Canad. Math. Bull. Vol. 17 (2), 1974.

STEEPEST DESCENT AND LEAST SQUARES SOLVABILITY

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Let T be a bounded linear operator defined on a Hilbert space H. An element $z \in H$ is called a least squares solution of the equation

$$(1) \qquad Tx = b$$

if $\|Tz-b\| = \inf\{\|Tx-b\| : x \in H\}$. It is easily shown that z is a least squares solution of (1) if and only if z satisfies the normal equation

$$(2) \qquad T^*Tx = T^*b,$$

where T^* is the adjoint of T.

If the range of T is closed, then (1) is always least squares solvable and Nashed [5] has shown that the steepest descent sequence $\{x_n\}$, given by

$$x_{n+1} = x_n - \alpha_n r_n, \qquad x_0 \in H$$

where $r_n = T^*Tx_n - T^*b$ and $\alpha_n = \|r_n\|^2/\|Tr_n\|^2$, converges to a least squares solution. Kammerer and Nashed [3] have established the convergence of $\{x_n\}$ to a least squares solution for operators with nonclosed range provided that the projection of b onto the closure of the range of T is in the range of TT^*.

In this note we give a necessary and sufficient condition for least squares solvability of (1) for operators with arbitrary range. This condition is reminiscent of Picard's criterion for solvability of linear integral equations of the first kind [7, p. 164] (see also [2], [4], and [6]) and is analogous to a theorem of Browder and Petryshyn [1] which characterizes the solvability of certain nonlinear operator equations in terms of the sequence of Picard iterates of the operator.

THEOREM. *Each weak limit point of* $\{x_n\}$ *is a least squares solution of* (1).

Proof. Suppose z is a weak limit point of $\{x_n\}$. Given $y \in H$, there is a subsequence $\{x_{n_j}\}$ with

$$\lim_j \langle z - x_{n_j}, y\rangle = 0$$

and

$$\lim \langle x_{n_j} - z, y - T^*Ty\rangle = 0.$$

However, $\langle x_{n_j} - z, y - T^*Ty\rangle = \langle x_{n_j} - z - r_{n_j} + T^*Tz - T^*b, y\rangle$, therefore addition of the limits gives

$$\langle T^*Tz - T^*b, y\rangle = \lim \langle r_{n_j}, y\rangle.$$

But Nashed [5] has shown that lim $r_n=0$ (the blanket assumption in [5] is that T has closed range, however the proof that lim $r_n=0$ does not use this assumption). Since y is arbitrary, it follows that z is a solution of (2), i.e. z is a least squares solution of (1).

In a Hilbert space every strongly bounded sequence contains a weakly convergent subsequence, therefore if $\{x_n\}$ is bounded for some $x_0 \in H$ the theorem above implies that (1) has a least squares solution. Conversely, if z is a least squares solution, then the argument of [3, 3.2] shows that $\{\|x_n-z\|\}$ is non-increasing and hence $\{x_n\}$ is bounded. Therefore we have:

COROLLARY. *Equation* (1) *is least squares solvable if and only if the steepest descent sequence* $\{x_n\}$ *is bounded for any* $x_0 \in H$.

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