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FINITELY PRESENTED INVERSE SEMIGROUPS WITH FINITELY MANY IDEMPOTENTS IN EACH $\mathcal{D}$-CLASS AND NON-HAUSDORFF UNIVERSAL GROUPOIDS

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Abstract

The complex algebra of an inverse semigroup with finitely many idempotents in each $\mathcal{D}$-class is stably finite by a result of Munn. This can be proved fairly easily using C^*-algebras for inverse semigroups satisfying this condition that have a Hausdorff universal groupoid, or more generally for direct limits of inverse semigroups satisfying this condition and having Hausdorff universal groupoids. It is not difficult to see that a finitely presented inverse semigroup with a non-Hausdorff universal groupoid cannot be a direct limit of inverse semigroups with Hausdorff universal groupoids. We construct here countably many nonisomorphic finitely presented inverse semigroups with finitely many idempotents in each $\mathcal{D}$-class and non-Hausdorff universal groupoids. At this time, there is not a clear C^*-algebraic technique to prove these inverse semigroups have stably finite complex algebras.

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1. Introduction

Given an inverse semigroup S, we denote by $E(S)$ its semilattice of idempotents and by $\leq$ its natural partial order (hence, $a \leq b$ if and only if $a = eb$ for some $e \in E(S)$). Given $s \in S$, let $\lambda_S(s) = \{t \in S \mid t \leq s\}$ and let $\mu_S(s)$ denote the set of maximal elements of $\lambda_S(s) \cap E(S)$ for $\leq$.

Paterson [5] associated an étale groupoid $\mathcal{G}(S)$ to every inverse semigroup S, called its universal groupoid, and showed that the C^*-algebra of the inverse semigroup is isomorphic to the C^*-algebra of its universal groupoid. The second author later

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generalized Paterson's result by showing that if K is any commutative ring with unit, then the semigroup algebra KS of S is isomorphic to a certain convolution algebra of K-valued functions on the groupoid $\mathcal{G}(S)$ [8]. There is now a well-developed theory of étale groupoid algebras which has proven useful for studying inverse semigroup algebras. Both the algebra and C^*-algebra of an étale groupoid are best behaved when the groupoid is Hausdorff. In particular, the theory of traces on groupoid C^*-algebras seems to only be well developed in the Hausdorff case where there is a faithful conditional expectation mapping to the algebra of continuous functions vanishing at infinity on the unit space. Universal groupoids of inverse semigroups are not always Hausdorff. It was shown in [8] that $\mathcal{G}(S)$ is Hausdorff if and only if $\lambda_S(s) \cap E(S)$ is finitely generated as an ideal in $E(S)$, that is, there is a finite set $F \subseteq \lambda_S(s) \cap E(S)$ such that if $e \in \lambda_S(s) \cap E(S)$, then $e \leq f$ for some $f \in F$. In particular, if $\mathcal{G}(S)$ is Hausdorff, then $\mu_S(s)$ is finite. For example, if G is a nontrivial group and E is the semilattice consisting of a zero element and a countably infinite set of orthogonal idempotents, then $S = G \cup E$ is an inverse monoid, where G acts trivially on the left and right of E, with a non-Hausdorff universal groupoid. Indeed, if $1 \neq g \in G$, then $\lambda_S(g) \cap E(S) = E$ which has infinitely many maximal elements. Note that S is a Clifford inverse monoid: each $\mathcal{D}$-class of S contains a single idempotent. However, S is not finitely generated and each of its finitely generated inverse subsemigroups does have a Hausdorff universal groupoid and hence S is a direct limit of Clifford monoids with Hausdorff universal groupoids.

The second author has recently initiated a study of stable finiteness of étale groupoid algebras [9] and, in particular, recovered a result of Munn showing that if S is an inverse semigroup whose $\mathcal{D}$-classes have finitely many idempotents, then KS is stably finite for any field K of characteristic 0 [4]. Recall that a ring R is stably finite if $M_n(R)$ does not contain a copy of the bicyclic monoid as a subsemigroup for any $n \geq 1$. In the case where S has a Hausdorff universal groupoid, this can be deduced using the theory of C^*-algebras, but in the non-Hausdorff case, one needs to work around this. However, since the stable finiteness result can be reduced to the case of finitely generated inverse semigroups with finitely many idempotents in each $\mathcal{D}$-class, it becomes of interest to know whether there are examples of such finitely generated inverse semigroups with non-Hausdorff universal groupoids. The easiest way to guarantee that each $\mathcal{D}$-class of S contains finitely many idempotents is to impose the stronger condition that each $\mathcal{R}$-class of S is finite (since each idempotent f in the $\mathcal{D}$-class of e is of the form $s^{-1}s$ with s in the $\mathcal{R}$-class of e).

This paper is then motivated by the following question.

PROBLEM 1.1. Is there a finitely generated inverse semigroup S such that:

(i) every $\mathcal{R}$-class of S is finite; and
(ii) $\mu_S(s)$ is infinite for some $s \in S$?

In Section 3, we present a construction that provides uncountably many nonisomorphic 3-generated inverse monoids satisfying the conditions of Problem 1.1.

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In Section 4, we present a construction which provides infinitely many nonisomorphic finitely presented 3-generated inverse monoids satisfying the conditions of Problem 1.1.

It is not difficult to show (see [9]) that if S is finitely presented and has a non-Hausdorff universal groupoid, then S cannot be written as a direct limit of inverse semigroups with Hausdorff universal groupoids. Since stable finiteness is preserved under direct limits, to really show that we cannot reduce stable finiteness of KS to the case that S has a Hausdorff universal groupoid, it is important to have a finitely presented inverse semigroup satisfying the conditions of Problem 1.1.

2. Preliminaries

The reader is assumed to have basic knowledge of inverse semigroup theory and automata theory, being respectively referred to [2, 6] for that purpose. Since the inverse semigroups we construct are actually inverse monoids, all the relevant definitions are presented in the monoid version.

2.1. Inverse automata. Given a finite alphabet A, we denote by A^{-1} a set of formal inverses of A and write $\widetilde{A} = A \cup A^{-1}$. An *inverse automaton* over the alphabet $\widetilde{A}$ is a structure of the form $\mathcal{A} = (Q, i, t, E)$, where:

- Q is the set of vertices;
- $i, t \in Q$ are the initial and terminal vertices, respectively;
- $E \subseteq Q \times \widetilde{A} \times Q$ is the set of edges,

satisfying the following properties:

- *deterministic*, $(p, a, q), (p, a, q') \in E \Rightarrow q = q'$;
- *involutive*, $(p, a, q) \in E \Leftrightarrow (q, a^{-1}, p) \in E$;
- *trim*, every vertex lies in some path from i to t.

If $i = t$, we refer to it as the *basepoint* of $\mathcal{A}$. If we do not specify the initial and terminal vertices, we have an *inverse graph*.

Assume now that $\mathcal{A}$ is involutive and trim, but not deterministic. A *folding* operation on $\mathcal{A}$ consists of identifying two distinct edges of the form $p \xleftarrow{a} q \xrightarrow{a} r$ (identifying also the inverse edges $p \xrightarrow{a^{-1}} q \xleftarrow{a^{-1}} r$). If $\mathcal{A}$ is finite and we fold enough edges, we end up obtaining a finite inverse automaton $\mathcal{A}'$, which we say is obtained from $\mathcal{A}$ by *complete folding*.

Is this operation confluent? That is, does the inverse automaton depend on the sequence of foldings? Given $u, v \in \widetilde{A}^*$, write $u \xrightarrow{*} v$ if v can be obtained from u by successively erasing factors of the form aa^{-1} $(a \in \widetilde{A})$. It is easy to check that

$$L(\mathcal{A}') = \{v \in \widetilde{A}^* \mid u \xrightarrow{*} v \text{ for some } u \in L(\mathcal{A})\}.$$

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Hence, the language of the inverse automaton $\mathcal{A}'$ is completely determined by $\mathcal{A}$. Since inverse automata are known to be minimal (see for example [1]), it follows that $\mathcal{A}'$ is itself determined by $\mathcal{A}$ and so the folding process is confluent.

A *Dyck word* on $\widetilde{A}$ is some $w \in \widetilde{A}^*$ satisfying $w \xrightarrow{*} 1$. These are the words representing the identity in the free group on A, and also play an important role in the theory of inverse semigroups as we soon see.

2.2. Free inverse monoids. Let A be a nonempty alphabet. We extend ${}^{-1}: A \to A^{-1} : a \mapsto a^{-1}$ to an involution on the free monoid $\widetilde{A}^*$ through

$$1^{-1} = 1, \quad (a^{-1})^{-1} = a, \quad (uv)^{-1} = v^{-1}u^{-1} \quad (a \in A;\ u, v \in \widetilde{A}^+).$$

The *free inverse monoid on* A is the quotient $FIM_A = \widetilde{A}^*/\rho$, where ρ is the congruence on $\widetilde{A}^*$ generated by the relation

$$\{(ww^{-1}w, w) \mid w \in \widetilde{A}^*\} \cup \{(uu^{-1}vv^{-1}, vv^{-1}uu^{-1}) \mid u, v \in \widetilde{A}^*\},$$

known as the *Wagner congruence* on $\widetilde{A}^*$.

Munn provided in [3] an elegant normal form for FIM_A using inverse automata (see also [7] by Scheiblich).

Given $w = a_1 \cdots a_n \in \widetilde{A}^*$ ($a_i \in \widetilde{A}$), let $\mathrm{Lin}(w)$ denote the *linear automaton* of w:

$$\longrightarrow q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n \longrightarrow$$

that contains also the inverse edges (to make it involutive). The *Munn tree* of w is the finite inverse automaton $\mathrm{MT}(w)$ obtained by completely folding $\mathrm{Lin}(w)$. This provides the following solution for the word problem of FIM_A.

THEOREM 2.1 [3]. *For all $u, v \in \widetilde{A}^*$, the following conditions are equivalent:*

(i) $u\rho = v\rho$;
(ii) $\mathrm{MT}(u) \cong \mathrm{MT}(v)$;
(iii) $u \in L(\mathrm{MT}(v))$ *and* $v \in L(\mathrm{MT}(u))$.

Such Munn trees are precisely those finite inverse automata on $\widetilde{A}$ whose underlying undirected graph is a tree (when we consider only the edges labeled by A).

It is easy to see that, given $w \in \widetilde{A}^*$,

$$w\rho \in E(FIM_A) \Leftrightarrow \mathrm{Lin}(w) \text{ has a basepoint } \Leftrightarrow w \text{ is a Dyck word.}$$

2.3. Inverse monoid presentations. A (finite) inverse monoid presentation is a formal expression of the form $\mathcal{P} = \langle A \mid R \rangle$, where A is a (finite) alphabet and R is a (finite) subset of $\widetilde{A}^* \times \widetilde{A}^*$. We usually describe the relations in R as formal equalities $r = s$.

Let $\tau = (\rho \cup R)^\sharp$ be the congruence on $\widetilde{A}^*$ generated by the relation $\rho \cup R$. The quotient $S = \widetilde{A}^*/\tau$ is the inverse monoid defined by $\mathcal{P}$. Equivalently, we might write $S = FIM_A/R^\sharp$, viewing R as a relation on FIM_A. We denote by $\varphi\colon \widetilde{A}^* \to S$ the canonical homomorphism.

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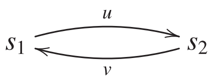
Stephen devised in [10] an approach which is the most useful tool known to date to deal with inverse monoid presentations. The *Cayley graph* of S with respect to the generating set A, denoted by $\mathrm{Cay}_A(S)$, has vertex set S and edges of the form $s \xrightarrow{a} s(a\varphi)$ for all $s \in S$ and $a \in \widetilde{A}$. This is not in general an involutive graph (it would be if S is a group). However, the strongly connected components of $\mathrm{Cay}_A(S)$ are actually inverse automata. Additionally, these correspond to the various $\mathcal{R}$-classes of S (so there exist paths



in $\mathrm{Cay}_A(S)$ for some $u, v \in \widetilde{A}^*$ if and only if $s_1 s_1^{-1} = s_2 s_2^{-1}$). The *Schützenberger graph* of $w \in \widetilde{A}^*$, denoted by $S\Gamma(w)$, is the strongly connected component of $\mathrm{Cay}_A(S)$ containing $w\varphi$ (that is, the induced subgraph having the $\mathcal{R}$-class of $w\varphi$ as a set of vertices). The *Schützenberger automaton* of $w \in \widetilde{A}^*$, denoted by $\mathcal{A}(w)$, is obtained from $S\Gamma(w)$ by setting $(ww^{-1})\varphi$ as initial vertex and $w\varphi$ as terminal vertex. We may also write $S\Gamma(u\varphi) = S\Gamma(u)$ or $\mathcal{A}(u\varphi) = \mathcal{A}(u)$ if it suits us.

Stephen proved the following theorems, which generalize those of Munn.

THEOREM 2.2 [10]. *For all $u, v \in \widetilde{A}^*$, the following conditions are equivalent:*

(i) $u\varphi \geq v\varphi$;
(ii) $u \in L(\mathcal{A}(v))$.

THEOREM 2.3 [10]. *For all $u, v \in \widetilde{A}^*$, the following conditions are equivalent:*

(i) $u\varphi = v\varphi$;
(ii) $\mathcal{A}(u) \cong \mathcal{A}(v)$;
(iii) $u \in L(\mathcal{A}(v))$ *and* $v \in L(\mathcal{A}(u))$.

Since $u\varphi \in E(S)$ if and only if $(uu^{-1})\varphi = u\varphi$, it follows that the idempotents of S are characterized by having a basepoint at their Schützenberger automaton.

It follows from Theorem 2.3 that the word problem is decidable for $\mathcal{P}$ if membership is decidable in the languages of its Schützenberger automata. In general, it is not. However, Stephen devised a procedure which brings positive results in many important cases. We describe it now.

Suppose that $\mathcal{A}$ is a finite involutive automaton and $r = s$ is a relation in R such that there exists in $\mathcal{A}$ a path of the form $p \xrightarrow{r} q$ but no path $p \xrightarrow{s} q$ (or *vice versa*). If we glue to $\mathcal{A}$ a 'path' $p \xrightarrow{s} q$ with the corresponding inverse edges (to keep it involutive), we say that this new automaton $\mathcal{A}'$ is obtained from $\mathcal{A}$ by an *expansion* (expanding through the relation $r = s$).

If $L(\mathcal{A}) \subseteq L(\mathcal{A}(w))$ and $\mathcal{A}'$ is obtained from $\mathcal{A}$ by either folding or through an expansion through some relation in R, then $L(\mathcal{A}') \subseteq L(\mathcal{A}(w))$. We call any finite involutive automaton $\mathcal{A}$ such that $w \in L(\mathcal{A}) \subseteq L(\mathcal{A}(w))$ an *approximate automaton* of w (for the presentation $\mathcal{P}$). This is the case of any finite inverse automaton

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obtained from $\mathrm{MT}(w)$ through a finite sequence of foldings and expansions. If $\mathcal{A}$ is an approximate automaton of w that admits neither foldings nor expansions, then $\mathcal{A} \cong \mathcal{A}(w)$.

Assume now that the $\mathcal{R}$-classes of S are all finite. Then the Schützenberger automata are all finite as well. Can we compute them? If R is finite, the answer is given through Stephen's sequence, which provides a systematic way of building approximate automata. Take $\mathcal{A}_1(w) = \mathrm{MT}(w)$. If $\mathcal{A}_n(w)$ is defined, let $\mathcal{A}'_{n+1}(w)$ be obtained by performing *simultaneously* all the possible expansions of $\mathcal{A}_n(w)$. Since R is finite, $\mathcal{A}'_{n+1}(w)$ is a finite involutive automaton. Now define $\mathcal{A}_{n+1}(w)$ by completely folding $\mathcal{A}'_{n+1}(w)$. We get then a sequence $(\mathcal{A}_n(w))_n$ of approximate automata of w. If $\mathcal{A}(w)$ is finite, it will eventually show up as a member of the sequence.

This is still true for infinite R if:

- we are sure of S having only finite $\mathcal{R}$-classes;
- only finitely many expansions can be applied to each $\mathcal{A}_n(w)$.

Anyway, in the general case of an arbitrary presentation, $v \in L(\mathcal{A}_T(u))$ implies that v is recognized by some approximate automaton of u; in fact, given any approximate automaton of u, one can perform a finite sequence of foldings and expansions to obtain an approximate automaton recognizing v.

The results in this subsection are used throughout the paper without further reference. The reader is referred to [10] for more details on Schützenberger automata and Stephen's sequences.

3. Finitely generated examples

In this section, we present a construction which provides uncountably many nonisomorphic 3-generated inverse monoids satisfying the conditions of Problem 1.1.

We start by proving the following proposition.

PROPOSITION 3.1. *Let S be an inverse monoid where every $\mathcal{R}$-class is finite. Let $\langle A \mid R\rangle$ be an inverse monoid presentation of S. Let $E \subseteq E(S) \setminus \{1\}$ and let $w_e \in \widetilde{A}^+$ be a Dyck word representing e for every $e \in E$. Let T be the inverse semigroup defined by the inverse monoid presentation*

$$\langle A \cup \{b\} \mid R,\ w_e b = w_e\ (e \in E)\rangle, \tag{3-1}$$

where b is a new letter. Then:

(i) *every $\mathcal{R}$-class of T is finite;*
(ii) *if no two distinct elements of E are $\mathcal{J}$-comparable in S and E is infinite, then $\mu_T(b)$ is infinite.*

PROOF. (i) Note that an idempotent e can always be represented by some Dyck word since $e = ee^{-1}$. Write $B = A \cup \{b\}$ and let $\psi\colon \widetilde{A}^* \to S$ and $\varphi\colon \widetilde{B}^* \to T$ be the canonical homomorphisms.

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We use the notation $\mathcal{A}_S(u)$ and $\mathcal{A}_T(u)$ to denote the Schützenberger automata of u relative to the presentation of S and T, respectively.

Let $u \in \widetilde{B}^*$. Then $\mathrm{MT}(u)$ is essentially the disjoint union of finitely many Munn trees for some words in $\widetilde{A}^*$, connected by b-trees to each other in a way that results in a tree. Let Q' be the set of vertices of $\mathrm{MT}(u)$ that admit a b-loop (at their image) in $\mathcal{A}_T(u)$. Let $\mathcal{A}_1$ be the finite inverse automaton obtained from $\mathrm{MT}(u)$ by adjoining a b-loop at each $q \in Q'$ followed by complete folding. Adding a b-loop and folding can absorb b-trees into that vertex creating possibly a wedge of Munn trees with $\widetilde{A}$-edges at that vertex. Such a wedge of $\widetilde{A}$-trees is itself an $\widetilde{A}$-tree, and hence an approximate automaton for some $\widetilde{A}$-word for FIM_A, folding therefore into a Munn tree. Thus, $\mathcal{A}_1$ is a finite inverse automaton consisting of Munn trees with edges in $\widetilde{A}$, say $\mathrm{MT}(u_1), \ldots, \mathrm{MT}(u_n)$, connected by b-edges and having some b-loops adjoined. Moreover, $\mathcal{A}_1$ is an approximate automaton of u with the property that each b-edge that is not a loop comes from $\mathrm{MT}(u)$.

We start now what we call the iterative process. Since every $\mathcal{R}$-class of S is finite, we can turn each of these Munn trees $\mathrm{MT}(u_i)$ inside $\mathcal{A}_1$ into a finite inverse automaton admitting no R-expansions by a finite series of R-expansions and foldings. The resulting automaton (which contains the same b-edges as $\mathcal{A}_1$) is a finite inverse automaton admitting no R-expansions, and is still an approximate automaton of u. However, vertices of the Munn trees $\mathrm{MT}(u_i)$ may have been identified in the process, so now folding of b-edges may become possible. After folding the b-edges, different $\widetilde{A}$-components may now share a vertex, but we claim that these new $\widetilde{A}$-components may still be viewed as approximate automata for some word on $\widetilde{A}$.

Indeed, assume that $\mathcal{B}$ and C are subautomata of $\mathcal{A}_1$ corresponding to approximate automata of $v, w \in \widetilde{A}^*$, respectively. If we identify the vertex p of $\mathcal{B}$ with the vertex q of C, we may modify v, w to assume that p and q were, respectively, the terminal and initial vertices of $\mathcal{B}$ and C. It follows easily that, by identifying p and q, we get an approximate automaton of $vw \in \widetilde{A}^*$.

We may now restart the process. At each iteration, the number of b-edges decreases, so we are bound to eventually halt by obtaining some finite inverse automaton $\mathcal{A}_2$ admitting no R-expansions, which is still an approximate automaton of u.

Now let $\mathcal{A}_3$ be obtained from $\mathcal{A}_2$ by adjoining a b-loop at each vertex p admitting a path $p \xrightarrow{w_e^{-1}} \cdots$ for some $e \in E$ (if it does not exist already). Adjoining these new b-loops does not allow any folding: if $p \xrightarrow{b^\varepsilon} q \neq p$ would be an edge of $\mathcal{A}_2$ for $\varepsilon = \pm 1$, then we have necessarily $p \in Q'$ because p is doomed to host a b-loop in $\mathcal{A}_T(u)$, and then we already have a b-loop at p in $\mathcal{A}_1$. Hence, $\mathcal{A}_3$ admits no folding and it certainly admits no expansions of any sort. Since $\mathcal{A}_3$ is an approximate automaton of u, it follows that $\mathcal{A}_3 = \mathcal{A}_T(u)$, which is therefore finite.

Therefore, $\mathcal{A}_T(u)$ must be finite in all cases and so is $\mathcal{R}_{u\varphi}$.

(ii) We start by noting that the homomorphism $\eta\colon S \to T$ extending the identity mapping on A is an embedding. It is immediate that we have a homomorphism $\theta\colon T \to S$ defined by

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$$b\theta = 1, \quad a\theta = a \ (a \in A).$$

Then $\theta\eta = 1_S$ (as it is the identity on A), and so η is an embedding of S into T.

Let $e \in E$. To construct $\mathcal{A}_T(w_e)$, we start by turning $\mathrm{MT}(w_e)$ into $\mathcal{A}_S(w_e)$, which is an approximate automaton of w_e for the presentation in Equation (3-1). Expanding through the relation $w_e b = w_e$ and subsequent folding produces a b-loop at the basepoint i and possibly at other vertices where a path labeled by w_e (necessarily a loop) can be read.

Suppose now that we have some path of the form $p \xrightarrow{w_{e'}} q$ in $\mathcal{A}_S(w_e)$ with $e' \in E \setminus \{e\}$. Then $uw_{e'}v \in L(\mathcal{A}_S(w_e))$ for some $u, v \in \widetilde{A}^*$ and so $(u\psi)e'(v\psi)e = e$, yielding $e' \geq_{\mathcal{J}} e$. This contradicts the assumption that no two distinct elements of E are $\mathcal{J}$-comparable in S, and hence $\mathcal{A}_S(w_e)$ admits no path of the form $p \xrightarrow{w_{e'}} q$ and so $\mathcal{A}_T(w_e)$ is indeed $\mathcal{A}_S(w_e)$ with a few b-loops attached (namely at the basepoint).

In view of the relation $w_e b = w_e$, we get $w_e\varphi \leq b\varphi$ and so $w_e\varphi \in \lambda_T(b\varphi) \cap E(T)$. Suppose now that $w_e\varphi \leq f \in \lambda_T(b\varphi) \cap E(T)$. Let $v \in \widetilde{B}^*$ be a Dyck word representing f.

On the one hand, $w_e\varphi \leq v\varphi$ implies $L(\mathcal{A}_T(v)) \subseteq L(\mathcal{A}_T(w_e))$. On the other hand, since $f \leq b\varphi$, we have $v\varphi = f = f(b\varphi) = (v\varphi)(b\varphi) = (vb)\varphi$. Thus, $\mathcal{A}_T(vb) \cong \mathcal{A}_T(v)$ and so there exists a b-loop at i in $\mathcal{A}_T(vb)$.

It follows that at some point in the Stephen's sequence of vb, we must have had some expansion through some relation of the form $w_{e'}b = w_{e'}$. Hence, $w_{e'}$ labels some path in $\mathcal{A}_T(v)$ and so $xw_{e'}y \in L(\mathcal{A}_T(v)) \subseteq L(\mathcal{A}_T(w_e))$ for some $x, y \in \widetilde{B}^*$.

We have seen above that there is no path in $\mathcal{A}_S(w_e)$ (and consequently none in $\mathcal{A}_T(w_e)$) labeled by some $w_{e'}$ for $e' \in E \setminus \{e\}$, and hence $\mathcal{A}_T(v)$ must admit some loop labeled by w_e at some vertex q_1.

Let $i = q_0 \xrightarrow{w} q_1$ be a path in $\mathcal{A}_T(v)$. For those familiar with the bicyclic monoid, if we put $x = (vww_e)\varphi$, then $xx^{-1} = f$ and $x^{-1}x = e(w^{-1}\varphi)f(w\varphi)e \leq e \leq f$. So if $e < f$, then x, x^{-1} generate a copy of the bicyclic monoid with identity f and hence the $\mathcal{R}$-class of f is infinite, contradicting part (i). For those not familiar with the bicyclic monoid, here is a direct proof. Here, $ww_ew^{-1} \in L(\mathcal{A}_T(v)) \subseteq L(\mathcal{A}_T(w_e))$ and so $(ww_ew^{-1})\varphi \geq w_e\varphi$. Assume that there exists a path

$$q_0 \xrightarrow{w} q_1 \xrightarrow{w} \cdots \xrightarrow{w} q_j \circlearrowleft w_e$$

in $\mathcal{A}_T(v)$ for some $j \geq 1$. Then $w^jw_ew^{-j} \in L(\mathcal{A}_T(v))$ yields $(w^jw_ew^{-j})\varphi \geq v\varphi$ and so

$$(w^{j+1}w_ew^{-(j+1)})\varphi = (w^j(ww_ew^{-1})w^{-j})\varphi \geq (w^jw_ew^{-j})\varphi \geq v\varphi.$$

Thus, $w^{j+1}w_ew^{-(j+1)} \in L(\mathcal{A}_T(v))$. By induction, it follows that we have a path of the above form in $\mathcal{A}_T(v)$ for every $j \geq 1$. Since $\mathcal{A}_T(v)$ is finite by part (i) and is an inverse automaton, we must have $q_j = q_0$ for some $j \geq 1$. However, then w_e labels a loop at $q_0 = i$ in $\mathcal{A}_T(v)$ and so $w_e\varphi \geq v\varphi = f$.

Thus, $f = w_e\varphi$ and so $w_e\varphi \in \mu_T(b\varphi)$ for every $e \in E$.

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Since E is infinite and $\eta\colon S \to T$ induced by the identity on A is an embedding, $\{w_e\varphi \mid e \in E\}$ is an infinite subset of $\mu_T(b\varphi)$. □

Now we can use Proposition 3.1 to produce examples which answer positively Problem 1.1.

EXAMPLE 3.2. Let $I \subseteq \mathbb{N} \setminus \{0\}$ be infinite and let T_I be defined by the inverse monoid presentation

$$\langle a, b, c \mid ab^i a = ab^i ac \ (i \in I)\rangle. \tag{3-2}$$

Then:

(i) all the $\mathcal{R}$-classes of T_I are finite;
(ii) $\mu_{T_I}(c)$ is infinite.

Indeed, let $S = FIM_{\{a,b\}}$, which has finite $\mathcal{R}$-classes [3]. Then

$$E_I = \{((ab^i a)^{-1}(ab^i a))\rho \mid i \in I\}$$

is an infinite subset of $E(S)$. If $i, j \in I$ are distinct, then there is no path labeled by $ab^i a$ in $\mathrm{MT}(ab^j a)$ and *vice versa*, so no two distinct elements of E_I are $\mathcal{J}$-comparable in S. Consider now the inverse monoid presentation

$$\langle a, b, c \mid (ab^i a)^{-1}(ab^i a) = (ab^i a)^{-1}(ab^i a)c \ (i \in I)\rangle.$$

In any inverse semigroup, $u = uc$ is equivalent to $u^{-1}u = u^{-1}uc$, and hence this presentation is equivalent to Equation (3-2).

Now it follows from Proposition 3.1 that all the $\mathcal{R}$-classes of T_I are finite and $\mu_{T_I}(c)$ is infinite.

We can now use Example 3.2 to prove the following proposition.

PROPOSITION 3.3. *There exist uncountably many nonisomorphic* 3*-generated inverse monoids satisfying the conditions of Problem 1.1.*

PROOF. For each infinite $I \subseteq \mathbb{N} \setminus \{0\}$, let T_I be defined by the inverse monoid presentation in Equation (3-2). We have shown in Example 3.2 that T_I satisfies the conditions of Problem 1.1. Since a countably infinite set contains uncountably many infinite subsets, it suffices to show that $T_I \ncong T_J$ for distinct $I, J \subseteq \mathbb{N} \setminus \{0\}$.

Out of symmetry, we may assume that $i \in I \setminus J$. Suppose that $\theta\colon T_I \to T_J$ is an isomorphism. It is easy to see that each generating set of T_J must contain:

- a or a^{-1};
- b or b^{-1};
- c or c^{-1}.

Hence, any minimal generating set of T_J is necessarily of the form $\{a^\varepsilon, b^\delta, c^\gamma\}$ with $\varepsilon, \delta, \gamma = \pm 1$. Thus, we may assume that θ is induced by some bijection $\{a, b, c\} \to \{a^\varepsilon, b^\delta, c^\gamma\}$.

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Since $ab^ia = ab^iac$ is a relation of the presentation of T_I, then $(ab^ia)\theta = (ab^iac)\theta$ holds in T_J. Hence, $c\theta$ labels a loop in $\mathcal{A}_{T_J}((ab^ia)\theta)$ and so $c\theta = c^\gamma$ necessarily.

Since $ab^ia = ab^iac$ holds in T_I, then $(ab^ia)\theta = (ab^ia)\theta c^\gamma$ must hold in T_J. The only way of enabling an expansion in $\mathrm{MT}((ab^ia)\theta)$ is if $a\theta = a$, $b\theta = b$, and $i \in J$, which is a contradiction. Therefore, $T_I \not\cong T_J$. □

4. Finitely presented examples

In this section, we present a construction which provides infinitely many nonisomorphic finitely presented 3-generated inverse monoids satisfying the conditions of Problem 1.1.

EXAMPLE 4.1. For each $t \geq 2$, let S_t be defined by the inverse monoid presentation

$$\langle a, b, c \mid ca = a, cb^{-t}c^{-1}b^t = cb^{-t}b^tc^{-1}\rangle.$$

Then:

(i) every $\mathcal{R}$-class of S_t is finite;
(ii) $\mu_{S_t}(aca^{-1})$ is infinite.

Let us check these facts.

(i) Write $S = S_t$ and $A = \{a, b, c\}$. Let $\varphi\colon \widetilde{A}^* \to S$ be the canonical homomorphism. For every $x \in A$, let $\pi_x\colon \widetilde{A}^* \to \mathbb{Z}$ be the homomorphism defined by

$$y\pi_x = \begin{cases} 1 & \text{if } y = x, \\ -1 & \text{if } y = x^{-1}, \\ 0 & \text{if } y \in \widetilde{A} \setminus \{x, x^{-1}\}. \end{cases}$$

Let $u \in \widetilde{A}^*$. It suffices to show that $S\Gamma(u)$ possesses only finitely many edges labeled by a, b, c.

If we perform an expansion involving the letter a, the subsequent folding prevents the number of a-edges to increase. Hence, the number of a-edges in $S\Gamma(u)$ is bounded by the number of a-edges in $\mathrm{MT}(u)$.

Dealing with the b-edges is harder because its number may increase through the Stephen's sequence of u. We start by proving a few remarks.

$$\text{If } v \text{ labels a loop in } S\Gamma(u), \text{ then } v\pi_b = 0. \tag{4-1}$$

Inspection of the defining relations shows that π_b induces a homomorphism $\overline{\pi}_b\colon S_t \to \mathbb{Z}$. If v reads a loop in $S\Gamma(u)$, then $s(v\varphi) = s$ for some $s \in S_t$, and so $s\overline{\pi}_b + v\pi_b = s\overline{\pi}_b$, whence $v\pi_b = 0$.

Let $v \in \widetilde{A}^+$. We call a vertex (respectively edge) of $S\Gamma(v)$ *original* if it corresponds to some vertex (respectively edge) of $\mathrm{MT}(v)$. We show that:

$$\text{Every vertex } p \text{ of } S\Gamma(u) \text{ admits some path } p \xrightarrow{b^m} q \text{ with } m \geq 0 \text{ and } q \text{ original.} \tag{4-2}$$

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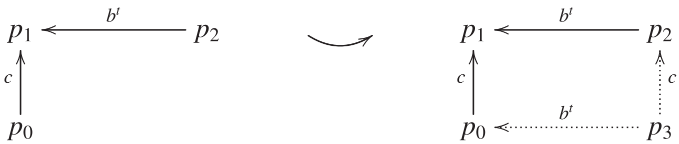
Once again, this property holds trivially for $\mathrm{MT}(u)$ and is preserved through folding. What about expansions? If we expand through the relation $ca = a$ and fold once to make the new c label a loop, the property still holds. Hence, we may assume that we are expanding through the relation $cb^{-t}c^{-1}b^t = cb^{-t}b^tc^{-1}$. The only way of getting new vertices is described by the following picture (where some appropriate folding is also assumed to limit the appearance of new vertices to a minimum):



If the property holds for the vertex p_0, it must also hold for the new vertices (p_4 and p_5 in the picture), thus the property is preserved through both sorts of expansions. It follows that the property holds throughout the whole Stephen's sequence of u. Since every vertex p of $S\Gamma(u)$ must originate from some term of the Stephen's sequence, then Equation (4-2) holds.

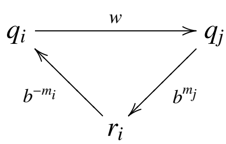
Let X be the set of b-edges of $S\Gamma(u)$. Let σ be the equivalence relation on X generated by relating edges

$$p \xrightarrow{b} q \quad \text{and} \quad p' \xrightarrow{b} q'$$

whenever $S\Gamma(u)$ admits a path

$$p \xrightarrow{b^s} \bullet \xrightarrow{c} \bullet \xrightarrow{b^{-s}} p'$$

for some $s \in \mathbb{Z}$. Since the unique expansions that increase the number of b-edges involve the relation $cb^{-t}c^{-1}b^t = cb^{-t}b^tc^{-1}$, it follows easily by induction on the usual expansion/folding scheme that every edge in X is σ-equivalent to some original edge. Thus, σ has only finitely many equivalence classes and we only need to show that the size of each equivalence class can be bounded. Let k be the number of vertices in $\mathrm{MT}(u)$. Suppose that some equivalence class of σ possesses $k+1$ different edges $p_i \xrightarrow{b} q_i$ for $i = 0, \ldots, k$. By Equation (4-2), for each $i = 0, \ldots, k$, there is a path $q_i \xrightarrow{b^{m_i}} r_i$ in $S\Gamma(u)$ with $m_i \geq 0$ and r_i original. Hence, there exist $0 \leq i < j \leq k$ such that $r_i = r_j$. Since $q_i \neq q_j$, we get $m_i \neq m_j$. However, it is easy to see that since $p_i \xrightarrow{b} q_i$ and $p_j \xrightarrow{b} q_j$ are σ-equivalent, there exists some path $q_i \xrightarrow{w} q_j$ in $S\Gamma(u)$ with $w\pi_b = 0$. However, then



is a loop in $S\Gamma(u)$ with $(wb^{m_j}b^{-m_i})\pi_b \neq 0$, contradicting Equation (4-1). Therefore, each equivalence class of σ has at most k elements and so $S\Gamma(u)$ has finitely many b-edges.

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It remains to bound the number of c-edges. Suppose that we remove all the a-edges and b-edges from $S\Gamma(u)$ to get the automaton $\mathcal{A}$. Each time we remove an edge, the number of connected components increases at most by one. Hence, the number of connected components of $\mathcal{A}$ is bounded and it suffices to show that all of them are finite.

Now for every edge $p_i \xrightarrow{c} q_i \neq p_i$ appearing for the first time in $\mathcal{A}_i(u)$ in the Stephen's sequence, there exist necessarily some edge $p_{i-1} \xrightarrow{c} q_{i-1} \neq p_{i-1}$ in $\mathcal{A}_{i-1}(u)$ and paths

$$\begin{array}{ccc} q_{i-1} & \xleftarrow{\;b^t\;} & q_i \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle c} \\ p_{i-1} & \xleftarrow[\;b^t\;]{} & p_i \end{array}$$

in $\mathcal{A}_i(u)$ because the expansion $ca = a$ could never produce the straight edge $p_i \xrightarrow{c} q_i$. It follows that, for every edge $p \xrightarrow{c} q \neq p$ in $S\Gamma(u)$, there exist some edge $p_1 \xrightarrow{c} q_1$ in $\mathrm{MT}(u)$ and paths

$$\begin{array}{ccc} q_1 & \xleftarrow{\;b^m\;} & q \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle c} \\ p_1 & \xleftarrow[\;b^m\;]{} & p \end{array}$$

in $\mathcal{A}(u)$ for some $m \geq 0$. Since $\mathrm{MT}(u)$ is finite and $S\Gamma(u)$ has finitely many b-edges, then we can bound the size of each connected component of $\mathcal{A}$. Thus, $S\Gamma(u)$ possesses only finitely many c-edges and we are done.

(ii) For every $n \geq 1$, let $e_n = ((ab^{tn}a)(ab^{tn}a)^{-1})\varphi$. We claim that

$$\overset{c\,\circlearrowleft}{p_0} \xrightarrow[a]{} \overset{c\,\circlearrowleft}{p_1} \xrightarrow[b^t]{} \overset{c\,\circlearrowleft}{p_2} \xrightarrow[b^t]{} \cdots \xrightarrow[b^t]{} \overset{c\,\circlearrowleft}{p_n} \xrightarrow[b^t]{} \overset{c\,\circlearrowleft}{p_{n+1}} \xrightarrow[a]{} p_{n+2} \qquad (4\text{-}3)$$

is $S\Gamma(e_n)$. Here an edge $p \xrightarrow{b^t} q$ is shorthand for a sequence of t edges in a straightline from p to q, labeled by b, (together with their inverse edges) and with no other edges incident on any vertex other than p, q.

It is clear that $\mathrm{MT}(e_n)$ is

$$\longleftrightarrow p_0 \xrightarrow[a]{} p_1 \xrightarrow[b^t]{} p_2 \xrightarrow[b^t]{} \cdots \xrightarrow[b^t]{} p_{n+1} \xrightarrow[a]{} p_{n+2}\ .$$

Expanding through the relation $ca = a$, we can produce the first and the last c-loops. Then we expand through the relation $cb^{-t}c^{-1}b^t = cb^{-t}b^tc^{-1}$ to successively produce the c-loops at $p_n, p_{n-1}, \ldots, p_1$. Since Equation (4-3) is an inverse automaton and admits no expansions, it must be $S\Gamma(e_n)$, which has therefore $tn + 3$ vertices.

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Now $\mathcal{A}(e_n)$ is obtained from $S\Gamma(e_n)$ by declaring p_0 the basepoint. Since $aca^{-1} \in L(\mathcal{A}(e_n))$, then $(aca^{-1})\varphi \geq e_n$ and so $e_n \in \lambda((aca^{-1})\varphi) \cap E(S)$. We show that $e_n \in \mu_S((aca^{-1})\varphi)$.

Suppose now that $f \in \lambda((aca^{-1})\varphi) \cap E(S)$ satisfies $e_n \leq f$. Let $v \in \widetilde{A}^*$ be a Dyck word representing f.

First we note that $f \in \lambda((aca^{-1})\varphi)$ implies $f \leq (aca^{-1})\varphi$. Hence, $aca^{-1} \in L(\mathcal{A}(v))$.

Write $B = \{b, c\}$. We show that

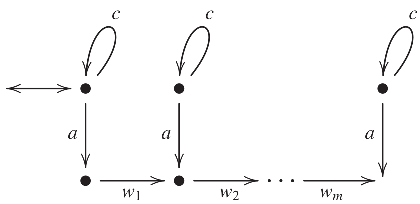
$$\text{If } w \in \widetilde{B}^*, \text{ then } S\Gamma(w) \text{ contains no } c\text{-loops.} \tag{4-4}$$

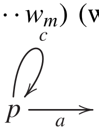
First note that if $u \in \widetilde{B}^*$, then u does not represent the same element of S_t as any word z containing a or a^{-1} since the relation $ca = a$ cannot be applied to any word in $\widetilde{B}^*$. It then follows that the inverse subsemigroup T_t of S_t generated by B is defined by the relation $cb^{-t}c^{-1}b^t = cb^{-t}b^tc^{-1}$. It also follows that a cannot label an edge in $S\Gamma(w)$ for any $w \in \widetilde{B}^*$ since if a labels an edge in $S\Gamma(w)$, then $w\varphi = z\varphi$ for some word $z \in \widetilde{A}^+$ containing an a or a^{-1}. Thus, the Schützenberger graphs of w in S_t and T_t coincide. From the presentation for T_t, $\pi_c|_{\widetilde{B}^*}$ factors through a homomorphism $T_t \to \mathbb{Z}$. Since $\mathbb{Z}$ is a group, it follows that if $z \in \widetilde{B}^*$ labels a loop in $S\Gamma(w)$, then $z\pi_c = 0$, and so Equation (4-4) holds.

Since $e_n \leq f = v\varphi$, then $v \in L(\mathcal{A}(e_n))$. Let $\mathcal{A}$ be obtained by removing the vertex p_{2n+2} from $\mathcal{A}(e_n)$ (in the notation given in Equation (4-3) for $S\Gamma(e_n)$). Suppose that $v \in L(\mathcal{A})$. We can factor

$$v = c^{i_0}(aw_1a^{-1}c^{i_1}) \cdots (aw_ma^{-1}c^{i_m})$$

with $m \geq 1$, $i_j \in \mathbb{Z}$, and $w_j \in B^*$. Expanding $\mathrm{Lin}(v)$ through the relation $ca = a$ and folding to get c-loops, we get the automaton



Applying expansions and folds to the part corresponding to $\mathrm{Lin}(w_1 \cdots w_m)$, the graph we get is $S\Gamma(w_1 \cdots w_m)$ (which contains no c-loops by Equation (4-4)) with finitely many subgraphs 

adjoined. No further expansion applies to this graph, and hence what we get is really $\mathcal{A}(v)$. However, we have remarked before that $aca^{-1} \in L(\mathcal{A}(v))$, contradicting the nonexistence of c-loops in $S\Gamma(w_1 \cdots w_m)$. Thus, $v \in L(\mathcal{A}(e_n)) \setminus L(\mathcal{A}')$.

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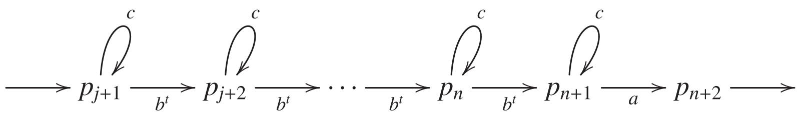
It follows that we can factor $v = v'v''$ so that v' labels a path of the form

$$p_0 \xrightarrow{v_0} p_0 \xrightarrow{a} p_1 \xrightarrow{v_1} p_1 \xrightarrow{v_2} p_2 \xrightarrow{v_3} p_2 \xrightarrow{v_4} p_3 \xrightarrow{v_5} \cdots \xrightarrow{v_{2n}} p_{n+1} \xrightarrow{v_{2n+1}} p_{n+1} \xrightarrow{a} p_{n+2}$$

in $\mathcal{A}(e_n)$ for some factorization $v' = v_0 a v_1 v_2 \cdots v_{2n+1} a$ such that $v_{2j}\,\rho\, b^t$ for $j = 1, \ldots, n$. And we may assume that

$$\text{The displayed edge } p_0 \xrightarrow{a} p_1 \text{ features the last occurrence of } p_0 \text{ in the path labeled by } v'. \qquad (4\text{-}5)$$

For $j = 0, \ldots, n$, let $\mathcal{A}_j$ denote the inverse automaton depicted by



We show that

$$\mathcal{A}(w) = \mathcal{A}_{n-j} \quad \text{for all } j = 0, \ldots, n \text{ and } w \in L(\mathcal{A}_{n-j}). \qquad (4\text{-}6)$$

We use induction on j. The case $j = 0$ is immediate in view of the relation $ca = a$. Hence, we assume that $j > 0$ and Equation (4-6) holds for $j - 1$.

Let $w \in L(\mathcal{A}_{n-j})$. Then we may write $w = w'xw''$ with $x\,\rho\, b^t$ and $w'' \in L(\mathcal{A}_{n-(j-1)})$. By the induction hypothesis, we get $\mathcal{A}(w'') = \mathcal{A}_{n-(j-1)}$. Folding the b-edges and expanding through the relation $cb^{-t}c^{-1}b^t = cb^{-t}b^tc^{-1}$, we obtain $\mathcal{A}(b^tw'') = \mathcal{A}_{n-j}$. However, $w \in L(\mathcal{A}_{n-j})$ and $xw''\rho = b^tw''\rho$, and thus $w\varphi \geq xw''\varphi$. Therefore,

$$w\varphi = (w'xw'')\varphi = (w'xw'')\varphi((xw'')\varphi)^{-1}(xw'')\varphi = w\varphi((xw'')^{-1}xw'')\varphi = (xw'')\varphi$$

and so $\mathcal{A}(w) = \mathcal{A}(b^tw'') = \mathcal{A}_{n-j}$, whence Equation (4-6) holds.

In particular, $\mathcal{A}(z) = \mathcal{A}_0$ for $z = v_1b^tv_3b^t \cdots b^tv_{2n+1}a$ in view of Equation (4-5). Expanding through the relation $ca = a$ and folding, it is easy to see that $\mathcal{A}(az)$ is just $\mathcal{A}(e_n)$ with p_{2n+2} as terminal vertex. Since v_0 labels a loop at p_0, we have $v_0\varphi \geq (az)(az)^{-1}\varphi$ and so $v_0az\varphi = az\varphi$. Thus, $\mathcal{A}(v') = \mathcal{A}(v_0az) = \mathcal{A}(az)$. Hence, $\mathcal{A}(v'(v')^{-1}) = \mathcal{A}(e_n)$ and so $(v'(v')^{-1})\varphi = e_n$. However, then

$$e_n = (v'(v')^{-1})\varphi \geq v\varphi \geq e_n$$

yields $v\varphi = e_n$ and so $e_n \in \mu_S((aca^{-1})\varphi)$ for every $n \geq 1$. Since we have established before that $S\Gamma(e_n)$ has $tn + 3$ vertices, it follows that $\mu_S((aca^{-1})\varphi)$ is infinite.

We can now use Example 4.1 to prove the following proposition.

PROPOSITION 4.2. *There exist infinitely many nonisomorphic finitely presented 3-generated inverse monoids satisfying the conditions of Problem 1.1.*

PROOF. We have shown in Example 4.1 that S_t satisfies the conditions of Problem 1.1 for every $t \geq 2$. Thus, it suffices to show that $S_t \not\cong S_n$ for distinct $t, n \geq 2$.

Suppose that $\theta\colon S_t \to S_n$ is an isomorphism. It is easy to see that each generating set of S_n must contain:

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- a^α with $\alpha = \pm 1$;
- b^ε with $\varepsilon = \pm 1$;
- c^δ with $\delta = \pm 1$.

Hence, any minimal generating set of S_n consists necessarily of three elements of this form. Thus, we may assume that θ is induced by some bijection $\{a, b, c\} \to \{a^\alpha, b^\varepsilon, c^\delta\}$.

Since $ca = a$ is a relation of the presentation of S_t, then $(c\theta)(a\theta) = a\theta$ holds in S_n, yielding successively $c\theta = c^\delta$ (since a and b never label loops), $a\theta = a^\alpha$, and $b\theta = b^\varepsilon$.

However, $cb^{-t}c^{-1}b^t = cb^{-t}b^tc$ is also a relation of the presentation of S_t, and hence $c^\delta b^{-\varepsilon t}c^{-\delta}b^{\varepsilon t} = c^\delta b^{-\varepsilon t}b^{\varepsilon t}c^{-\delta}$ holds in S_n. Since $\mathrm{MT}(c^\delta b^{-\varepsilon t}c^{-\delta}b^{\varepsilon t})$ admits no expansion for the presentation of S_n, we have reached a contradiction. Therefore, $S_t \not\cong S_n$. □

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