$$

we say that A and B *θ-doubly commute*.

The following lemma is proved by repeated applications of the commutation rule from θ. It will be used liberally in the sequel.

Lemma 3.1 *Let $A = [A_1,\dots,A_m]$ and $B = [B_1,\dots,B_n]$ be θ-commuting row operators. For each $k,l \geq 1$, θ determines a permutation $\theta_{k,l} \in S_{m^k \times n^l}$ so that*

$$A_u B_w = B_{w'} A_{u'}$$

when $\theta_{k,l}(u,w) = (u',w')$.

Any 2-graph with a single vertex, in the sense of [11], is uniquely determined by a single permutation. Thus, two θ-commuting row contractions A and B determine a contractive representation of single vertex 2-graph. This is the perspective θ-commuting row operators are studied from in, e.g., [4, 5, 7].

Definition 3.2 Let $S = [S_1,\dots,S_m]$ and $T = [T_1,\dots,T_n]$ be θ-commuting row isometries on a Hilbert space H. We say that S and T have a *Słociński–Wold decomposition* if H decomposes into

$$H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss},$$

where H_{uu}, H_{us}, H_{su}, and H_{ss} are both S-reducing and T-reducing subspaces satisfying:

(i) $S|_{H_{uu}}$ and $T|_{H_{uu}}$ are both row unitaries.
(ii) $S|_{H_{us}}$ is a row unitary, and $T|_{H_{us}}$ is an n-shift.
(iii) $S|_{H_{su}}$ is an m-shift, and $T|_{H_{su}}$ is a row unitary.
(iv) $S|_{H_{ss}}$ is an m-shift, and $T|_{H_{ss}}$ is an n-shift.

The following general lemma will be used throughout our analysis.

Lemma 3.2 *$S = [S_1,\dots,S_m]$ is a row isometry which θ-commutes with a row operator $A = [A_1,\dots,A_l]$. Let $H = H_u \oplus H_s$ be the Wold decomposition of S. Then H_u is A-invariant.*

Proof Take $h \in H_u$ and fix $k \geq 0$. Since S is a row unitary on H_u,

$$h = \sum_{|w|=k} S_w S_w^* h.$$

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Choose an A_i, $1 \le i \le l$. For each w with $|w| = k$, there is a w' with $|w'| = k$, and i_w with $1 \le i_w \le l$ so that $A_i S_w = S_{w'} A_{i_w}$. Thus,

$$\begin{aligned} A_i h &= A_i \sum_{|w|=k} S_w S_w^* h \\ &= \sum_{|w|=k} S_{w'} A_{i_w} S_w^* h \in \sum_{|w|=k} S_w H. \end{aligned}$$

Since this holds for all $k \ge 0$, $A_i H_u \subseteq H_u$ by Theorem 2.2. ∎

We can now give a general statement on the existence of Słociński–Wold decompositions. The case when $m = n = 1$ is covered in [19, Proposition 3].

Proposition 3.3 *Let $S = [S_1, \dots, S_m]$ and $T = [T_1, \dots, T_n]$ be θ-commuting row isometries on H. Then S and T have a Słociński–Wold decomposition if and only if:*

(i) *if $H = H_u^S \oplus H_s^S$ is the Wold decomposition of S, then H_u^S reduces T; and*
(ii) *if $H_s^S = H_u^T \oplus H_s^T$ is the Wold decomposition of $T|_{H_s^S}$, then H_u^T reduces S.*

Proof If S and T have a Słociński–Wold decomposition, then conditions (i) and (ii) are clearly satisfied.

Suppose now that conditions (i) and (ii) are satisfied. Let $H = H_u^S \oplus H_s^S$ be the Wold decomposition for S. Let $H_u^S = K_u^T \oplus K_s^T$ be the Wold decomposition of H_u^S from the restriction of T to H_u^S. By Lemma 3.2, K_u^T is S-invariant. Take any $1 \le i \le m$, and $h \in K_u^T$. Recall, by Lemma 3.1, for each $k \ge 1$, there is a permutation $\theta_{1,k}$ on $S_{m \times n^k}$ so that for $1 \le i \le m$ and $w \in \mathbb{F}_n^+$, $S_i T_w = T_{w'} S_{i'}$ when $\theta_{1,k}(i, w) = (i', w')$. Hence, for every $k \ge 1$,

$$\begin{aligned} S_i^* h &= S_i^* \sum_{|w|=k} T_w T_w^* h \\ &= \sum_{|w|=k} S_i^* T_w T_w^* h \\ &= \sum_{|w|=k} \sum_{l=1}^{m} S_i^* T_w S_l S_l^* T_w^* h \\ &= \sum_{|w|=k} \sum_{\theta_{1,k}(i,w_i)=(l,w)} T_{w_i} S_l^* T_w^* h \\ &\in \bigoplus_{|w|=k} T_w H_u^S, \end{aligned}$$

where the fact that S is a row unitary on H_u^S is used in the third equality. It follows from Theorem 2.2 that $S_i^* h \in K_u^T$. Hence, K_u^T is S-reducing.

Letting $H_s^S = H_u^T \oplus H_s^T$ be the Wold decomposition of $T|_{H_u^S}$, we have that $H_{uu} = K_u^T$, $H_{us} = K_s^T$, $H_{su} = H_u^T$, and $H_{ss} = H_s^T$ gives the desired Słociński–Wold decomposition. ∎

Skalski and Zacharias studied Wold decompositions of isometric representations of product systems of C*-correspondences [18]. The following is a special case of one of their results.

Theorem 3.4 (Cf. [18, Theorem 2.4]) *If S and T are θ-double commuting row isometries, then they have a Słociński–Wold decomposition.*

Proof Let $H = H_u^S \oplus H_s^S$ be the Wold decomposition of H from S. We will show that H_u^S is T-reducing. Lemma 3.2 gives that H_u^S is T-invariant, so it only remains to show that H_u^S is T^*-invariant. Take $1 \le j \le n$ and $h \in H_u^S$. Using the condition that S and T θ-doubly commute and that S is a row unitary on H_u^S, we have, for every $k \ge K$,

$$\begin{aligned} T_j^* h &= \sum_{|w|=k} T_j^* S_w S_w^* h \\ &= \sum_{\theta_{k,1}(w_k, j)=(w, j_w)} S_{w_k} T_{j_w}^* S_w^* h \\ &\in \sum_{|w|=k} S_w H. \end{aligned}$$

Thus, $T_j^* h \in H_u^S$ by Lemma 2.2.

Now, let $H_s^S = H_u^T \oplus H_s^T$ be the Wold decomposition of $T|_{H_s^S}$. The same calculation as above, with the roles of S and T swapped, shows that H_u^T is S-reducing. Thus, S and T have a Słociński–Wold decomposition by Proposition 3.3. ∎

Remark 3.5 As described in [18], the Słociński–Wold decomposition for θ-doubly commuting row isometries has additional structure on the shift part H_{ss}. On H_{ss}, S and T are not just both (m and n) shifts. The operators S and T work as shifts *together*, giving an ampliation of the left-regular representation of the unital semigroup

$$F_\theta^+ = \langle i_1, \dots, i_m, j_1, \dots, j_n \colon i_k j_l = j' i' \text{ when } \theta(i_k, j_l) = (i', l') \rangle.$$

Explicitly, if $M = \bigcap_{i=1}^m \ker S_i^* \cap \bigcap_{j=1}^n \ker T_j^*$, then

$$H_{ss} = \bigoplus_{u \in \mathbb{F}_m^+,\ w \in \mathbb{F}_n^+} S_u T_w M.$$

Theorem 3.4 generalizes Theorem 3 of [19]. In the rest of this note, we will give analogues of Theorems 4 and 5 of [19] for θ-commuting row isometries. That is, we will give sufficient conditions for the existence of a Słociński–Wold decomposition for θ-commuting row isometries based on the Lebesgue decomposition of their unitary parts.

Lemma 3.6 *Let $S = [S_1, \dots, S_m]$ be a row isometry on H with $m \ge 2$, and let P be the structure projection for S. If $T = [T_1, \dots, T_n]$ is a row isometry on H which θ-commutes with S. Then PH is T^*-invariant.*

Proof By Theorem 2.2, S is absolutely continuous on $P^\perp H$. Thus, by [10, Corollary 4.17], $P^\perp H$ is spanned by wandering vectors for S. Recall that a vector $h \in H$ is wandering for S if $\langle S_w h, h \rangle = 0$ for all $w \in \mathbb{F}_m^+$, $w \ne \varnothing$. Let h be a wandering vector for S. Then, for any $1 \le j \le n$ and $w \in \mathbb{F}_n^+$, $|w| \ge 1$, we have

$$\langle S_w T_j h, T_j h \rangle = \langle S_{w'} h, T_{j'}^* T_j h \rangle,$$

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where w' and j' satisfy $S_w T_j = T_{j'} S_{w'}$. If $j' \neq j$, then $T_{j'}^* T_j = 0$, in which case $\langle S_w T_j h, T_j h \rangle = 0$. If $j' = j$, then

$$\langle S_w T_j h, T_j h \rangle = \langle S_{w'} h, h \rangle = 0,$$

since h is wandering for S and $|w'| = |w| \geq 1$. Hence, $T_j h$ is wandering for S, and so $T_j h \in P^\perp H$. It follows that $T_j P^\perp H \subseteq P^\perp H$, and hence PH is T^*-invariant. ■

Let V be an isometry on a Hilbert space H, and let $N \in B(H)$ be an operator commuting with V. Let $H = H_{\text{abs}} \oplus H_{\text{sing}} \oplus H_s$ be the Lebesgue–Wold decomposition of V. It then follows from [12, Theorem 2.1] that H_{sing} reduces N. Thus, if $H_{\text{abs}} = \{0\}$, the unitary part of V reduces N. In Proposition 3.7, we show that if S and T are θ-commuting row isometries and the unitary part of S is singular, then the Wold decomposition of S reduces T.

Proposition 3.7 *Let $S = [S_1, \dots, S_m]$ and $T = [T_1, \dots, T_n]$ be θ-commuting row isometries on H. Let $H = H_u \oplus H_s$ be the Wold decomposition for S. If the unitary part of S is singular, then H_u reduces T*

Proof When $m = 1$, the result follows from [12, Theorem 2.1] (see [19, Remark 2]). Otherwise, we have $H_u = PH$ where P is the structure projection for S. The result follows from Lemmas 3.2 and 3.6. ■

We now give a row-isometry analog of [19, Theorem 4].

Theorem 3.8 *Let $S = [S_1, \dots, S_m]$ and $T = [T_1, \dots, T_n]$ be θ-commuting row isometries on a Hilbert space H. Furthermore, suppose that the unitary parts of S and T are singular. Then S and T have a Słociński–Wold decomposition.*

Proof The result follows immediately from Propositions 3.3 and 3.7. ■

The following lemma generalizes [19, Lemma 2] to row isometries. It is notable that the conditions are less restrictive for the row-isometry case than they are in single-isometry case dealt with in [19].

Lemma 3.9 *Let S be an m-shift of finite multiplicity on a Hilbert space H. Let $T = [T_1, \dots, T_n]$ be a row unitary on H which θ-commutes with S. If*

(1) *$n \geq 2$, or*

(2) *$n = 1$ and T has empty point spectrum,*

then $H = \{0\}$.

Proof Let $L = \bigcap_{i=1}^m \ker S_i^*$. By assumption, L is finite-dimensional. Since T and S θ-commute, it is clear that L is T^*-invariant. As T is a row unitary, if $h \in L$ and $1 \leq i \leq m$, we have that

$$S_i^* T_j h = \sum_{k=1}^n T_k T_k^* S_i^* T_j h = \sum_{\theta(i,k)=(i_k,j)} T_k S_{i_k}^* h = 0,$$

and so L is T-reducing.

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If $n \geq 2$, then $T_1|_L, \ldots, T_n|_L$ are isometries with pairwise orthogonal finite-dimensional ranges. If $n = 1$, then $T|_L$ is a unitary on a finite-dimensional space and so has an eigenvalue. In either case, we see that we must have $L = \{0\}$ and hence $H = \{0\}$. ∎

We end with the following generalization of [19, Theorem 5].

Theorem 3.10 *Let $S = [S_1, \ldots, S_m]$ and $T = [T_1, \ldots, T_n]$ be θ-commuting row isometries on a Hilbert space H. Assume that the unitary part of S is singular, and that the shift part of S has finite multiplicity, then S and T have a Słociński–Wold decomposition if*

(i) *$n \geq 2$; or*

(ii) *$n = 1$ and θ is the identity permutation.*

Proof Let $H = H_u^S \oplus H_s^S$. As S has only singular unitary part, H_u^S reduces T by Proposition 3.7. Let $H_s^S = K_u^T \oplus K_s^T$ be the Wold decomposition of the restriction of T to H_s^S. Lemma 3.2 says that K_u^T is S-invariant. As S is an m-shift of finite multiplicity on H_s^S, the restriction of S to K_u^T is an m-shift of finite multiplicity. When $m = 1$, this is [9, Lemma 4]; when $m \geq 2$, it follows from [15, Theorem 3.1] and [15, Theorem 3.2].

When $n \geq 2$, it follows from Lemma 3.9 that $K_u^T = \{0\}$ and hence S and T have a Słociński–Wold decomposition by Proposition 3.3. When $n = 1$ and T is an isometry commuting with each S_i, the proof follows as in [19, Theorem 4]. ∎

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