EIGENVALUES OF COMPLEX TRIDIAGONAL MATRICES

by P. M. GIBSON†

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Results of Arscott (1) and Jayne (3) on real matrices are generalized to obtain bounds for the real parts of the eigenvalues of certain complex tridiagonal matrices, and bounds for the imaginary parts of the eigenvalues of other tridiagonal matrices are given. It is shown that analogous results hold for zeros of the permanent of certain characteristic matrices.

If A is an n-square matrix and $0 \leqq j<k \leqq n$, let $A(j, k]$ denote the $(k-j)$-square principal submatrix lying in rows (and columns) $j+1, \ldots, k$ of A. Let Ω be a complex n-square tridiagonal matrix,

$$\Omega=\begin{bmatrix} b_1 & c_1 & 0 \ldots 0 & & 0 \\ a_1 & b_2 & c_2 \ldots 0 & & 0 \\ 0 & a_2 & b_3 \ldots 0 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & \ldots a_{n-1} & b_n \end{bmatrix},$$

where $a_k c_k$ is real, $k=1, \ldots, n-1$.

Lemma. *If $b_j=0$ for $j=1, \ldots, n$, and $a_k c_k<0$ for $k=1, \ldots, n-1$, then there exists an n-square non-singular diagonal matrix $D=[d_j]$ with $d_1=1$ such that $D^{-1} \Omega D$ is skew-hermitian.*

Proof. The proof is inductive. Clearly the lemma is true for $n=1$. Suppose that $n \geqq 2$. Define the n-square diagonal matrix $P=[p_j]$ by letting

$$p_2=\sqrt{|a_1/c_1|}, \quad p_j=1 \text{ if } j \neq 2.$$

If

$$\Gamma=[\gamma_{jk}]=P^{-1} \Omega P$$

then

$$\gamma_{1j}=-\bar{\gamma}_{j1}, \quad j=1, \ldots, n,$$

and the $(n-1)$-square tridiagonal matrix $\Gamma(1, n]$ satisfies the hypotheses of the lemma. By the inductive assumption there exists an $(n-1)$-square non-singular

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diagonal matrix $Q = [q_j]$ with $q_j = 1$ such that $Q^{-1}\Gamma(1, n]Q$ is skew-hermitian. Let

$$R = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}, \quad D = PR.$$

Then $D = [d_j]$ is an n-square non-singular diagonal matrix with $d_1 = 1$, and $D^{-1}\Omega D$ is skew-hermitian.

Theorem 1. *If λ is an eigenvalue of Ω and $a_k c_k \leqq 0$ for $k = 1, \ldots, n-1$, then*

$$\min\{\mathrm{Re}\,(b_j) \mid j = 1, \ldots, n\} \leqq \mathrm{Re}\,(\lambda) \leqq \max\{\mathrm{Re}\,(b_j) \mid j = 1, \ldots, n\}. \tag{1}$$

Proof. First suppose that $a_k c_k < 0$ for $k = 1, \ldots, n-1$. Let B be the n-square diagonal matrix with diagonal equal to the diagonal of Ω. According to the lemma, there is an n-square non-singular diagonal matrix D and a skew-hermitian matrix C such that

$$D^{-1}(\Omega - B)D = C.$$

Then λ is an eigenvalue of $B+C$, and

$$B+C+(B+C)^* = B+\bar{B}.$$

Hence, if μ and ν are, respectively, the smallest and the largest eigenvalues of $(B+\bar{B})/2$ then (**4**, p. 142)

$$\mu \leqq \mathrm{Re}\,(\lambda) \leqq \nu.$$

However,

$$\min\{\mathrm{Re}\,(b_j) \mid j = 1, \ldots, n\} = \mu,$$

$$\nu = \max\{\mathrm{Re}\,(b_j) \mid j = 1, \ldots, n\}.$$

Hence, (1) holds, if $a_k c_k < 0$ for $k = 1, \ldots, n-1$. Now suppose that

$$1 \leqq k_1 < k_2 < \ldots < k_m \leqq n-1,$$

$$a_k c_k = 0 \text{ if } k \in \{k_1, \ldots, k_m\},$$

$$a_k c_k < 0 \text{ if } k \notin \{k_1, \ldots, k_m\},$$

$k = 1, \ldots, n-1$. Let

$$k_0 = 0, \quad k_{m+1} = n.$$

We have

$$\det(zI - \Omega) = \prod_{\alpha=0}^{m} \det(zI - \Omega(k_\alpha, k_{\alpha+1}]).$$

Hence, since λ is an eigenvalue of Ω, it is an eigenvalue of some $\Omega(k_\alpha, k_{\alpha+1}]$. Therefore, by the first part of the proof,

$$\min\{\mathrm{Re}\,(b_j) \mid j = k_\alpha+1, \ldots, k_{\alpha+1}\} \leqq \mathrm{Re}\,(\lambda) \leqq \max\{\mathrm{Re}\,(b_j) \mid j = k_\alpha+1, \ldots, k_{\alpha+1}\}.$$

Hence, (1) is true.

Arscott (1) proved that (1) holds if λ is real, Ω is real, and $a_k c_k < 0$ for $k = 1, \ldots, n-1$. Jayne (3) generalized Arscott's result by removing the requirement that λ be real. Our method of proving Theorem 1 is quite different from the techniques used by Arscott and Jayne.

Theorem 2. *If λ is an eigenvalue of Ω and $a_k c_k \geqq 0$ for $k = 1, \ldots, n-1$, then*

$$\min \{\mathrm{Im}\,(b_j) \mid j = 1, \ldots, n\} \leqq \mathrm{Im}\,(\lambda) \leqq \max \{\mathrm{Im}\,(b_j) \mid j = 1, \ldots, n\}. \tag{2}$$

Proof. Apply Theorem 1 to the tridiagonal matrix $-i\Omega$.

Theorem 3. *If λ is a zero of* per $(zI-\Omega)$ *and $a_k c_k \geqq 0$ for $k = 1, \ldots, n-1$, then* (1) *is satisfied.*

Proof. It follows from (2) that if Γ is the tridiagonal matrix obtained from Ω by replacing each c_k by $-c_k$ then

$$\mathrm{per}\,(zI-\Omega) = \det\,(zI-\Gamma).$$

Hence, since λ is a zero of per $(zI-\Omega)$, λ is an eigenvalue of Γ. Therefore, by Theorem 1, (1) holds.

Similarly it can be shown that Theorem 2 implies the following.

Theorem 4. *If λ is a zero of* per $(zI-\Omega)$ *and $a_k c_k \leqq 0$ for $k = 1, \ldots, n-1$, then* (2) *is satisfied.*

REFERENCES

(1) F. M. Arscott, Latent roots of tri-diagonal matrices, *Edinburgh Math. Notes* **44**, 5-7 [in *Proc. Edinburgh Math. Soc.* (2) **12** (1961)].

(2) P. M. Gibson, An identity between permanents and determinants, *Amer. Math. Monthly* **76** (1969), 270-271.

(3) J. W. Jayne, An exclusion theorem for tri-diagonal matrices, *Proc. Edinburgh Math. Soc.* (2) **16** (1969), 251-253.

(4) M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities* (Allyn and Bacon, Boston, 1964).

University of Alabama in Huntsville
Huntsville, Alabama 35807
U.S.A.