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INFINITELY MANY ARBITRARILY SMALL SOLUTIONS FOR SINGULAR ELLIPTIC PROBLEMS WITH CRITICAL SOBOLEV–HARDY EXPONENTS

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Abstract Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that $0 \in \Omega$, $N \geqslant 3$, $2^*(s) = 2(N-s)/(N-2)$, $0 \leqslant s < 2$, $0 \leqslant \mu < \bar{\mu} = \frac{1}{4}(N-2)^2$. We obtain the existence of infinitely many solutions for the singular critical problem $-\Delta u - \mu(u/|x|^2) = (|u|^{2^*(s)-2}/|x|^s)u + \lambda f(x,u)$ with Dirichlet boundary condition for suitable positive number λ.

Keywords: singular elliptic equation; multiple solutions; critical Sobolev–Hardy growth; compactness

2000 *Mathematics subject classification:* Primary 35J60, 35J25

1. Introduction and main result

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that $0 \in \Omega$, $N \geqslant 3$, $2^*(s) = 2(N-s)/(N-2)$, $0 \leqslant s < 2$, $0 \leqslant \mu < \bar{\mu} = \frac{1}{4}(N-2)^2$. For $\lambda \in \mathbb{R}$ consider the elliptic problem

$$\left.\begin{aligned} -\Delta u - \mu\frac{u}{|x|^2} &= \frac{|u|^{2^*(s)-2}}{|x|^s}u + \lambda f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $f(x,0) \equiv 0$ and $f(x,u)$ is a lower-order perturbation of $u^{2^*(s)-1}$, in the sense that $f(x,u)/|u|^{2^*(s)-2}u \to 0$ as $|u| \to +\infty$, $s \in [0,2)$. A typical case of problem (1.1), $f(x,u) = u$, i.e. the problem

$$\left.\begin{aligned} -\Delta u - \mu\frac{u}{|x|^2} &= \frac{|u|^{2^*(s)-2}}{|x|^s}u + \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

has recently been studied by Kang and Peng [**19**, **20**]. In [**20**], they studied the existence of positive solutions of (1.2) by a mountain-pass argument [**27**], and in [**19**] they obtained

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a sign-changing solution by minimax methods. Problem (1.2) with $s = 0$ has been studied by many authors in recent years (see, for example, [**2**, **6**, **9**, **12**–**15**, **17**, **24**, **28**, **32**]). Early results for (1.2) with $\mu = 0$ and $s = 0$ can be found, for instance, in [**5**, **7**, **11**, **21**, **26**, **29**].

When $\mu = 0$ and $s = 0$, problem (1.1) reduces to

$$\left.\begin{aligned} -\Delta u &= |u|^{2^*-2}u + \lambda f(x,u) \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.3}$$

where $2^* = 2^*(0) = 2N/(N-2)$. In [**29**], Sang obtained finitely many solutions for (1.3) by using Bartolo *et al.*'s critical-point theorem if $f(x,u)$ is odd in u. Li and Zou [**21**] again studied problem (1.3) and obtained infinitely many solutions in the case when $f(x,u)$ satisfies the following conditions:

(i) $f(x,u) \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $f(x,-u) = -f(x,u)$ for all $u \in \mathbb{R}$;

(ii) $\lim_{|u|\to\infty} f(x,u)/|u|^{2^*-2}u = 0$ uniformly for $x \in \Omega$;

(iii) $\lim_{u\to 0^+} f(x,u)/u = \infty$ uniformly for $x \in \Omega$;

(iv) $\frac{1}{2}f(x,u)u - F(x,u) \geqslant a - b|u|^{2^*}$ for almost every $x \in \Omega$ and $u \in \mathbb{R}$, where

$$F(x,u) = \int_0^u f(x,t)\,\mathrm{d}t, \quad b \geqslant 0,\ a \leqslant 0.$$

In [**25**], Pádua *et al.* investigated (1.3) and proved the existence of two positive solutions with a sublinear term at the origin. For the case of a p-Laplacian involving critical Sobolev exponents, we refer the reader to [**1**, **8**, **16**, **30**] and the references therein. Recently, via linking arguments, Zou [**33**] has proved the existence of infinitely many sign-changing solutions for a class of perturbed Schrödinger equations with critical Sobolev growth in $\mathbb{R}^N$ (the parameter λ appears in the critical exponent term).

In this paper we are interested in searching for infinitely many solutions of problem (1.1); we show that there exists a sequence of infinitely many arbitrarily small solutions converging to zero by using a new version of the symmetric mountain-pass lemma due to Kajikiya [**18**]. To the best of our knowledge, there is no such result on singular elliptic problems with critical Sobolev exponents and Hardy terms.

The main result of this paper is as follows.

Theorem 1.1. *Suppose that $f(x,u)$ satisfies (i) and (iii) above, and*

(ii′) $\lim_{|u|\to\infty} f(x,u)/(|u|^{2^*(s)-2}u) = 0$ *uniformly for $x \in \Omega$ and $0 \leqslant s < 2$.*

There then exists $\lambda^* > 0$, such that, for any $\lambda \in (0, \lambda^*)$, problem (1.1) has a sequence of non-trivial solutions $\{u_n\}$ tending to zero as $n \to \infty$.

Remark 1.2. In [**21**], the authors proved the existence of infinitely many solutions for (1.2), but did not give any further information on the sequence of solutions. In this paper we shall prove that this sequence of solutions may converge to 0.

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Remark 1.3. In our result, the nonlinearity $f(x,u)$ need not satisfy condition (iv) as in [**21**]. Hence, we relax the restrictions imposed on the nonlinearity $f(x,u)$. Moreover, we consider more general nonlinearity than is considered in [**19**, **20**]. Hence, we make a remarkable improvement of the main results of [**19**–**21**].

2. Preliminary results

We start this section by recalling the variational framework for problem (1.1). Considering the Sobolev space $H_0^1(\Omega)$ endowed with the equivalent norm for $0 \leqslant \mu < \bar{\mu}$, we have

$$\|u\| \triangleq \left(\int_\Omega |\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right)^{1/2} \quad \forall u \in H_0^1(\Omega). \tag{2.1}$$

By Hardy's inequality, this norm is equivalent to the usual norm in $H_0^1(\Omega)$. Define the best Sobolev–Hardy constant by

$$A_s(\Omega) \triangleq \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{(\int_\Omega |u|^{2^*(s)}/|x|^s)^{2/2^*(s)}}.$$

Then $A_s(\Omega)$ is independent of Ω [**20**]. The linear elliptic operator $L \triangleq -\Delta - \mu(1/|x|^2)$ is positive and has discrete spectrum σ_μ in $H_0^1(\Omega)$ if $0 \leqslant \mu < \bar{\mu}$. The first eigenvalue of the operator L in $H_0^1(\Omega)$ is given by

$$\lambda_1(\mu) \triangleq \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\int_\Omega |u|^2}.$$

We denote the norm of an $L^p(\Omega)$ space as $|u|_p$ and various positive constants as c. The functional associated with (1.1) is given by

$$\Phi(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} - \lambda \int_\Omega F(x,u) \quad \forall u \in H_0^1(\Omega).$$

Standard arguments [**27**] show that, under assumption (ii′), Φ belongs to $C^1(H_0^1(\Omega), \mathbb{R})$. Furthermore, the (weak) solutions of (1.1) are precisely the critical points of this functional. The following result is proved in [**20**].

Proposition 2.1 (Kang and Peng [20]). *Suppose that* $0 \leqslant s < 2$, $2 \leqslant q \leqslant 2^*(s)$ *and* $0 \leqslant \mu < \bar{\mu}$. *Then*

(i) $\displaystyle\int_\Omega (u^2/|x|^2) \leqslant (1/\bar{\mu}) \int_\Omega |\nabla u|^2 \quad \forall u \in H_0^1(\Omega)$,

(ii) *there exists a constant* $C > 0$ *such that*

$$\left(\int_\Omega |u|^q/|x|^s\right)^{1/q} \leqslant C\|u\| \quad \forall u \in H_0^1(\Omega),$$

(iii) *the map* $u \to u/|x|^{s/q}$ *from* $H_0^1(\Omega)$ *into* $L^q(\Omega)$ *is compact for* $q < 2^*(s)$.

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We recall that, given E a real Banach space and $I \in C^1(E,\mathbb{R})$, we say that I satisfies the Palais–Smale condition on the level $c \in \mathbb{R}$, denoted by $(\mathrm{PS})_c$, if every sequence $\{u_n\} \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

Since the norm defined in (2.1) is equivalent to the usual norm by the Hardy inequality (Proposition 2.1 (i)), we now enumerate the concentration–compactness principle due to Lions [**22**, **23**], which is similar to that of [**8**, **28**, **30**, **31**].

Lemma 2.2. *Let $\{u_n\} \subset H_0^1(\Omega)$ be a bounded sequence. There then exist two non-negative and bounded measures on $\bar{\Omega}$, γ, ν, and there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that*

$$|\nabla u_n|^2 - \mu \frac{u_n^2}{|x|^2} \rightharpoonup \gamma, \qquad \frac{|u_n|^{2^*(s)}}{|x|^s} \rightharpoonup \nu$$

weakly in the sense of measures.

Lemma 2.3. *Let $\{u_n\} \subset H_0^1(\Omega)$ be such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and*

$$|\nabla u_n|^2 - \mu \frac{u_n^2}{|x|^2} \rightharpoonup \gamma, \qquad \frac{|u_n|^{2^*(s)}}{|x|^s} \rightharpoonup \nu$$

weakly in the sense of measures, where γ and ν are non-negative and bounded measures on $\bar{\Omega}$. There then exist some at most countable index set J and a family $\{x_j : j \in J\}$ of points in $\bar{\Omega}$ such

(a) *$\nu = |u|^{2^*(s)}/|x|^s + \sum_{j\in J} \nu_j \delta_{x_j}$ that, where $\{\nu_j : j \in J\}$ is a family of positive numbers,*

(b) *$\gamma \geqslant |\nabla u|^2 - \mu(u^2/|x|^2) + \sum_{j\in J} \gamma_j \delta_{x_j}$, where $\{\gamma_j : j \in J\}$ is a family of positive numbers satisfying $A_s(\nu_j)^{2/2^*(s)} \leqslant \gamma_j$ for all $j \in J$. In particular, $\sum_{j\in J}(\nu_j)^{2/2^*(s)} < \infty$, $s \in [0,2)$.*

Under condition (ii′), we have

$$f(x,u)u = o\biggl(\frac{|u|^{2^*(s)}}{|x|^s}\biggr), \qquad F(x,u) = o\biggl(\frac{|u|^{2^*(s)}}{|x|^s}\biggr),$$

which means that, for all $\varepsilon > 0$, there exist $a(\varepsilon)$ and $b(\varepsilon) > 0$ such that

$$|f(x,u)u| \leqslant a(\varepsilon) + \varepsilon \frac{|u|^{2^*(s)}}{|x|^s}, \tag{2.2}$$

$$|F(x,u)| \leqslant b(\varepsilon) + \varepsilon \frac{|u|^{2^*(s)}}{|x|^s}. \tag{2.3}$$

Hence,

$$F(x,u) - \tfrac{1}{2} f(x,u)u \leqslant c(\varepsilon) + \varepsilon \frac{|u|^{2^*(s)}}{|x|^s} \tag{2.4}$$

for some $c(\varepsilon) > 0$.

Now we have the following lemma about the local (PS) condition.

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Lemma 2.4. *Assume condition* (ii$'$) *holds. Then, for any* $\lambda > 0$, *the functional* Φ *satisfies the local (PS) condition in*

$$\Big(-\infty, \frac{2-s}{2(N-s)}A_s^{(N-s)/(2-s)} - \lambda h\Big)$$

in the following sense: if

$$\Phi(u_n) \to c < \frac{2-s}{N-s}A_s^{(N-s)/(2-s)} - \lambda h$$

and $\Phi'(u_n) \to 0$ *for some sequence in* $H_0^1(\Omega)$, *then* $\{u_n\}$ *contains a subsequence converging strongly in* $H_0^1(\Omega)$, *where* $h = c((2-s)/4(N-s)\lambda)|\Omega|$.

Proof. Let $\{u_n\}$ be a sequence in $H_0^1(\Omega)$ such that

$$\Phi(u_n) \to c < \frac{2-s}{2(N-s)}A_s^{(N-s)/(2-s)} - \lambda h, \tag{2.5}$$

$$\Phi'(u_n) \to 0. \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned}\Phi(u_n) - \tfrac{1}{2}\langle \Phi'(u_n), u_n\rangle &= \Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big)\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} - \lambda\int_\Omega [F(x,u) - \tfrac{1}{2}f(x,u_n)u_n] \\ &= c + o(1)\|u_n\|,\end{aligned}$$

i.e.

$$\frac{2-s}{2(N-s)}\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} = \lambda\int_\Omega [F(x,u_n) - \tfrac{1}{2}f(x,u_n)u_n] + c + o(1)\|u_n\|.$$

Then by (2.4) and Hölder inequality, we get

$$\Big(\frac{2-s}{2(N-s)} - \lambda\varepsilon\Big)\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} \leqslant \lambda c(\varepsilon)|\Omega| + c + o(1)\|u_n\|.$$

Setting $\varepsilon = (2-s)/4(N-s)\lambda$, we have

$$\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} \leqslant M + o(1)\|u_n\|, \tag{2.7}$$

where $o(1) \to 0$ and $M = M(N, \lambda, |\Omega|)$ is some positive number. On the other hand,

$$\begin{aligned}\Phi(u_n) &= \frac{1}{2}\int_\Omega \Big(|\nabla u_n|^2 - \mu\frac{u_n^2}{|x|^2}\Big) - \frac{1}{2^*(s)}\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} - \lambda\int_\Omega F(x,u_n) \\ &= c + o(1).\end{aligned}$$

Combining (2.3) and (2.7), we deduce that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Therefore, we can assume that $u_n \to u$ weakly in $H_0^1(\Omega)$. By Lemma 2.2, there exist measures γ and ν

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such that the conclusions of Lemma 2.3 hold. We claim that $\nu_j = 0$ for all $j \in J$. Indeed, following the same idea of [**21**], let x_j be singular point of measures γ and ν, we define a cut-off function $\phi \in C_0^1(\mathbb{R}^N)$ such that $\phi(x) \equiv 1$ on $B(x_j, \varepsilon)$, $\phi(x) = 0$ on $\mathbb{R}^N \setminus B(x_j, 2\varepsilon)$ and $|\nabla\phi(x)| \leqslant 2/\varepsilon$ on $\mathbb{R}^N$. Obviously, $\langle \Phi'(u_n), u_n\phi\rangle \to 0$, i.e.

$$\int_\Omega \biggl(\nabla u_n \nabla(u_n\phi) - \mu\frac{u_n^2\phi}{|x|^2}\biggr) - \int_\Omega \frac{|u_n|^{2^*(s)}\phi}{|x|^s} - \lambda\int_\Omega f(x,u_n)u_n\phi \to 0 \quad \text{as } n \to \infty.$$

Therefore, by Lemma 2.2 and the Hölder inequality, we obtain

$$\begin{aligned}
-\lim_{n\to\infty}\int_\Omega u_n\nabla u_n\nabla\phi &= \int_\Omega \phi\,\mathrm{d}\gamma - \int_\Omega \phi\,\mathrm{d}\nu - \lambda\int_\Omega f(x,u)u\phi,\\
\lim_{n\to\infty}\biggl|\int_\Omega u_n\nabla u_n\nabla\phi\biggr| &\leqslant \lim_{n\to\infty}\sup\int_\Omega |u_n|\,|\nabla u_n|\,|\nabla\phi|\\
&\leqslant \lim_{n\to\infty}\sup\biggl(\int_\Omega |\nabla u_n|^2\biggr)^{1/2}\biggl(\int_\Omega |u_n|^2|\nabla\phi|^2\biggr)^{1/2}\\
&\leqslant c'\lim_{n\to\infty}\biggl(\int_\Omega |u_n|^2|\nabla\phi|^2\biggr)^{1/2}\\
&= c'\biggl(\int_\Omega |u|^2|\nabla\phi|^2\biggr)^{1/2}\\
&\leqslant c'\biggl(\int_\Omega |u|^{2\times(2^*(s)/2)}\biggr)^{1/2^*(s)}\\
&\qquad\times\biggl(\int_\Omega |\nabla\phi|^{2(2^*(s))/(2^*(s)-2)}\biggr)^{((2^*(s)-2)/2^*(s))/2}\\
&\leqslant c'\biggl(\int_{B(x_j,2\varepsilon)} |u|^{2^*(s)}\biggr)^{1/2^*(s)} \to 0 \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

where the c' denotes various generic positive constants.

Combining this with Lemma 2.3, we obtain $\nu_j = \gamma_j(x_j) \geqslant \gamma_j \geqslant A_s\nu_j^{2/2^*(s)}$. This result implies that $\nu_j = 0$ or $\nu_j \geqslant A_s^{(N-s)/(2-s)}$. If the second case, $\nu_j \geqslant A_s^{(N-s)/(2-s)}$, holds for some $j \in J$, then, by using Lemma 2.3 and the Hölder inequality, we have that

$$\begin{aligned}
c &= \lim_{n\to\infty}(\Phi(u_n) - \tfrac{1}{2}\langle\Phi'(u_n), u_n\rangle)\\
&= \lim_{n\to\infty}\biggl\{\biggl(\frac{1}{2} - \frac{1}{2^*(s)}\biggr)\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} - \lambda\int_\Omega [F(x,u_n) - \tfrac{1}{2}f(x,u_n)u_n]\biggr\}\\
&= \frac{2-s}{2(N-s)}\int_\Omega \mathrm{d}\nu - \lambda\int_\Omega [F(x,u) - \tfrac{1}{2}f(x,u)u]\\
&\geqslant \frac{2-s}{2(N-s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} + \frac{2-s}{2(N-s)}A_s^{(N-s)/(2-s)} - \lambda c(\varepsilon)|\Omega| - \lambda\varepsilon\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\\
&= \biggl(\frac{2-s}{2(N-s)} - \lambda\varepsilon\biggr)\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} + \frac{2-s}{2(N-s)}A_s^{(N-s)/(2-s)} - \lambda c(\varepsilon)|\Omega|
\end{aligned}$$

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$$\geqslant \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)} - \lambda c\left(\frac{2-s}{4(N-s)\lambda}\right)|\Omega|$$
$$= \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)} - \lambda h,$$

where $\varepsilon = (2-s)/(4(N-s)\lambda)$. This is impossible. Consequently, $\nu_j = 0$ for all $j \in J$ and hence

$$\int_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s} \to \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}.$$

Now $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and the Brezis–Lieb Lemma [**4**] implies that

$$\lim_{n\to\infty} \int_\Omega |x|^{-s}|u_n - u|^{2^*(s)} = 0.$$

Finally, we show that $u_n \to u$ in $H_0^1(\Omega)$, since

$$\|u_n - u\|^2 = \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle + \int_\Omega |x|^{-s}(|u_n|^{2^*(s)-2}u_n - |u|^{2^*(s)-2}u)(u_n - u) + \lambda \int_\Omega (f(x,u_n) - f(x,u))(u_n - u).$$

Therefore,

$$\begin{aligned} A &:= \left| \int_\Omega |x|^{-s}(|u_n|^{2^*(s)-2}u_n - |u|^{2^*(s)-2}u)(u_n - u)\right| \\ &\leqslant \int_\Omega |x|^{-s}|u_n|^{2^*(s)-1}|u_n - u| + \int_\Omega |x|^{-s}|u|^{2^*(s)-1}|u_n - u| \\ &:= A_1 + A_2, \\ A_1 &\leqslant \left(\int_\Omega |x|^{-s}|u_n - u|^{2^*(s)}\right)^{1/2^*(s)} \left(\int_\Omega |x|^{-s}|u_n|^{2^*(s)}\right)^{(2*(s)-1)/2^*(s)}. \end{aligned}$$

It follows from the Sobolev–Hardy inequality, the boundedness of $\{u_n\}$ in $H_0^1(\Omega)$ and the above arguments that $A_1 \to 0$ as $n \to \infty$. Similarly, $A_2 \to 0$ as $n \to \infty$. In view of the Strauss Lemma [**3**] and $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, we deduce that $\|u_n - u\| \to 0$ as $n \to \infty$. The proof is complete. □

3. A sequence of arbitrarily small solutions

In this section we prove the existence of infinitely many solutions of (1.1) which tend to zero. Let E be a Banach space and let

$$\Gamma = \{A \subset E : A \text{ is closed in } E \text{ symmetric with respect to the origin}\}.$$

For $A \in \Gamma$, define $\gamma(A) = \inf\{m \in \mathbb{N} : \exists \varphi \in C(A, \mathbb{R}^m) \setminus \{0\},\ \varphi(x) = -\varphi(-x)\}$; if there is no mapping φ as above for any $m \in \mathbb{N}$, then $\gamma(A) = +\infty$. Let Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geqslant k$. We list the following main properties of genus (see [**18**, Lemma 2.6] or [**10**]).

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Proposition 3.1. *Let A and B be closed symmetric subsets of E which do not contain the origin. Then the following hold.*

(1) *If there exists an odd continuous mapping from A to B, then $\gamma(A) \leqslant \gamma(B)$.*

(2) *If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$.*

(3) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geqslant \gamma(A) - \gamma(B)$.*

(4) *The n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk–Ulam Theorem.*

(5) *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$, such that $N_\delta(A) \in \Gamma$ and $\gamma(N_\delta(A)) = \gamma(A)$, where $N_\delta(A) = \{x \in E : \|x - A\| \leqslant \delta\}$.*

The following version of the symmetric mountain-pass lemma is due to Kajikiya [**18**].

Lemma 3.2. *Let E be an infinite-dimensional Banach space and $I \in \boldsymbol{C}^1(E, \mathbb{R})$ and suppose the following conditions hold.*

(A1) *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the local Palais–Smale condition, i.e. for some $c^* > 0$, in the case when every sequence $\{u_k\}$ in E satisfying $\lim_{k\to\infty} I(u_k) = c < c^*$ and $\lim_{k\to\infty} \|I'(u_k)\|_{E^*} = 0$ has a convergent subsequence.*

(A2) *For each $k \in N$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$.*

Then either (i) or (ii) below holds.

(i) *There exists a sequence $\{u_k\}$ such that $I'(u_k) = 0$, $I'(u_k) < 0$ and $\{u_k\}$ converges to zero.*

(ii) *There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $I'(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$; $I'(v_k) = 0$, $I(v_k) < 0$, $\lim_{k\to\infty} I(v_k) = 0$ and $\{v_k\}$ converges to a non-zero limit.*

Remark 3.3. In [**18**], the functional $I(u)$ is required to satisfy the Palais–Smale condition in global. However, by a careful examination of the proof of the main results of [**18**], we find it is sufficient that the functional $I(u)$ satisfies the local Palais–Smale condition with the critical value levels c below zero. So, Kajikiya's conclusion, i.e. [**18**, Theorem 1], remains true for Lemma 3.2 in this paper.

Remark 3.4. From Lemma 3.2 we have a sequence $\{u_k\}$ of critical points such that $I(u_k) \leqslant 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

In order to get infinitely many solutions we need some lemmas. Under the assumptions of Theorem 1.1, we take $\varepsilon = 1/\lambda_1(\mu)$ in (2.3) (where $\lambda_1(\mu)$ is the first eigenvalue of the $-\Delta - \mu(1/|x|^2)$ with zero Dirichlet boundary data defined in § 2), then by the definition

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of A_s and Proposition 2.1, for $\lambda \in (0, \lambda_1(\mu))$ we have

$$\begin{aligned}\Phi(u) &\geqslant \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right) - \frac{1+\lambda\varepsilon 2^*(s)}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} - \lambda b(\varepsilon)|\Omega| \\ &\geqslant \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right) - \frac{1+2^*(s)}{2^*(s)}\left(\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)\right)^{2^*(s)/2} A_s^{-2^*(s)/2} \\ &\qquad - \lambda b\left(\frac{1}{\lambda_1(\mu)}\right)|\Omega| \\ &= A\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right) - B\left(\int_\Omega |\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)^{2^*(s)/2} - \lambda C,\end{aligned}$$

where

$$A = \tfrac{1}{2}, \qquad B = \frac{1+2^*(s)}{2^*(s)}A_s^{-2^*(s)/2}, \qquad C = b\left(\frac{1}{\lambda_1(\mu)}\right)|\Omega|,$$

and A, B, C are positive constants independent of λ.

Let $g(t) = At^2 - Bt^{2^*(s)} - \lambda C$. Then

$$\Phi(u) \geqslant g(\|u\|).$$

Furthermore, there exists

$$\lambda_1^* = \min\left\{\lambda_1(\mu), \frac{A(2-s)}{C(N-s)}\left(\frac{2A}{2^*(s)B}\right)^{2/(2^*(s)-2)}\right\} > 0$$

such that, for $\lambda \in (0, \lambda_1^*)$, $g(t)$ attains its positive maximum, that is, there exists

$$R_1 = \left(\frac{2A}{2^*(s)B}\right)^{1/2^*(s)}$$

such that $e_1 = g(R_1) = \max_{t\geqslant 0} g(t) > 0$. Hence, for $e_0 \in (0, e_1)$, we can find $R_0 < R_1$ such that $g(R_0) = e_0$. If we define

$$\tau(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant R_0, \\ \dfrac{At^2 - \lambda C - e_1}{Bt^{2^*(s)}} & \text{for } t \geqslant R_1, \\ C^\infty, \tau(t) \in [0,1] & \text{for } R_0 \leqslant t \leqslant R_1. \end{cases}$$

Then it is easy to see that $\tau(t) \in [0,1]$ for all $t \geqslant 0$ and $\tau(t)$ is C^∞. Let $\varphi(u) = \tau(\|u\|)$ and consider the truncated functional

$$J(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right) - \frac{\varphi(u)}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} - \lambda\varphi(u)\int_\Omega F(x,u).$$

Then

$$\begin{aligned}J(u) &\geqslant A\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right) - B\varphi(u)\left(\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)\right)^{2^*(s)/2} - \lambda C \\ &\triangleq \bar{g}(\|u\|),\end{aligned}$$

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where $\bar{g}(t) = At^2 - B\tau(t)t^{2^*(s)} - \lambda C$, and

$$\bar{g}(t) = \begin{cases} g(t), & t \leqslant R_0, \\ e_1, & t \geqslant R_1. \end{cases}$$

From the above arguments, J has the following properties.

Lemma 3.5.

(1) *$J \in C^1(H_0^1(\Omega), \mathbb{R})$ and J is even and bounded from below.*

(2) *If $J(u) < e_0$, then $\bar{g}(\|u\|) < e_0$; consequently, $\|u\| < R_0$ and $\Phi(u) = J(u)$.*

(3) *There exists λ^* such that, for $\lambda \in (0, \lambda^*)$, J satisfies a local (PS) condition for*

$$c < e_0 \in \left(0, \min\left\{e_1, \frac{2-s}{2(N-s)} A_s^{(N-s)/(2-s)} - \lambda h\right\}\right).$$

Proof. Conditions (1) and (2) are evident. For (3), let us choose

$$\lambda^* \in \left(0, \min\left\{\frac{2-s}{2h(N-s)} A_s^{(N-s)/(2-s)}, \lambda_1^*\right\}\right).$$

Condition (3) of the lemma holds in view of Lemma 2.4 and the construction of J. □

Lemma 3.6. *Assume that* (iii) *holds. Then, for any $k \in \mathbb{N}$, there exist $\delta = \delta(k) > 0$ such that $\gamma(\{u \in H_0^1(\Omega) : J(u) \leqslant \delta(k)\} \setminus \{0\}) \geqslant k$.*

Proof. Firstly, by (iii), for any fixed $u \in H_0^1(\Omega), u \neq 0$, we have

$$F(x, \rho u) \geqslant M(\rho)(\rho u)^2 \quad \text{with } M(\rho) \to \infty \text{ as } \rho \to 0.$$

Secondly, given any $k \in \mathbb{N}$, let E_k be a k-dimensional subspace of $H_0^1(\Omega)$. There then exist constant α_k such that

$$\|u\| \leqslant \alpha_k |u|_2 \quad \forall u \in E_k.$$

Therefore, for any $u \in E_k$ with $\|u\| = 1$ and ρ small enough, we have

$$\begin{aligned} J(\rho u) &\leqslant \frac{\rho^2}{2} \int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) - \frac{\rho^{2^*(s)}}{2^*(s)} \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} - \lambda M(\rho)\rho^2 \int_\Omega |u|^2 \\ &\leqslant \left(\frac{1}{2} - \frac{\lambda M(\rho)}{\alpha_k^2}\right)\rho^2 \\ &\triangleq -\delta(k) < 0, \end{aligned}$$

that is, $\{u \in E_k : \|u\| = \rho\} \subset \{u \in H_0^1(\Omega) : J(u) \leqslant -\delta(k)\} \setminus \{0\}$. This completes the proof. □

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Remark 3.7. In Lemma 3.6 we showed that $A = \{u \in E_k : \|u\| = \rho\} \subset \{u \in H_0^1(\Omega) : J(u) \leqslant -\delta(k)\} \setminus \{0\}$. Therefore, $\gamma(\{u \in H_0^1(\Omega) : J(u) \leqslant -\delta(k)\} \setminus \{0\}) \geqslant \gamma(\{u \in E_k : \|u\| = \rho\}) = \gamma(A)$. Since $A = \{u \in E_k : \|u\| = \rho\}$ is a sphere with radius ρ in E_k, a k-dimensional subspace of E, so $\gamma(A) = k$ by Proposition 3.1 (4). As for an odd application $\varphi : A \to R^k \setminus \{0\}$, we may consider the odd mapping $\varphi : x \to \varphi(x) = (x_1, x_2, \dots, x_k) \in R^k \setminus \{0\}$, where $x = x_1e_1 + x_2 + \dots + x_ke_k \in A$, $x_i \in R$, $i = 1, 2, \dots, k$, and $e_1, e_2, \dots, e_k$ is a basis of E_k. This is a well known fact. For more properties about the genus, we refer the reader to [**27**].

Now we are in the position to prove Theorem 1.1 by Lemma 3.2.

Proof of Theorem 1.1. Recall that $\Gamma_k = \{A \in H_0^1(\Omega) \setminus \{0\} : A \text{ is closed and } A = -A,\ \gamma(A) \geqslant k\}$ and define

$$c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} J(u).$$

By Lemmas 3.5 (1) and 3.6, we know that $-\infty < c_k < 0$. Therefore, assumptions (A1) and (A2) of Lemma 3.2 are satisfied. This means that J has a sequence of solutions $\{u_n\}$ converging to zero. Hence, Theorem 1.1 follows by Lemma 3.5 (2).

Note that, when $\mu = 0$ and $s = 0$, our result is similar to that of [**21**] without the restriction (iv). When $0 < s < 2$, $0 < \mu \leqslant \bar{\mu}$, our result is new. □

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