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THE WORK OF L. G. KOVÁCS ON REPRESENTATION THEORY

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In memoriam Laci Kovács

Abstract

We discuss some of the work of Laci Kovács on representation theory and related topics.

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1. Introduction

The work of (Laci) Kovács on representation theory is wide and varied. An article of this nature necessitates some selectivity, so I hope that the choices I have made leave a representative account, doing proper justice to his work, and that of his many collaborators. It was my privilege to be one of those collaborators, and I record here that I learned and gained much from this experience. I saw at first hand the breadth and depth of his mathematical knowledge. I also saw how he would allow ideas to gestate after taking a fresh and original approach to a problem often taking several steps back to rethink the proper and most general context in which to view the problem. Laci always strove for definitiveness, as well as clarity and accuracy. If he wanted to bound a quantity, he would aim for the best possible bound, providing examples to illustrate that this bound could not be improved upon. If he was involved in some classification project, he would insist that the classification was as comprehensive as possible, and totally reliable for future users. If he was involved in the proof of a theorem, he liked to probe the boundaries of validity, and often gave examples to demonstrate how weakening the hypotheses would invalidate the conclusion.

Nothing should be inferred from the order in which the material below is presented. It seems more appropriate to group the results discussed according to subject matter, rather than chronological order, since some themes recur several times throughout Laci's research career.

2. Characteristic-free proofs

There are several recurring themes in Laci's work on representation theory: often, he would take a result which was known in characteristic zero, and then provide a characteristic-free approach to prove a more general result. One example of this was the 1971 paper [8] with Roger Bryant, in which it is proved that if α is a difference of characters (that is, trace functions afforded by finite-dimensional representations) of a (not necessarily finite) group G, and n is an integer, then the function $\alpha^{(n)}$ given by $\alpha^{(n)}(g) = \alpha(g^n)$ is also a difference of such characters. One motivation for this result was a conjecture of Ballew and Higgins concerning the behaviour of the function counting the number of n-roots of elements of a finite group G. The characteristic-zero version of the main result would have been sufficient to settle that conjecture, but, typically, the necessary result on generalized characters was placed in a more general context. The proper context is when $G = \mathrm{GL}(n, F)$ for a field F and α is the character afforded by its natural module V. The necessary result is then derived by short and elegant argument by decomposing the n-fold tensor power of V as a module for $G \times S_n$. The use of eigenvalues is avoided altogether.

Another example is furnished by a letter to Curtis, which appeared *verbatim* as the article [21], and became quite well known among specialists, partly due to the unusual mode of its publication, apparently at the suggestion of Peter Neumann. The very fact that a letter to another mathematician was already written with sufficient care and clarity to be published in a journal as it stood says much about the precision of Laci's mathematical writing in any context. The letter is about a page and a half, so is written with economy, yet it is liberally sprinkled with references and background information.

The mathematical issue addressed in the letter is this: Brauer [6] had published a proof that if P and Q are permutation matrices with $P^{-1}MQ = M$ for some $M \in \mathrm{GL}(n, \mathbb{C})$, then the permutations associated to P and Q, say σ and τ, have the same cycle structure. This result was known to Burnside, and Brauer was partly motivated by the case that M is the complex character table of some group (or the table of Brauer characters). Brauer has left a footnote in [6] indicating that the same result was true (with a modified proof, not given) replacing $\mathbb{C}$ by any field.

Brauer's proof (which works for any field of characteristic zero) used the fact that $MQM^{-1} = P$, from which it follows that $\mathrm{trace}(P^k) = \mathrm{trace}(Q^k)$ for every integer k, so permutation equivalence readily follows. The key point is that in this setting, the trace of a permutation matrix reveals the number of fixed points of the associated permutation, a fact which is no longer true in characteristic $p > 0$. Brauer's proof would carry over (with minor modifications using Brauer characters) to characteristic p for permutation matrices of multiplicative order prime to p, but not for general permutations (consider the case when P and Q have order p^m for some m).

Laci made the observation that (in any characteristic) we can easily count the number of orbits (including those of length 1) of σ and τ (and likewise for all their powers) from the matrix similarity of P and Q, for there is one dimension of fixed points on the underlying vector space for each orbit of σ, including orbits of length 1,

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while the dimension of the fixed point space is clearly invariant under matrix similarity. He then used a Möbius inversion type argument, citing a (never-vanishing) determinant introduced in the nineteenth century by Smith, to deduce in a clean, characteristic-free, manner that σ and τ have the same cycle type.

A further notable example occurs in the joint paper [9], again with Bryant, where a substantial strengthening of earlier results of Blichfeldt [2] and Brauer [7] is obtained by a very direct and short argument. Let G be a finite group, say $G = \{g_1, g_2, \ldots, g_n\}$, where $g_1 = 1_G$. Let K be a field and, for each i, let V_i be a KG-module on which g_i does not act as a scalar. Then the free KG-module of rank 1 is a direct summand of the tensor product $\bigotimes_{i=2}^{n} V_i$. This readily implies that if we have KG-modules W_i such that g_i acts nontrivially on W_i for $2 \le i \le n$, then the KG-module $\bigoplus_{J\subseteq\{2,\ldots,n\}} W_J$ has a free summand of rank 1, where $W_J = \bigotimes_{j\in J} W_j$. The Krull–Schmidt theorem then implies that each projective indecomposable KG-module is isomorphic to a direct summand of some W_J (in fact, somewhat stronger statements are proved later in the paper). The transition between the two results is simple and neat. It is noted that $g \in G$ acts nontrivially on U if and only if g does not act as a scalar on $K \oplus U$, where K is the trivial KG-module.

This may be compared with results of Blichfeldt and Brauer at the level of complex characters: if G is a finite group, and χ is a faithful complex character of G which takes t different values on nonidentity elements of G, then each irreducible character μ of G occurs with nonzero multiplicity in χ^r for some integer r with $1 \le r \le t+1$. There is a fairly complicated proof of a weaker version of this result in Burnside's book [12], but Brauer gives a shorter proof by a Van der Monde determinant argument, and Blichfeldt's proof is a rather short argument using characters.

The proof of the rather stronger result in the paper under discussion is extremely elegant and short: it is clear that for $2 \le i \le n$, we can find a nonzero $v_i \in V_i$ which is not an eigenvector of g_i (in its action on V_i). Setting $w = v_2 \otimes v_3 \cdots \otimes v_n$, it is quickly demonstrated that $\{wg_i : 1 \le i \le n\}$ is linearly independent, so these vectors span a KG-submodule isomorphic to the regular module.

3. Number of generators of permutation groups and linear groups

Laci always maintained an interest in the question of how many elements it took to generate particular kinds of groups. When attempting to bound the number of generators of permutation groups it is often necessary to bound the number of generators of linear groups along the way (and *vice versa*), so representation theoretic questions naturally arise.

In the paper [26], joint with Newman, it was proved that there is a constant c such that every nilpotent transitive subgroup of S_d can be generated by $\lfloor (cd)/\sqrt{\log d} \rfloor$ elements and that this general type of bound cannot be improved.

The paper [15], joint with Dixon, considered the minimum number of generators of finite p-groups P with a faithful irreducible representation of degree d over a field $\mathbb{F}$ of characteristic different from p. It is shown that there are constants a, b

which both depend on $\mathbb{F}$ and p, and a constant c (independent of $\mathbb{F}$ and p), such that P can be generated by $ad + b\lfloor (cd)/\sqrt{\log d}\rfloor + 2$ elements. It is also noted that no bound of the form $\lfloor (c'd)/\sqrt{\log d}\rfloor$ could be improved upon, essentially in view of the Kovács–Newman result. It is interesting to note that the behaviour of the integral $\int_0^{\pi/2} (\sin(px)/(p\sin x))^n\, dx$ plays a role in the proof.

In the paper [11], joint with Bryant and the present author, the result with Dixon is improved from p-groups to the wider class of groups which are the central product of a solvable normal subgroup and various components (quasi-simple subnormal subgroups). This allows a similar generalization of the above result of Kovács and Newman on transitive permutation groups. Along the way, analogues of both the Kovács–Newman permutation group result and the Dixon–Kovács linear group result are proved for solvable groups (with bounds of the form $\lfloor (c'd)/\sqrt{\log d}\rfloor$, where d is the degree). It is interesting that this result, and the results of the earlier papers discussed, depend on a (previously known) upper bound for the length of an antichain in a poset which is a Cartesian product of $d \geq 2$ chains. This is the origin of the appearance of the quantity $d/\sqrt{\log d}$.

In a somewhat different direction, the paper [27], joint with the present author, proved that every finite completely reducible linear group of degree d can be generated by $\lfloor 3d/2\rfloor$ or fewer elements. Such a result had previously been proved for finite p-groups by Isaacs in [20], and already in the case of 2-groups it is not generally possible to improve the $\lfloor 3d/2\rfloor$ bound. The proof of the result for the general finite group uses Clifford theoretic results proved in a rather more general context than usual for finite group theory, and some of these are proved separately in Laci's paper [23], to be discussed below.

4. Bounding the number of conjugacy classes of linear groups and permutation groups

A classical theorem of Jordan bounds the index of an Abelian normal subgroup of a finite complex linear group of degree d as a function of d. Evidence suggests that if we content ourselves with bounding the number of conjugacy classes, as opposed to the order, somewhat sharper and more uniform bounds should be attainable (again, modulo Abelian normal subgroups). Laci became interested in this question.

The number of conjugacy classes of a finite group G will be denoted by $k(G)$. In the paper [28], joint with the present author, bounds were obtained on $k(G)$ when G is a finite permutation group of degree d or a completely reducible linear group of degree d, the two questions being inextricably intertwined.

These results were motivated, in part at least, by the p-solvable case of Brauer's $k(B)$-problem from modular representation theory (which, in general, is to bound, given a prime p, the number of irreducible characters in a p-block B by the order of its defect group. In the particular case of p-solvable groups, this question was shown by Nagao to be equivalent to proving that when V is a minimal normal p-subgroup of a finite group H with $[H : V]$ coprime to p, then $k(H) \leq |V|$. The latter problem became known as the $k(GV)$-problem).

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One difficulty with the $k(GV)$-problem is that it is does not seem directly amenable to an inductive treatment, particular problems being caused when V is an imprimitive module. In the paper under discussion, a strategy is developed to try to circumvent this difficulty. While the strategy, as it stood, would not, in itself, directly lead to a solution of the problem, it did enable partial progress and led to new types of bound for the various quantities under discussion.

It was proved in this paper that when G is a solvable subgroup of the symmetric group S_d, we have $k(G) \leq 3^{(d-1)/2}$ and that, if G is a solvable completely reducible subgroup of $\mathrm{GL}(V)$ for V a d-dimensional vector space over the field of p elements, then $k(GV) \leq 3^{d-1}p^d$. As a consequence of these results, it was proved that if B is a p-block of defect d of a finite solvable group, then $k(B) \leq 3^{d-1}p^d$. It was also proved that when G is an arbitrary subgroup of the symmetric group S_d, then $k(G) \leq 5^{d-1}$ and that there is a constant c, independent of p, such that if B is a p-block of defect d of a finite p-solvable group, then $k(B) \leq c^{d-1}p^d$. Also, a reduction was given of the proof that $k(G) \leq 2^{d-1}$ for each subgroup G of S_d to the case that G was almost simple, and this was later proved (and improved upon) by later authors (see, for example, Maróti [33], where the $3^{(d-1)/2}$ bound is established for all subgroups of S_d). These results depended on another result on linear groups, proved in the paper (and dependent on the classification of the finite simple groups for nonsolvable groups) that there is a constant c such that whenever G is a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$, we have $k(H) \leq c^{n-1}$ for each subgroup H of $G/F(G)$, and that the constant c may be replaced by 3 for solvable G.

The $k(GV)$-problem was finally solved in 2004 [18], and refinements of the results of the paper presently under discussion, and some of the techniques introduced there, played a role in several of the papers written in final stages of that solution, particularly for smaller primes (the $k(GV)$-problem was first solved for very large primes in [34] in 'generic' style, but the verification for smaller primes turned out to be extremely delicate).

5. Action on polynomial algebras, semigroup representations

In the paper [1], joint with Alperin, the action of $G = \mathrm{SL}(2, k)$ on the polynomial algebra $k[x, y]$ is considered when k is a finite field of characteristic p and cardinality q. Then V_n is the kG-module afforded by homogeneous polynomials of degree n, and is the dual of a Weyl module for G. The interesting point noted here, which improved a result of Glover [17], treating the case $q = p$, is that the sequence (W_n) is periodic of period $q(q-1)$, where W_n is the maximal projective-free summand of V_n.

The proof is short: it is proved that $W_n = 0$ whenever $n > 0$ is divisible by q and that $W_{q(q-1)+1}$ is the trivial module. From this, and the exact sequence

$$0 \to V_{n-1} \otimes V_{m-1} \to V_n \otimes V_n \to V_{m+n-1} \to 0,$$

valid for all $m, n > 1$, which is due to G. E. Wall, the result follows. This exact sequence is also employed in the inductive proof that $W_{mq} = 0$, having established that V_q is the

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Steinberg module, which is isomorphic to the tensor product of the first r Frobenius twists of V_p, where $q = p^r$.

In the article [21], Laci considered the (graded) module structure of the polynomial algebra $U = F[x_1, x_2, \dots, x_r]$, where F is a field of prime characteristic p having the (possibly infinite) field F_q as a subfield. The algebra U is considered as a module for any of M, the multiplicative semigroup of $r \times r$ matrices with entries in F_q, $G = \mathrm{GL}(r, F_q)$ or $S = \mathrm{SL}(r, F_q)$. It had become apparent from work of Krop [30, 31] that considering representations of semigroups, rather than groups, was sometimes advantageous in this context, and Laci had taken much interest in Krop's work.

The main result of [22] considered the module $V = U/U^{p^e}U$ and so the quotient of U modulo the ideal generated by p^eth powers, where $p^e \le q$, with its inherited grading. A natural grading is placed on the monomials forming the most obvious basis for the dth graded component V_d. After a careful and detailed analysis, Laci proved that the S-endomorphisms generate a subalgebra of $\mathrm{End}_F(V_d)$ spanned by endomorphisms (b, b'), where b and b' are basis elements such that b dominates b' in the partial order, and (b, b') sends b to B' and annihilates the other basis elements. Furthermore, it is proved that this subalgebra contains the full image of M in the endomorphism ring.

In a later paper [24], Laci considered the semigroup algebra $K[M]$, where M is the multiplicative semigroup $M_n(F)$, F being a finite field, and K being a commutative ring with identity such that $|F|$ is a unit in K. In what he calls Faddeev's proposition, he proved that $K[M]$ is the direct sum of $n + 1$ algebras, the rth of which is a full $m_r \times m_r$ matrix algebra over the group ring $K[\mathrm{GL}(r, F)]$, where m_r is the number of r-dimensional subspaces of an n-dimensional vector space over F.

A complete and self-contained proof is given, and it is noted that this is a more general version of a result announced by Faddeev, a proof of which had not previously been published.

This result has recently been used by Kuhn [32] to determine the structure of the category $\mathrm{Rep}(F; K)$ of functors from finite-dimensional F-vector spaces to K-modules, which is of some interest in algebraic topology.

6. Factoring group algebras

The paper [14], joint with Carlson, considered natural 'factorization' questions related to algebras and to regular modules for groups. It is proved that if G is a finite Abelian p-group and the group algebra FG may be expressed in the form $A \otimes B$ for subalgebras A and B, then there are subgroups X and Y of G such that $A \cong FX$ and $B \cong FY$.

A result of similar nature (stated in somewhat more precise form in Theorem 3.1 of the paper under discussion) is that if F is any field and G is a finite Abelian group such that the regular matrix representation is equivalent to the Kronecker product of two matrix representations, then FG admits an algebra factorization $FG = B \otimes C$ such that the regular representation of G is equivalent to the Kronecker product of the regular matrix representations of the algebras B and C.

7. Other work

In the paper [23], Laci proved that if a finite-dimensional faithful irreducible representation ρ over a field F of a group G admits no proper tensor factorization (when viewed as a projective representation) and G contains a subgroup K which commutes with all its other G-conjugates such that some irreducible component of $\mathrm{Res}^G_K(\rho)$ is absolutely irreducible, then the representation ρ (viewed as a projective representation) is tensor induced from a representation of $H = N_G(K)$. Results of this nature are useful in Clifford theory, though, as remarked earlier, this theorem is stated in a greater than usual degree of generality, which is exploited in the paper [27] discussed earlier.

In the paper [16], joint with Glasby, it is proved that if H is a normal subgroup of prime index in a finite group G, and F is a field not of characteristic zero and not algebraically closed, then there are generally six possibilities when an irreducible FH-module is induced to G. This is in contrast to the usual two possibilities when F is either algebraically closed or of characteristic zero.

In the paper [25], joint with Leedham-Green, a class of p-groups is exhibited which demonstrates that there can be no bound on the derived length of finite groups whose complex irreducible characters are all induced from linear characters of normal subgroups (so-called nM-groups).

In the paper [10], joint with Bryant, it is proved that whenever p is a prime and H is a subgroup of $\mathrm{GL}(d, p)$, then there is a d-generator finite p-group P such that $\mathrm{Aut}(P) \cong H$ in its action on the Frattini factor $P/\Phi(P)$.

In the paper [19], joint with Howlett, it is proved that $\dim_F H^1(G, V) \leq \dim_F H^1(N, U)$ whenever V is an irreducible FG-module for a field F, N is a subnormal subgroup acting nontrivially on V and U is an irreducible FN-submodule of V. Such results are useful in the study of maximal subgroups of finite groups.

Laci wrote three joint papers (see [3–5] with Bovdi, Bovdi and Mihovski and Bovdi and Sehgal, respectively) concerned with units in modular group algebras. He also wrote a joint paper [13], with Butler and Campbell, on infinite-rank integral representations of groups and orders, as well as a joint paper [29] with Sim giving a rather precise classification of certain finite metacyclic linear groups.

I do not discuss Laci's several papers about group representations associated to action on Lie powers with various coauthors, since these are discussed in the article by Marianne Johnson.

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