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A CENTRAL LIMIT THEOREM FOR MULTIPLICATIVE SYSTEMS

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Introduction. The central limit theorem was originally proved for independent random variables. The independence is a very strong notion and hard to check. There are various efforts to prove different theorems on independent variables (e.g. strong law of large numbers, central limit theorem, the law of iterated logarithm, convergence theorem of Kolmogorov) under weaker conditions, like mixing, martingale-difference, orthogonality. Among these concepts the weakest one is orthogonality, but this ensures only the validity of law of large numbers. A useful concept of this type was introduced by Alexits [1] namely that of multiplicative systems, defined as a sequence of random variables $\xi_1, \xi_2, \ldots$ satisfying the condition

$$E\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k} = 0 \qquad k = 1, 2, \ldots; i_1 < i_2 < \cdots < i_k. \tag{1}$$

If, in addition, it satisfies

$$E\xi_i^2\xi_j^2 = E\xi_i^2E\xi_j^2 \qquad (i \neq j) \tag{2}$$

it will be called an MS. A multiplicative system is called strongly multiplicative, if for $i_1<i_2<\cdots<i_k$

$$E\xi_{i_1}^2\xi_{i_2}^2\cdots\xi_{i_k}^2 = E\xi_{i_1}^2E\xi_{i_2}^2\cdots E\xi_{i_k}^2. \tag{3}$$

Let us mention that a martingale-difference, i.e. a sequence satisfying

$$E\xi_1 = 0, \qquad E(\xi_n \mid \xi_1, \ldots, \xi_{n-1}) = 0 \tag{4}$$

is multiplicative, supposing that the expectation on the left-hand side of (1) exists. Extending the central limit theorem for martingales, Doob [2] used the condition

$$E\xi_1^2 = \sigma_1^2, \qquad E(\xi_n^2 \mid \xi_1, \ldots, \xi_{n-1}) = \sigma_n^2. \tag{5}$$

Obviously a system satisfying (4) and (5) is strongly multiplicative, supposing that the expectations on the left of (1) and (3) exist.

An example for a multiplicative system, which is not martingale-difference, is the trigonometric sequence $\cos n_kx$ (on $[0, 2\pi]$ with Lebesgue measure) under the lacunarity condition $n_{k+1}/n_k\geq 2$. If it satisfies the stronger condition $n_{k+1}/n_k\geq 3$, it is strongly multiplicative.

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Alexits [3] showed that a bounded multiplicative system ξ_n is a convergence system, i.e. the convergence of $\sum c_k^2$ implies the a.e. convergence of the series $\sum c_k \xi_k$.(1)

Révész [4] and Gapuskin [5] proved the central limit theorem for bounded MS. Takahashi [6] proved that a bounded MS obeys the upper part of the law of iterated logarithm and Révész [7] showed that this is not true for bounded multiplicative systems, i.e. condition (2) cannot be omitted.

Neither can it be omitted in the case of central limit theorem, as it can be seen easily using the following remark. If a sequence $\xi_1, \xi_2, \ldots$ is multiplicative then so is the sequence $\eta\xi_1, \eta\xi_2, \ldots$ if only η is independent of the sequence ξ_n.

In this paper we are going to prove a central limit theorem for strongly multiplicative systems satisfying the Lindeberg condition. We remark that this is the first theorem on multiplicative systems not assuming uniformly boundedness. Actually we do not exhaust the full strength of condition (3), we need condition (2) and a bound on the product expectations:

$$E\xi_{i_1}^2\xi_{i_2}^2 \cdots \xi_{i_k}^2 \le C^k \cdot E\xi_{i_1}^2 E\xi_{i_2}^2 \cdots E\xi_{i_k}^2. \tag{6}$$

(6) is obviously satisfied in all the above mentioned cases (independence, martingales, boundedness), thus our theorem will contain all the above mentioned central limit theorems.

1. **The statement of the theorem.** For the sake of generality we state our theorem for double arrays. We say that a double array $X_{n,j}$ ($n=1, 2, \ldots; j=1, 2, \ldots, N_n$) satisfies one of the above conditions if this condition holds within each row.

THEOREM. *If a double array $X_{n,j}$ satisfies conditions* (1), (2), (6) *and the Lindeberg condition:*

$$\lim_{n\to\infty} \sum_{j=1}^{N_n} \int_{|X_{n,j}|>\eta} X_{n,j}^2 \, dP = 0 \quad \textit{for any} \quad \eta > 0, \tag{7}$$

and if $\sum_{j=1}^{N_n} EX_{n,j}^2 = 1$ for all n, then the distribution function of $S_n = \sum_{j=1}^{N_n} X_{n,j}$ tends to the standard normal distribution function.

The structure of the proof basically agrees with that of Révész's proof [4] and can be formulated in the following way. We prove the following proposition, and check the validity of its assumptions.

(1) In the meanwhile, Révész and the author proved that it is sufficient if (1) holds for only $k=4$, and the boundedness can be substituted by $E\xi_n^4 \le K$.

PROPOSITION. *If a sequence ξ_n satisfies the following conditions*

$$E\prod_{k=1}^{n}\left(1+\frac{it\xi_k}{\sqrt{n}}\right)\to 1 \tag{A}$$

$$\frac{\xi_1^2+\cdots+\xi_n^2}{n}\to 1 \quad \textit{in probability} \tag{B}$$

$$E\prod_{k=1}^{n}\left(1+\frac{t^2\xi_k^2}{n}\right) < K = K(t), \tag{C}$$

then it follows the central limit theorem, i.e. the distribution function of $1/\sqrt{n}\sum_{k=1}^{n}\xi_k$ *tends to the standard normal distribution function.*

While checking the validity of the assumptions we are going to use a Lemma and the following Lebesgue-type theorem. If a sequence ξ_n of uniform integrable random variables converges to 0 in measure, then $E|\xi_n|\to 0$. In particular $E\xi_n\to 0$.

Recall that a sequence ξ_n of real or complex valued random variables is called uniformly integrable if for any $\varepsilon>0$ there is a positive number $A>0$ such that

$$\int_{|\xi_n|>A}|\xi_n|\,dP<\varepsilon \quad \text{for all } n.$$

(Actually this definition is slightly different from the usual one.)

Clearly $E|\xi_n|^2<K$ $(n=1,2,\ldots)$ implies the uniform integrability of the sequence ξ_n.

LEMMA. *If the double array $\xi_{n,j}$ $(n=1,2,\ldots;\ j=1,2,\ldots,N_n)$ of pairwise independent random variable satisfies the following conditions:*

(i) $\displaystyle\sum_{j=1}^{N_n}\int_{|\xi_{n,j}|>1}|\xi_{n,j}|\,dP\to 0$

as $n\to\infty$;

(ii) $\displaystyle\sum_{j=1}^{N_n}\int_{|\xi_{n,j}|\leq 1}|\xi_{n,j}|^2\,dP\to 0$

as $n\to\infty$, then taking

$$\xi_n=\sum_{j=1}^{N_n}\xi_{n,j}$$

we have

$$\xi_n-E\xi_n\to 0$$

in probability.

If the variables are nonnegative, and $E\xi_n$ is bounded, then the pairwise independence of the variables can be replaced by uncorrelatedness.

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REMARK. (C) is implied by (6) but it can be ensured by different types of conditions. (6) is a condition on the dependence of the squares of the variables but if the variables themselves are limited in some way, we do not need further limitations on their relation. The most obvious example is the case of uniformly bounded variables, since $|\xi_n|<K$ implies (6) (and, of course, also the Lindeberg condition) but we do not need that strong bound.

In the case of identically distributed variables it is sufficient if they have moment-generating function, in the general case these functions should obviously be uniform in some way.

We call a sequence exponentially bounded if for each real t there is a finite number $K=K(t)$ for which

$$Ee^{t\xi_n} < K.$$

This condition implies both (C) and the Lindeberg condition, thus an exponentially bounded MS satisfies the central limit theorem.

2. **Proof of the Theorem and the Lemma.** We are making use of the following simple expansion:

$$e^{ix} = (1 + ix)e^{-(x^2/2)+r(x)}$$

where $|r(x)|\leq|x|^3$ for x real. By this expansion we have for given t

$$\begin{aligned} e^{itS_n} &= \prod_{j=1}^{N_n}(1+itX_{n,j})\exp\left(-\frac{t^2}{2}\sum_j X_{n,j}^2+\sum_j r(tX_{n,j})\right) \\ &= e^{-t^2/2}\prod_{j=1}^{N_n}(1+itX_{n,j})+A_n, \end{aligned}$$

where

$$A_n = \prod_j(1+itX_{n,j})\left[\exp\left(-\frac{t^2}{2}\sum_j X_{n,j}^2+\sum_j r(tX_{n,j})\right)-e^{-t^2/2}\right].$$

We will use the notation B_n for the product $\prod_{j=1}^{N_n}(1+itX_{n,j})$ and C_n for the square bracketed expression in the last equality. I.e. $A_n=B_n\cdot C_n$.

By (1) we have for the characteristic function of S_n

$$Ee^{itS_n} = e^{-t^2/2}+EA_n,$$

therefore it remains to show that

$$EA_n \to 0 \quad \text{as} \quad n\to\infty,$$

and according to the above mentioned Lebesgue-type theorem it is sufficient to show that $A_n\to 0$ in measure and A_n is uniformly integrable.

Since

$$A_n = e^{itS_n}-e^{-t^2/2}\cdot B_n,$$

and e^{itS_n} is bounded, if B_n is uniformly integrable, then so is A_n. According to (6)

$$E|B_n|^2=E\left|\prod_j(1+itX_{n,j})\right|^2=E\prod_j(1+t^2X_{n,j}^2)$$
$$\le\prod_j(1+Ct^2EX_{n,j}^2)\le\exp\left(\sum_j Ct^2EX_{n,j}^2\right)=e^{Ct^2}<\infty,$$

and hence B_n (and A_n) is uniformly integrable.

We can complete the proof by showing

$$A_n\to 0\quad\text{in measure.}$$

We will show first that

(8) $$\sum_j X_{n,j}^2\to 1\quad\text{in probability and}$$

(9) $$\sum_j r(tX_{n,j})\to 0\quad\text{in probability,}$$

which imply that $C_n\to 0$ in probability.

Since

$$|B_n|^2=\prod_j(1+t^2X_{n,j}^2)\le\exp\left(t^2\sum_j X_{n,j}^2\right)\to e^{t^2}\text{ in probability,}$$

we have

$$P(|B_n|>e^{t^2/2}+1)\to 0\quad\text{as}\quad n\to\infty$$

(we can say it is "bounded in probability").

Therefore (8) and (9) and $A_n=B_nC_n$ imply

$$P(|A_n|>\varepsilon)\le P(|B_n|>e^{t^2/2}+1)+P\left(|C_n|>\frac{\varepsilon}{e^{t^2/2}+1}\right)$$

tends to 0 as $n\to\infty$.

I.e., it is sufficient to prove (8) and (9).

We show that the lemma is applicable for the variables $\xi_{n,j}=X_{n,j}^2$. They are nonnegative, uncorrelated and

$$E\sum_{j=1}^{N_n}X_{n,j}^2=1,$$

so it is enough to check (i) and (ii).

$$\sum_j\int_{|X_{n,j}|>1}X_{n,j}^2\,dP\to 0$$

by (7) (the Lindeberg condition). Choose $0<\varepsilon<1$ arbitrary.

$$\sum_j\int_{|X_{n,j}|\le 1}X_{n,j}^4\,dP=\sum_j\int_{|X_{n,j}|\le\eta}X_{n,j}^4\,dP+\sum_j\int_{\eta<|X_{n,j}|\le 1}X_{n,j}^4\,dP$$
$$\le\eta^2\sum_j\int X_{n,j}^2\,dP+\sum_j\int_{\eta<|X_{n,j}|}X_{n,j}^2\,dP<\varepsilon$$

if we take $\eta=\sqrt{\varepsilon/2}$ and n is large enough (according to (7) again). Thus the lemma implies (8).

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For proving (9) we use the estimation $|r(x)| \leq |x|^3$.

$$\left|\sum_j r(tX_{n,j})\right| \leq |t|^3 \sum_j |X_{n,j}|^3 \leq |t|^3 \max_{1\leq j\leq N_n} |X_{n,j}| \sum_j X_{n,j}^2.$$

Because of (8), it is enough to show that $\max_{1\leq j\leq N_n} |X_{n,j}| \to 0$ in probability as $n\to\infty$.

$$P\left(\max_j |X_{n,j}| > \varepsilon\right) \leq \sum_j P(|X_{n,j}| > \varepsilon) \leq \frac{1}{\varepsilon^2} \sum_j \int_{|X_{n,j}|>\varepsilon} X_{n,j}^2 \, dP \to 0$$

by (7), proving our theorem.

Proof of the Lemma. Define

$$\xi_{n,j}^* = \begin{cases} \xi_{n,j} & \text{if } |\xi_{n,j}| \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\xi_n^* = \sum_{j=1}^{N_n} \xi_{n,j}^*.$$

(10) $$P(\xi_n \neq \xi_n^*) \to 0 \quad \text{by (i).}$$

(11) $$E(\xi_n^* - E\xi_n^*)^2 = \sum_j \operatorname{Var}(\xi_{n,j}^*) \leq \sum_j E(\xi_{n,j}^*)^2 = \sum_j \int_{|\xi_{n,j}|\leq 1} \xi_{n,j}^2 \, dP \to 0 \quad \text{by (ii).}$$

Therefore $\xi_n^* - E\xi_n^* \to 0$ in probability, which, together with (10) implies that

$$\xi_n - E\xi_n^* \to 0 \quad \text{in probability.}$$

It remains to show that

$$E\xi_n - E\xi_n^* \to 0.$$

$$|E\xi_n - E\xi_n^*| \leq \sum_j \left| \int \xi_{n,j} \, dP - \int \xi_{n,j}^* \, dP \right| \leq \sum_j \int_{|\xi_{n,j}|>1} |\xi_{n,j}| \, dP \to 0$$

by (i), proving the first part of the lemma.

If $\xi_{n,j} \geq 0$, $E\xi_n$ is bounded and the pairwise independence is replaced by

$$E\xi_{n,j}\xi_{n,k} = E\xi_{n,j}E\xi_{n,k} \qquad (j \neq k),$$

the proof remains the same, except (11) which is to change as follows:

$$\begin{aligned} E(\xi_n^* - E\xi_n^*)^2 &= E\left(\sum_j \xi_{n,j}^*\right)^2 - (E\xi_n^*)^2 \\ &= \sum_j E(\xi_{n,j}^*)^2 + \sum_{\substack{1\leq j,k\leq N_n \\ 1\neq k}} E\xi_{n,j}^*\xi_{n,k}^* - (E\xi_n^*)^2 \\ &\leq \sum_j E(\xi_{n,j}^*)^2 + \sum_{j,k=1}^{N_n} E\xi_{n,j}\xi_{n,k} - (E\xi_n^*)^2 \\ &= \sum_j E(\xi_{n,j}^*)^2 + (E\xi_n + E\xi_n^*)(E\xi_n - E\xi_n^*), \end{aligned}$$

which tends to 0 as we have already proved (using only (i) and (ii)).

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