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CLASSIFICATION OF SUBPENCILS FOR HYPERPLANE SECTIONS ON CERTAIN K3 SURFACES

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Abstract We classify the subpencils of complete linear systems for the hyperplane sections on K3 surfaces obtained as the complete intersection of a hyperquadric and a hypercubic. The classification is done from three points of view, namely, the type of a general fibre, the base locus and the Horikawa index of the essential member. This classification shows the distinct phenomenons depending on the rank of the hyperquadrics containing the surface.

Keywords: hyperplane sections; nonhyperelliptic curves; pencils; Horikawa index

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1. Introduction

Throughout the paper, all the varieties are defined over the field $\mathbb{C}$ of complex numbers.

Let S be a minimal resolution of a projective surface S' obtained as a complete intersection of a hyperquadric and a hypercubic of $\mathbb{P}^4$. Assume that S' has at most rational double points as the singularities. Then S is a K3 surface, and a general member of the complete linear system of the hyperplane sections is a nonhyperelliptic curve of genus 4.

Denote by Λ the complete linear system of the hyperplane sections of S, and let $\mathcal{P} \subset \Lambda$ be a subpencil. Assume that a general member of $\mathcal{P}$ is smooth. Let $\nu : \widetilde{S} \to S$ be a blow-up such that the complete linear system of the proper transform of the member of $\mathcal{P}$ is base point free. We assume that ν is the shortest among the blow-up with the above property. Then there exists a surjective morphism $f : \widetilde{S} \to \mathbb{P}^1$ whose general fibre is a nonhyperelliptic curve of genus 4. Furthermore, f is relatively minimal. For each fibre $\mathcal{F}$ of f, the invariant $\mathrm{Ind}(\mathcal{F})$ named Horikawa index (H-index) is defined as we mention in the next section. $\mathrm{Ind}(\mathcal{F})$ is a non-negative rational number, and we have $\mathrm{Ind}(\mathcal{F}) = 0$ except for a finite number of fibres of f.

Nonhyperelliptic curve C of genus 4 is obtained as the complete intersection of a hyperquadric $\mathfrak{Q}_0$ and a hypercubic $\mathfrak{Y}$ in $\mathbb{P}^3$. Since the defining equation of $\mathfrak{Q}_0$ is given as a quadratic form of the homogeneous coordinates of $\mathbb{P}^3$, the rank $\mathrm{rk}(\mathfrak{Q}_0)$ is defined, which is equal to 3 or 4. $\mathfrak{Q}_0$ is expressed as follows:

(I) If $\mathrm{rk}(\mathfrak{Q}_0) = 4$, we have $\mathfrak{Q}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the hyperplane section of $\mathfrak{Q}_0$ is a diagonal divisor. If we consider C as a divisor of $\mathfrak{Q}_0$, then C is linearly equivalent to the triple of the diagonal. According to [1], in this case, C is called as *Eisenbud–Harris general* (*EH-general*).

(II) If $\mathrm{rk}(\mathfrak{Q}_0) = 3$, $\mathfrak{Q}_0$ is a cone over a smooth conic. If we denote by Δ_0 the tautological divisor of the Hirzebruch surface $\Sigma_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$, then $\mathfrak{Q}_0$ is the image of Σ_2 by the morphism $\Phi_{|\Delta_0|}$ defined by the complete linear system $|\Delta_0|$. If Γ is a fibre of the ruling $\mu : \Sigma_2 \to \mathbb{P}^1$ and if Δ_∞ is a section of μ with $\Delta_\infty \sim \Delta_0 - 2\Gamma$, then Δ_∞ is contracted to the vertex of $\mathfrak{Q}_0$. Since C does not go through the vertex, we may consider as $C \subset \Sigma_2$. If we consider that C is a divisor of Σ_2, then we have $C \sim 3\Delta_0$. In this case, let us call C *Eisenbud–Harris special* (*EH-special*).

If a general fibre of f above is EH-general (respectively, EH-special), f is called the EH-general fibration (respectively, the EH-special fibration), $\mathcal{P}$ the EH-general subpencil (respectively, the EH-special subpencil). The definition of H-index depends on whether f is EH-general or EH-special. The sum of H-indices for the fibration we investigate in the present paper is as follows (see § 2 for details):

- When f is EH-general, then the sum is equal to $1/2$.
- When f is EH-special, then the sum is equal to $6/7$.

Each case is divided into several cases as follows:

(I) The case where f is EH-general.
  - (I-i) There are two fibres with H-index $1/4$.
  - (I-ii-a) There is a fibre with H-index $1/2$, and the rank of the hyperquadric containing the fibre is 3.
  - (I-ii-b) There is a fibre with H-index $1/2$, and the rank of the hyperquadric containing the fibre is 2.

(II) The case where f is EH-special.
  - (II-i-a) There are three fibres with H-index $2/7$.
  - (II-i-b) There is a fibre with H-index $2/7$ and a fibre with H-index $4/7$.
  - (II-i-c) There is a fibre with H-index $6/7$, and the rank of the hyperquadric containing the fibre is 3.
  - (II-ii-a) There are two fibres with H-index $3/7$.
  - (II-ii-b) There is a fibre with H-index $6/7$, and the rank of the hyperquadric containing the fibre is 2.
  - (II-ii-c) There is a fibre with H-index $6/7$, and the rank of the hyperquadric containing the fibre is 1.

Let $Q \subset \mathbb{P}^4$ be the hyperquadric containing S'. As in the case of $\mathfrak{Q}_0$, the rank $\mathrm{rk}(Q)$ is also defined, and we have that $\mathrm{rk}(Q)$ is one of 3, 4 and 5. As we will see in § 4, if $\mathrm{rk}(Q) = 3$, then Q has a singular curve ℓ as the compound rational double point of

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type A_1, and S' has rational double points on ℓ. We restrict our arguments to the generic case as follows:

Assumption 1. *When* $\mathrm{rk}(Q) = 5$, *we assume* $S' = S$. *When* $\mathrm{rk}(Q) = 4$, *we assume* S' *does not go through the vertex of* Q *and that* $S' = S$ *holds. When* $\mathrm{rk}(Q) = 3$, *we assume that* S' *does not have singularities except for the intersection points with* ℓ.

The main result of the present paper is as follows:

Theorem 1. *Let the notation and the conditions be as above. The classification of subpencils of* Λ *without fixed component is as follows:*

(1) The case where $\mathrm{rk}(Q) = 3$

Type	*The type of a general member*	*The fixed points (the most generic case)*	*The type of fibres with positive H-index*
R3-1	*EH-special*	*6 simple base points*	*(II-i-a), (II-i-b), (II-i-c)*
R3-2	*EH-special*	*6 simple base points*	*(II-ii-b)*
R3-3	*EH-special*	*3 base points with intersection multiplicity 2*	*(II-ii-c)*

(2) The case where $\mathrm{rk}(Q) = 4$

Type	*The type of a general member*	*The fixed points (the most generic case)*	*The type of fibres with positive H-index*
R4-1	*EH-general*	*6 simple base points*	*(I-ii-a)*
R4-2	*EH-general*	*6 simple base points*	*(I-ii-b)*
R4-3	*EH-special*	*6 simple base points*	*(II-ii-a)*
R4-4	*EH-special*	*3 base points with intersection multiplicity 2*	*(II-ii-b)*

(3) The case where $\mathrm{rk}(Q) = 5$

Type	*The type of a general member*	*The fixed points (the most generic case)*	*The type of fibres with positive H-index*
R5-1	*EH-general*	*6 simple base points*	*(I-i)*
R5-2	*EH-general*	*6 simple base points*	*(I-ii-a)*
R5-3	*EH-special*	*3 base points with intersection multiplicity 2*	*(II-i-a), (II-i-b), (II-i-c)*

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We set the notation as follows:

Notation 1. $\sim$ means the linear equivalence of two divisors. For a non-negative integer d, denote by $\mu : \Sigma_d := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) \to \mathbb{P}^1$ the Hirzebruch surface. Let Δ_0 be the tautological divisor of Σ_d, and Δ_∞ the section of μ with $\Delta_\infty^2 = -d$ and $\Delta_0\Delta_\infty = 0$. For a linear system Ω of divisors for some variety, denote by Φ_Ω the rational map defined by Ω. For a global section s of an invertible sheaf, denote by (s) the divisor defined by $s = 0$. For a linear system Ω, denote by $\mathrm{Bs}\Omega$ the base locus of Ω.

2. Horikawa index

In this section, we review the H-index for nonhyperelliptic fibrations of genus 4 for both EH-general case and EH-special case. Let $f : S \to B$ be a nonhyperelliptic fibration of genus 4 over a smooth projective curve B. Assume that f is relatively minimal and that f is not isotrivial. Let $K_{S/B} = K_S - f^*K_B$ be the relative canonical divisor and $\omega_{S/B} := \mathcal{O}_S(K_{S/B})$ the relative dualizing sheaf. Then $E := f_*\omega_{S/B}$ is a locally free sheaf of rank 4 over B. If we put $\chi_f := \deg E$, then it is known that the inequalities $K_{S/B}^2 > 0$ and $\chi_f > 0$ both hold. Furthermore, the following inequalities are known:

Theorem 2. ([4], [11], [12]). *Under the above conditions, we have*

$$K_{S/B}^2 \geq \frac{24}{7}\chi_f.$$

Moreover, if we assume that a general fibre of f is EH-general, then

$$K_{S/B}^2 \geq \frac{7}{2}\chi_f$$

holds.

Remark 1. Similar results to Theorem 2 are obtained as follows:

(i) ([17], [11]) If a general fibre of f is a hyperelliptic curve of genus $g\,(\geq 2)$, then we have

$$K_{S/B}^2 \geq \frac{4(g-1)}{g}\chi_f.$$

(ii) ([9], [11], [13]) If a general fibre of f is a nonhyperelliptic curve of genus 3, then we have

$$K_{S/B}^2 \geq 3\chi_f.$$

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Let us return to the case of genus 4 fibrations. For each fibre $\mathcal{F}$ of *f*, denote by $\omega_{\mathcal{F}}$ the dualizing sheaf. The multiplication map

$$\mathrm{Sym}^2 H^0(\omega_{\mathcal{F}}) \to H^0(\omega_{\mathcal{F}}^{\otimes 2})$$

defines the multiplication map

$$\varphi : \mathrm{Sym}^2 E \to f_* \omega_{S/B}^{\otimes 2}.$$

By our assumption and Max–Noether's theorem, φ is generically surjective. We obtain the following exact sequence:

$$0 \to L \to \mathrm{Sym}^2 E \to f_* \omega_{S/B}^{\otimes 2} \to \mathcal{T} \to 0,$$

where L is a line bundle and $\mathcal{T}$ is a sheaf supported over finitely many points of B. Denote by $\pi : \mathfrak{W} := \mathbb{P}(E) \to B$ the $\mathbb{P}^3$-bundle defined by E and by T the tautological divisor of $\mathfrak{W}$. The natural morphism $f^*E \to \omega_{S/B}$ defines the rational map $\psi : S \cdots > \mathfrak{W}$ over B. ψ is called the *relative canonical map*, and the image $\psi(S) \subset \mathfrak{W}$ is called the *relative canonical image* of S by ψ.

Lemma 1. **([10]).** *Let the notation and conditions be as above. Then there exists an irreducible relative hyperquadric* $\mathfrak{Q} \in |2T - \pi^* L|$ *containing* $\psi(S)$.

Notation 2. Denote by $\pi_{\mathfrak{Q}}$ the restriction of π to $\mathfrak{Q}$.

2.1. H-index for EH-general case

Assume that f is the EH-general fibration. Let $q \in H^0(\mathcal{O}_W(2T - \pi^* L))$ be the global section defining $\mathfrak{Q}$. Since q can be considered as an element of $H^0(B, (\mathrm{Sym}^2 E) \otimes L^{-1})$, q defines the morphism $q : E^\vee \to E \otimes L^{-1}$. By considering the determinant map $\det q : \det E^\vee \to \det(E \otimes L^{-1})$, we can consider that $\det q$ is an element of $H^0(B, (\det E)^{\otimes 2} \otimes L^{-4})$. Since f is EH-general, we have $\det q \neq 0$, and hence, $\det q$ defines an effective divisor $\mathrm{Discr}(\mathfrak{Q})$ over B. A general fibre of $\pi_{\mathfrak{Q}} : \mathfrak{Q} \to B$ is of rank 4, while for any point $p \in \mathrm{suppDiscr}(\mathfrak{Q})$, the rank of the fibre $\pi_{\mathfrak{Q}}^{-1}(p)$ is less than 4. $\mathrm{Discr}(\mathfrak{Q})$ is called the *discriminant locus* of $\mathfrak{Q}$. The following is known:

Theorem 3. **([10]).** *For* $p \in B$, *denote by* $\mathrm{mult}_p \mathrm{Discr}(\mathfrak{Q})$ *the coefficient of* p *in* $\mathrm{Discr}(\mathfrak{Q})$ *and by* $\mathcal{T}_p$ *the restriction of* $\mathcal{T}$ *to* p. *If we put*

$$\mathrm{Ind}(f^{-1}(p)) := \frac{1}{4}\mathrm{mult}_p \mathrm{Discr}(\mathfrak{Q}) + \mathrm{length}\mathcal{T}_p, \tag{2.1}$$

then the following equality holds:

$$K_{S/B}^2 = \frac{7}{2}\chi_f + \sum_{p \in B} \mathrm{Ind}(f^{-1}(p)) \tag{2.2}$$

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Remark 2. The value (2.1) is called the *H-index* of the fibre $f^{-1}(p)$ in the case of EH-general fibration. Furthermore, the equality (2.2) is called *the slope equality* for the EH-general case.

2.2. H-index for EH-special case

In this subsection, we review the H-index for EH-special case. Although H-index is defined in [5] without any special condition, we use another definition in [14]. H-index in the latter case is defined under the assumption that the multiplication map $\mathrm{Sym}^2 E \to f_*\omega_{S/B}^{\otimes 2}$ is surjective, and as we will see in § 4, the fibrations we consider in this paper satisfy this assumption.

Since a general fibre of $\pi_{\mathfrak{Q}} : \mathfrak{Q} \to B$ is a quadric cone, we obtain the relative vertex $B_0 \subset \mathfrak{Q}$. Since B_0 is a section of π, we have the short exact sequence

$$0 \to E_0 \to E \to M \to 0$$

defining the embedding $B_0 \subset \mathfrak{W}$. E_0 is a locally free sheaf of rank 3 over B, and M is an invertible sheaf over B.

If $\rho : \widetilde{\mathfrak{W}} \to \mathfrak{W}$ is a blow-up along B_0, we obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathfrak{W}} & \xrightarrow{\ \rho\ } & \mathfrak{W} \\ {\scriptstyle \widetilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{P}(E_0) & \xrightarrow[\ \zeta\ ]{} & B. \end{array}$$

Put $\mathbb{E} := \rho^{-1}(B_0)$ and let $\widetilde{Q}$ be the proper transform of $\mathfrak{Q}$ by ρ and T_{E_0} the tautological divisor of $\mathbb{P}(E_0)$. Since $\rho^*T \sim \widetilde{\pi}^*T_{E_0} + \mathbb{E}$, we have

$$\widetilde{Q} \sim \rho^*\mathfrak{Q} - 2\mathbb{E} \sim \widetilde{\pi}^*(2T_{E_0} - \zeta^*L),$$

namely, there exists a conic bundle $Q_0 \in |2T_{E_0} - \zeta^*L|$ such that $\widetilde{Q} = \widetilde{\pi}^{-1}(Q_0)$. If $q_0 \in H^0(\mathcal{O}_{\mathbb{P}(E_0)}(2T_{E_0} - \zeta^*L))$ defines Q_0, we can define the discriminant locus $\mathrm{Discr}(Q_0)$ as in the previous subsection. For any $p \in \mathrm{suppDiscr}(Q_0)$, the fibre $\pi_{\mathfrak{Q}}^{-1}(p)$ is a hyperquadric of $\mathbb{P}^3$ with rank less than 3.

Note that the restriction of the relative canonical image S' to B_0 defines an effective divisor δ over B.

The following theorem is proved:

Theorem 4. ([14]). *Let the notation and the conditions be as above. For any $p \in B$, put*

$$\mathrm{Ind}(f^{-1}(p)) = \frac{2}{7}\mathrm{mult}_p\delta + \frac{3}{7}\mathrm{mult}_p\mathrm{Discr}(Q_0). \tag{2.3}$$

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Then we have the following equality:

$$K_{S/B}^2 = \frac{24}{7}\chi_f + \sum_{p\in B} \mathrm{Ind}(f^{-1}(p)). \tag{2.4}$$

Remark 3. The value (2.3) is called the *H-index* of the fibre $f^{-1}(p)$ in the case of EH-special fibration. Furthermore, the equality (2.4) is called *the slope equality* for the EH-special case.

3. Fibres with positive H-index

Let S be our K3 surface and $f : \widetilde{S} \to \mathbb{P}^1$ as in § 1. We have $K^2_{\widetilde{S}/\mathbb{P}^1} = 18$ and $\chi_f = 5$. Let us investigate the fibres of f with positive H-index. We use the same notation as in the previous section for the fibration $f : \widetilde{S} \to \mathbb{P}^1$.

3.1. EH-general case

If f is EH-general, then the sum of H-indices is $1/2$ by Equation (2.2). Hence, if $\mathrm{Ind}(f^{-1}(p)) > 0$ for $p \in \mathbb{P}^1$, we have $\mathcal{T}_p = 0$ and $\mathrm{mult}_p\mathrm{Discr}(\mathfrak{Q}) = 1$ or 2 by Equation (2.1). In either case, $\pi_{\mathfrak{Q}}^{-1}(p)$ is a hyperquadric with rank less than 4.

If $\mathrm{rk}(\pi_{\mathfrak{Q}}^{-1}(p)) = 3$ and if S is sufficiently general, then $f^{-1}(p)$ is an EH-special non-hyperelliptic curve of genus 4. If $\mathrm{rk}(\pi_{\mathfrak{Q}}^{-1}(p)) = 2$, then $\pi_{\mathfrak{Q}}^{-1}(p)$ is a sum of two distinct hyperplanes, and if S is sufficiently general, $f^{-1}(p)$ is the sum of two elliptic curves intersecting at three points transversally.

We have the following three cases:

(I-i) $\mathrm{Discr}(Q) = p_1 + p_2$ for some $p_1, p_2 \in \mathbb{P}^1$.
(I-ii-a) $\mathrm{Discr}(Q) = 2p_1$ for some $p_1 \in \mathbb{P}^1$ and $\mathrm{rk}(\pi_Q^{-1}(p_1)) = 3$.
(I-ii-b) $\mathrm{Discr}(Q) = 2p_1$ for some $p_1 \in \mathbb{P}^1$ and $\mathrm{rk}(\pi_Q^{-1}(p_1)) = 2$.

3.2. EH-special case

Assume f is EH-special. We obtain $K^2_{\widetilde{S}/B} - (24/7)\chi_f = 6/7$ from Equation (2.4). Hence, by [15, Theorem 1.5], the multiplication map is surjective. The sum of the H-indices is $6/7$. Hence, by considering Equation (2.3), we obtain the following six possibilities:

(II-i-a) $\delta = p_1 + p_2 + p_3\,(p_1, p_2, p_3 \in \mathbb{P}^1, p_i \neq p_j \Leftrightarrow i \neq j\,(i, j = 1, 2, 3))$, and $\mathrm{Discr}(Q_0) = 0$.
(II-i-b) $\delta = p_1 + 2p_2\,(p_1, p_2 \in \mathbb{P}^1,\ p_1 \neq p_2)$ and $\mathrm{Discr}(Q_0) = 0$.
(II-i-c) $\delta = 3p_1\,(p_1 \in \mathbb{P}^1)$ and $\mathrm{Discr}(Q_0) = 0$.
(II-ii-a) $\delta = 0$ and $\mathrm{Discr}(Q_0) = p_1 + p_2\,(p_1, p_2 \in \mathbb{P}^1,\ p_1 \neq p_2)$.
(II-ii-b) $\delta = 0$ and $\mathrm{Discr}(Q_0) = 2p_1\,(p_1 \in \mathbb{P}^1)$ with $\mathrm{rk}(\pi_{\mathfrak{Q}}^{-1}(p)) = 2$.
(II-ii-c) $\delta = 0$ and $\mathrm{Discr}(Q_0) = 2p_1\,(p_1 \in \mathbb{P}^1)$ with $\mathrm{rk}(\pi_{\mathfrak{Q}}^{-1}(p)) = 1$.

We investigate the details for each case. Note that the direct image $E := f_*\omega_{\widetilde{S}/\mathbb{P}^1}$ satisfies

$$E \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3},$$

from the results of [7]. See also [2].

3.2.1. *The case where the discriminant locus is 0*

First, consider the cases (II-i-a), (II-i-b) and (II-i-c). If E_0 and M are as in the previous section, we obtain $E_0 \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}(1)$, $M \cong \mathcal{O}_{\mathbb{P}^1}(2)$ and $L \cong \mathcal{O}_{\mathbb{P}^1}(2)$ by considering the argument of [14, § 1]. It is easily proved that $Q_0 \cong \Sigma_0$ and $T_{E_0}|_{Q_0} \sim 2\Delta_0 + \Gamma$, where Γ is the restriction of a fibre of $\mathbb{P}(E_0) \to \mathbb{P}^1$ to Q_0, and Δ_0 is a fibre of the natural projection, which is different from the one whose fibre is Γ. (See Notation 1 also for Δ_0.) Since $\widetilde{W} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \oplus \mathcal{O}_{\mathbb{P}(E_0)}(2F'))$, where F' is a fibre of ζ, we have

$$\widetilde{Q} \cong \mathbb{P}(\mathcal{O}_{\Sigma_0}(2\Delta_0 + \Gamma) \oplus \mathcal{O}_{\Sigma_0}(2\Gamma)),$$

and hence,

$$\widetilde{Q} \cong \mathbb{P}(\mathcal{O}_{\Sigma_0}(2\Delta_0 - \Gamma) \oplus \mathcal{O}_{\Sigma_0}). \tag{3.1}$$

If $T_{\widetilde{Q}}$ is the tautological divisor of $\widetilde{Q}$ under the consideration of Equation (3.1), and if $\widetilde{F}$ is a fibre of $\widetilde{Q} \to \mathbb{P}^1$, we have

$$S_1 \sim 3T_{\widetilde{Q}} + 3\widetilde{F},$$

where S_1 is the preimage of S in $\widetilde{Q}$. Put $\mathbb{E}_0 := \mathbb{E}|_{\widetilde{Q}}$. Then $\mathbb{E}_0 \cong Q_0$, and furthermore, the restriction $S_1|_{\mathbb{E}_0}$ consists of fibres of $(\zeta \circ \widetilde{\pi})|_{\mathbb{E}_0}$, and the image of the sum of these fibres by $\widetilde{Q} \to \mathbb{P}^1$ is δ. Namely, for a point $p \in \operatorname{supp}\delta$, the fibre of $f : \widetilde{S} \to \mathbb{P}^1$ over p is the one with positive H-index, and its value is one of $2/7$, $4/7$ and $6/7$. If we consider the fibre of $\widetilde{Q} \to \mathbb{P}^1$ over p as Σ_2, then the restriction of $\mathbb{E}_0$ to Σ_2 is Δ_∞, and hence, the restriction $S_1|_{\Sigma_2}$ is of the form of $\Delta_\infty + C$ with $C \sim 2\Delta_0 + 2\Gamma$. Hence, if $\mathcal{P}$ is generic in $\mathrm{Gr}(3, 1)$, this fibre consists of a rational curve and a hyperelliptic curve of genus three intersecting at two points.

3.2.2. *The case where the discriminant locus is not 0*

Next, consider the case (II-ii-a), (II-ii-b) and (II-ii-c). We have $E_0 = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$, $M = \mathcal{O}_{\mathbb{P}^1}(1)$ and $L = \mathcal{O}_{\mathbb{P}^1}(2)$. We use the notation of the previous section under these consideration. Then we have

$$\widetilde{W} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(E_0)}(T_{E_0}) \oplus \mathcal{O}_{\mathbb{P}(E_0)}(F')), \tag{3.2}$$

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where F' is a fibre. Let us consider the structure of Q_0 ($\subset \mathbb{P}(E_0)$). It is easily proved that Q_0 is disjoint to the section $\widetilde{B} := \mathbb{P}(E_0/\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, and if $\widetilde{\mathbb{P}(E_0)} \to \mathbb{P}(E_0)$ is the blow-up along $\widetilde{B}$, we obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathbb{P}(E_0)} & \longrightarrow & \mathbb{P}(E_0) \\ \downarrow & & \downarrow \\ \Sigma_1 & \longrightarrow & \mathbb{P}^1. \end{array}$$

We may consider $Q_0 \subset \widetilde{\mathbb{P}(E_0)}$. We have $\widetilde{\mathbb{P}(E_0)} \cong \mathbb{P}(\mathcal{O}_{\Sigma_1}(\Delta_0 + \Gamma) \oplus \mathcal{O}_{\Sigma_1}(\Gamma))$, and it is easily proved that Q_0 is a double cover of Σ_1 branched along a divisor $\mathcal{B}$ that is linearly equivalent to $2\Delta_0$. Note that the inverse image of Δ_∞ by the double cover is the sum of two (-1) curves unless Δ_∞ is contained in $\widetilde{B}$. Furthermore, we have $K^2_{Q_0} = 6$.

3.2.2.1 If $\mathcal{B}$ is smooth, then Q_0 is isomorphic to the blown-up surface at two points q_1 and q_2 of Σ_1 contained in some section $\Delta_0' \in |\Delta_0|$. (Namely, Q_0 has the structure of the branched double cover over Σ_1 and the structure of the blown-up surface of Σ_1.) By the adjunction formula, we have $K_{Q_0} \sim -T_{E_0}|_{Q_0}$. Hence, by considering Equation (3.2), if we denote by $\nu : Q_0 \to \Sigma_1$ the above blow-up, and if we put $\mathbb{E}_i = \nu^{-1}(q_i)$ $(i = 1, 2)$, we have

$$\widetilde{Q} \cong \mathbb{P}(\mathcal{O}_{Q_0}(\nu^*(2\Delta_0) - \mathbb{E}_1 - \mathbb{E}_2) \oplus \mathcal{O}_{Q_0}). \tag{3.3}$$

If $p_i \in \mathbb{P}^1$ is the image of q_i $(i = 1, 2)$, then we have $\mathrm{Discr}(Q_0) = p_1 + p_2$, and $\widetilde{S}$ has singular fibres of type (II-ii-a). The fibre of $\mathfrak{Q} \to \mathbb{P}^1$ over p_i is of rank 2, namely the fibre is a sum of two distinct hyperplanes, and hence, $f^{-1}(p_i)$ is a sum of two elliptic curves intersecting at three points transversally if $\mathcal{P}$ is sufficiently general in the Grasmannian $\mathrm{Gr}(3, 1)$.

3.2.2.2 We consider the case where $\mathcal{B}$ ($\subset \Sigma_1$) is written as $\mathcal{B} = \Delta_0 + \Delta_0'$ for $\Delta_0, \Delta_0' \in |\Delta_0|$ with $\Delta_0 \neq \Delta_0'$. Let q be the intersection point of Δ_0 and Δ_0'. Then Q_0 has a rational double point of type A_1 over q. Hence, $\widetilde{Q}$ has a singular locus along the fibre over the singularity of Q_0. Here, we consider the normalization of $\widetilde{Q}$.

Let $\widetilde{Q}_0$ be the minimal resolution of Q_0. If $p_1 \in \mathbb{P}^1$ is the image of q, then the fibre of $\widetilde{Q}_0 \to \mathbb{P}^1$ over p_1 consists of three rational curves ℓ_1, ℓ_2 and ℓ_3. We may assume $\ell_1^2 = \ell_3^2 = -1$, $\ell_2^2 = -2$, $\ell_1\ell_3 = 0$ and $\ell_1\ell_2 = \ell_3\ell_2 = 1$. Moreover, if we denote by $\widetilde{\Delta}_\infty + \widetilde{\Delta}'_\infty$ the inverse image of Δ_∞ by the double cover $\widetilde{Q}_0 \to \Sigma_1$ with $\widetilde{\Delta}_\infty\widetilde{\Delta}'_\infty = 0$, we may assume $\ell_1\widetilde{\Delta}_\infty = 1$ and $\ell_3\widetilde{\Delta}'_\infty = 1$. Namely, $\widetilde{Q}_0$ is obtained as follows:

For some point $q_1 \in \Sigma_1 \setminus \Delta_\infty$, let Γ_1 be a fibre containing $q_1, \nu_1 : Q_1 \to \Sigma_1$ a blow-up at q_1 and $\widetilde{\Gamma}_1$ the proper transform of Γ_1. Put $\mathbb{E}_1 := \nu_1^{-1}(q_1)$ and let q_2 be a point of $\mathbb{E}_1 \setminus \widetilde{\Gamma}_1$. Then by blowing-up at q_2, we obtain $\widetilde{Q}_0$. Let $\nu_2 : \widetilde{Q}_0 \to Q_1$ be the blow-up and put $\mathbb{E}_2 := \nu_2^{-1}(q_2)$. Put $\widehat{Q} := \widetilde{Q} \times_{Q_0} \widetilde{Q}_0$. Then by the same argument as 3.2.2.1, we obtain

$$\widehat{Q} \cong \mathbb{P}(\mathcal{O}_{\widetilde{Q}_0}(\nu_2^*(\nu_1^*(2\Delta_0) - \mathbb{E}_1) - \mathbb{E}_2) \oplus \mathcal{O}_{\widetilde{Q}_0}). \tag{3.4}$$

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Note that if $T_{\widehat{Q}}$ is the tautological divisor of $\widehat{Q}$, then we have the preimage of $\widetilde{S}$ in $\widehat{Q}$ is linearly equivalent to $3T_{\widehat{Q}}$ since $\delta = 0$.

We have $\mathrm{Discr}(Q_0) = 2p_1$. Moreover, the rank of the fibre of $Q_0 \to \mathbb{P}^1$ over p_1 is 2, and hence, the fibre of $\mathfrak{Q} \to \mathbb{P}^1$ over p_1 is a sum of two hyperplanes. Namely, the fibre of $f : \widetilde{S} \to \mathbb{P}^1$ over p_1 is of type (II-ii-b).

Let $\Pi : \widehat{Q} \to \widetilde{Q}_0$ be the natural morphism ($\mathbb{P}^1$-bundle) and put $\mathbb{F}_i = \Pi^{-1}(\ell_i)$ $(i = 1, 2, 3)$. Then we have $\mathbb{F}_1 \cong \mathbb{F}_3 \cong \Sigma_1$ and $\mathbb{F}_2 \cong \Sigma_0$. Furthermore, if S_1 is the preimage of $\widetilde{S}$ in $\widehat{Q}$, and if $\mathcal{P}$ is sufficiently general, then the restrictions $S_1|_{\mathbb{F}_1}$ and $S_1|_{\mathbb{F}_3}$ are both elliptic curves and $S_1|_{\mathbb{F}_2}$ is a sum of disjoint three rational curves that are (-2)-curves as the curves of S_1. Namely, the singular fibre of f over p_1 is of the form $C_1 + C_2 + \sum_{i=1}^{3} E_i$, where C_j $(j = 1, 2)$ is an elliptic curve with $C_1 \cap C_2 = \emptyset$, and E_i $(i = 1, 2, 3)$ is a (-2)-curve with $E_i \cap E_{i'} = \emptyset \Leftrightarrow i \neq i'$ and $C_j E_i = 1$ as curves of $\widetilde{S}$.

3.2.2.3 Assume $\widetilde{B}$ is of the form $\widetilde{B} = \Delta_0 + \Delta_\infty + \Gamma$ for some $\Delta_0 \in |\Delta_0|$ and some fibre Γ. Let q_1' and q_2' be the points such that $\Delta_0 \cap \Gamma = \{q_1'\}$ and $\Delta_\infty \cap \Gamma = \{q_2'\}$. Furthermore, let $\gamma : \widetilde{\Sigma}_1 \to \Sigma_1$ be the blow-up at q_1' and q_2', and $\widetilde{\Delta}_0, \widetilde{\Delta}_\infty$ and $\widetilde{\Gamma}$ the proper transforms of Δ_0, Δ_∞ and Γ, respectively. Put $\mathcal{E}_i = \gamma^{-1}(q_i')$ $(i = 1, 2)$. If $\widetilde{Q}_0$ is a normalization of $\widetilde{\Sigma}_1 \times_{\Sigma_1} Q_0$ and if $p_1 \in \mathbb{P}^1$ is the image of q_i' by $Q_0 \to \mathbb{P}^1$, then the fibre of $\widetilde{Q}_0 \to \mathbb{P}^1$ over p_1 consists of three rational components ℓ_1, ℓ_2 and ℓ_3, where we may assume that ℓ_1 dominates $\mathcal{E}_1$, ℓ_2 dominates $\mathcal{E}_2$ and ℓ_3 dominates $\widetilde{\Gamma}$. ℓ_1 and ℓ_2 are (-2)-curves and ℓ_3 is a (-1)-curve. The curve that dominates $\widetilde{\Delta}_\infty$ is a (-1)-curve, and the self-intersection number of the curve dominating $\widetilde{\Delta}_0$ is 0. Hence, $\widehat{Q}_0$ is obtained as follows: For a point $q_1 \in \Sigma_1 \setminus \Delta_\infty$, let $\nu_1 : Q_1 \to \Sigma_1$ be the blow-up at q_1 and $\widehat{\Gamma}$ the proper transform of the fibre containing q_1. Put $\mathbb{E}_1 := \nu_1^{-1}(q_1)$, and let q_2 be the intersection point of $\widehat{\Gamma}$ and $\mathbb{E}_1$. By blowing-up at q_2, we obtain $\widehat{Q}_0$. Let $\nu_2 : \widehat{Q}_0 \to Q_1$ be the blow-up, and put $\mathbb{E}_2 := \nu_2^{-1}(q_2)$. Then by the same argument as in 3.2.2.1, if we put $\widehat{Q} := \widetilde{Q} \times_{Q_0} \widehat{Q}_0$, we obtain

$$\widehat{Q} \cong \mathbb{P}(\mathcal{O}_{\widetilde{Q}_0}(\nu_2^*(\nu_1^*(2\Delta_0) - \mathbb{E}_1) - \mathbb{E}_2) \oplus \mathcal{O}_{\widetilde{Q}_0}). \tag{3.5}$$

Note that $\widehat{Q}_0$ is the minimal resolution of $Q_0\,(\subset \mathbb{P}(E_0))$ that has two rational double points of type A_1. If S_1 is the preimage of $\widetilde{S}$ in $\widehat{Q}$, and if $\mathcal{P}$ is sufficiently general, then the fibre of $S_1 \to \mathbb{P}^1$ over p_1 can be written as $2C + \sum_{i=1}^{6} E_i$, where C is an elliptic curve and E_i is (-2)-curve such that $CE_i = 1$ and $E_i E_j = 0$ $(i \neq j)$.

We have $\mathrm{Discr}(Q_0) = 2p_1$. Moreover, the rank of the fibre of $Q_0 \to \mathbb{P}^1$ is 1, and hence, the fibre of $\mathfrak{Q} \to \mathbb{P}^1$ over p_1 is a double of a hyperplane. Namely, the fibre of $f : \widetilde{S} \to \mathbb{P}^1$ over p_1 is of type (II-ii-c).

Remark 4. In either case of 3.2.2.1, 3.2.2.2 and 3.2.2.3, the singular fibre with positive H-index has the components of elliptic curves. Let C be one of the elliptic curves. By Zariski's lemma (cf. e.g., [3, (8.2) Lemma]), $C\mathcal{F} = 0$ for a fibre of f, which leads us to $C^2 = -3$. By the adjunction formula, we have $K_{\widetilde{S}} C = 3$. On the other hand, since S is a K3 surface, $K_{\widetilde{S}}$ consists of six exceptional curves of the blow-up $\widetilde{S} \to S$. Namely, C intersects with 3 of them.

4. Classification of subpencils

Let S' be the surface obtained as the complete intersection of a hyperquadric Q and a hypercubic Y. Assume S' satisfies the Assumption 1. Let S be a desingularization of S'.

For Q, the following is known (see [6], [8]):

(4-1) The case where $\mathrm{rk}(Q) = 3$.

Put $E_3 := \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, and let $\pi_3 : W_3 := \mathbb{P}(E_3) \to \mathbb{P}^1$ be the $\mathbb{P}^2$-bundle, T_3 the tautological divisor of W_3 and F a fibre of π. For the rational map $\Phi_{|T_3|} : W_3 \to \mathbb{P}^4$, we have $Q \cong \Phi_{|T_3|}(W_3)$. Let $T_{3,0} \subset W_3$ be the relative hyperplane with $T_{3,0} \sim T_3 - 2F$. Then we have $T_{3,0} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $\beta_i : T_{3,0} \to \mathbb{P}^1$ $(i = 1, 2)$ be the natural projection. We may assume $\beta_1 = \pi_3|_{T_{3,0}}$. If we put $Z := \Phi_{|T_3|}(T_{3,0})$, then Z is a line in $\mathbb{P}^4$, and we have $\Phi_{|T_3|}|_{T_{3,0}} = \beta_2$. Z is a compound rational double point of type A_1 of Q.

(4-2) The case where $\mathrm{rk}(Q) = 4$.

Put $E_4 := \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}$, and let $\pi_4 : W_4 := \mathbb{P}(E_4) \to \mathbb{P}^1$ be the $\mathbb{P}^2$-bundle, T_4 the tautological divisor of W_4 and F a fibre of π_4. We have $Q \cong \Phi_{|T_4|}(W_4)$. For a section $B_0 = \mathbb{P}(E_4/\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$ of π_4, we have that $\Phi_{|T_4|}(B_0)$ is a point. If we put $q_0 := \Phi_{|T_4|}(B_0)$, then Q has a three-dimensional rational double point of type A_1 at q_0.

(4-3) The case where $\mathrm{rk}(Q) = 5$.

For a point $q \in \mathbb{P}^4 \setminus Q$, let $\nu : \widetilde{P} \to \mathbb{P}^4$ be the blow-up at q. Then we have $\widetilde{P} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3})$. Let $\Pi : \widetilde{P} \to \mathbb{P}^3$ be the $\mathbb{P}^1$-bundle, $\widetilde{H}$ the tautological divisor of $\widetilde{P}, H_0 \subset \mathbb{P}^3$ a hyperplane and put $\mathbb{E} := \nu^{-1}(q)$. We have $\widetilde{H} \sim \mathbb{E} + \Pi^* H_0$. If we consider $Q \subset \widetilde{P}$, we have $Q \sim 2\widetilde{H}$. If we put $\Pi_Q := \Pi|_Q$, then the morphism $\Pi_Q : Q \to \mathbb{P}^3$ is a double cover branched along some smooth hyperquadric.

Π_Q is also considered as follows: For a hyperplane section $\widetilde{H}_Q$ of Q, we have $\dim |\widetilde{H}_Q| = 4$. There exists a base point free three-dimensional subspace $V \subset |\widetilde{H}_Q|$ such that $\Pi_Q = \Phi_V$. For a hyperplane $H_0 \subset \mathbb{P}^4$, $\Pi_Q^* H_0$ is a hyperplane section of Q. However, all the hyperplane sections cannot be written as above. On the other hand, we have the following:

Lemma 2. *For any hyperplane section $\widetilde{H}_Q \in |\widetilde{H}_Q|$, there exist a base point free three-dimensional subspace $V \subset |\widetilde{H}_Q|$ and a hyperplane $H_0 \subset \mathbb{P}^3$ such that $\widetilde{H}_Q = \Phi_V^* H_0$.*

Proof. If we take V as $\widetilde{H}_Q \in V$ and if we put H_0 the image of $\widetilde{H}_Q$, then we obtain the desired equality. □

4.1. The case rk(Q) = 3 (The proof for (1) of Theorem 1)

Let $Q \subset \mathbb{P}^4$ be a hyperquadric of rank 3. Let the notation be as in (4-1). If $Y \subset \mathbb{P}^4$ is a general hypercubic, then we may assume that Y is smooth and that the intersection $S' := Q \cap Y$ has rational double points over Z and no other singularity. Let $S_1 \subset W_3$ be the preimage of S_0 by $\Phi_{|T_3|}$. We have $S_1 \sim 3T_3$, and there exists a divisor δ_0 of degree

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3 over $\mathbb{P}^1$ such that $S_1|_{T_{3,0}} \sim \beta_2^*\delta_0$. If Y is sufficiently general, then δ_0 is reduced and S_1 is nonsingular, namely, if S is the minimal resolution of S', then we have $S_1 = S$. In order to classify subpencils of the complete linear system of the hyperplane section, it is sufficient to classify those of S_1. The hyperplane section of S_1 is written as $T_3|_{S_1}$. Therefore, in order to achieve our goal, it is sufficient to classify the subpencil of $|T_3|$.

Let X_0, $X_1 \in H^0(\mathcal{O}_{W_3}(T_3))$ and $X_2 \in H^0(\mathcal{O}_{W_3}(T_3 - 2F))$ be global sections defining the homogeneous coordinates of each fibre of π_3. Any member $\Psi \in H^0(\mathcal{O}_{W_3}(T_3))$ can be written as

$$\Psi = c_0X_0 + c_1X_1 + \psi_2X_2, \quad (c_0,\ c_1 \in \mathbb{C},\ \ \psi_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(2))). \tag{4.1}$$

The divisor (Ψ) is irreducible if and only if $(c_0, c_1) \neq (0, 0)$, and we obtain the following:

Lemma 3. *There exist three types of subpencil $\widehat{\mathcal{P}}$ of $|T_3|$ as follows:*

(3-i) Any member of $\widehat{\mathcal{P}}$ is irreducible.
(3-ii) $\widehat{\mathcal{P}}$ has only one reducible member.
(3-iii) Any member of $\widehat{\mathcal{P}}$ is reducible.

There exists no other type of subpencils.

Proof. Let T_3, $T_3' \in |T_3|$ be two distinct members. Assume that the global sections defining T_3 and T_3' are written as $c_0X_0 + c_1X_1 + \psi_2X_2$ and $c_0'X_0 + c_2'X_1 + \psi_2'X_2$, respectively. Let $\widehat{\mathcal{P}} \subset |T_3|$ be the subpencil generated by T_3 and T_3'. If $c_0c_2' \neq c_1c_0'$, then any member of $\widehat{\mathcal{P}}$ is irreducible. If $c_0c_1' = c_1c_0'$ and that at least one of (c_0, c_1) and (c_0', c_1') is not equal to $(0, 0)$, then reducible members of $\widehat{\mathcal{P}}$ is only the one defined by the global section of the form $\psi_2''X_2$. If $(c_0, c_1) = (c_0', c_1') = (0, 0)$, then any member of $\widehat{\mathcal{P}}$ is reducible. □

If T_3 is irreducible, then we have $T_3 \cong \Sigma_2$ as a variety. If we put $\Delta_0 := T_3|_{T_3}$, then we have $\Delta_0^2 = 2$. If Γ is a fibre of the ruling $T_3 \to \mathbb{P}^1$, and if Δ_∞ is the section with $\Delta_\infty \sim \Delta_0 - 2\Gamma$, then we have $T_{3,0}|_{T_3} \sim \Delta_\infty$.

Assume that T_3, $T_3' \in |T_3|$ are irreducible and that the restriction $T_3'|_{T_3}$ is written as $\Delta_\infty + \Gamma_1 + \Gamma_2$. In this case, we have $T_3|_{T_{3,0}} = T_3'|_{T_{3,0}}$, and this is the case (3-ii) of Lemma 3. Hence, in the case of (3-i), the restriction $T_3'|_{T_3}$ is irreducible.

Since $S_1|_{T_3} = Y|_{T_3} \sim 3\Delta_0$, a general member of $\mathcal{P}$ is EH-special nonhyperelliptic curve of genus 4 in the cases where $\widehat{\mathcal{P}}$ is of type(3-i) or (3-ii).

4.1.1. Type (3-i)

Let $\widehat{\mathcal{P}}$ be a subpencil of type (3-i). Then the base locus $\mathrm{Bs}\widehat{\mathcal{P}}$ is an irreducible rational curve. For the pencil $\mathcal{P}$ over S corresponding to $\widehat{\mathcal{P}}$, the base locus $\mathrm{Bs}\mathcal{P}$ consists of six points scheme theoretically. If $\mathcal{P}$ is generic in $\mathrm{Gr}(3, 1)$, then $\mathrm{Bs}\mathcal{P}$ consists of six points set-theoretically also, and distinct two members of $\mathcal{P}$ intersect at these points transversally.

If $\eta_3 : \widetilde{W}_3 \to W_3$ is a blow-up along $\mathrm{Bs}\widehat{\mathcal{P}}$, we obtain the following commutative diagram:

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$$\begin{array}{ccc}
\widetilde{W}_3 & \xrightarrow{\eta_3} & W_3 \\
{\scriptstyle \xi_3}\big\downarrow & & \big\downarrow{\scriptstyle \pi_3} \\
T_{3,0} \cong \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow[\beta_1]{} & \mathbb{P}^1 \\
{\scriptstyle \beta_2}\big\downarrow & & \\
\mathbb{P}^1. & &
\end{array}$$

If we denote by Δ a fibre of β_1 and by Γ a fibre of β_2, then we have

$$\widetilde{W}_3 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\Gamma) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2\Delta)).$$

If S_2 is the proper transform of S_1 by η_3, then $\beta_1 \circ \xi_3 : S_2 \to \mathbb{P}^1$ is an elliptic fibration, and $\beta_2 \circ \xi_3 : S_2 \to \mathbb{P}^1$ is an EH-special nonhyperelliptic fibration of genus 4. If $\mathbb{E}$ is an exceptional divisor of η_3, then we have $\mathbb{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$. We may consider that $T_{3,0}$ is contained in $\widetilde{W}_3$. The morphism $\Phi_{|2\eta_3^* T_3 - \mathbb{E}|}$ maps $T_{3,0}$ onto a rational curve by contracting each fibre of β_2 onto a point. We obtain the following commutative diagram:

$$\begin{array}{ccc}
\widetilde{W}_3 & \xrightarrow{\eta_3} & W_3 \\
\big\downarrow & & \big\downarrow{\scriptstyle \Phi_{|T_3|}} \\
Q' & \longrightarrow & Q,
\end{array}$$

where $Q' \to Q$ is the blow-up along the image of $\mathrm{Bs}\widehat{\mathcal{P}}$ by $\Phi_{|T_3|}$. Q' has a structure of a quadric cone bundle $\widetilde{\xi}_3 : Q' \to \mathbb{P}^1$. Furthermore, if we consider $\mathbb{E} \cong T_{3,0}$ ($\cong \mathbb{P}^1 \times \mathbb{P}^1$), the restriction of a fibre of $\widetilde{\xi}_3$ to $\mathbb{E}$ is a fibre of β_2. We have $\widetilde{W}_3 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2\Delta - \Gamma))$, which coincides with Equation (3.1). Namely, the morphism $\widetilde{W}_3 \to Q'$ coincides with the morphism $\widetilde{Q} \to \mathfrak{Q}$ in § 2.2. If $\widetilde{T}_3$ is the tautological divisor of $\widetilde{W}_3$ under this consideration, we obtain $S_2 \sim 3\widetilde{T}_3 + 3\xi_3^* \Gamma$. Therefore, the image S_3 of S_2 in Q' intersects with the relative vertex of Q' at three points scheme theoretically. If $\widetilde{S} \to S_3$ is the minimal resolution, and if $f : \widetilde{S} \to \mathbb{P}^1$ is the naturally obtained EH-special nonhyperelliptic fibration of genus 4, then f has singular fibres of type (II-i-a), (II-i-b) or (II-i-c), and no other fibre with positive H-index. $S' \subset Q$ is the image of S_3 by $Q' \to Q$, and it is proved that the complete linear system of the hyperplane section of the minimal resolution S of S' has the subpencil of type R3-1 of Theorem 1.

4.1.2. Type (3-ii)

Let us consider the case (3-ii). We use the same notation as in the proof of Lemma 3.

Since $c_0 c_1' = c_1 c_0'$, we can change one of the basis of the two-dimensional subspace of $H^0(\mathcal{O}_{W_3}(T_3))$ defining $\mathcal{P}$ to one of the form $\widetilde{\psi}_2 X_2$ ($\widetilde{\psi}_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(2))$). Assume that another basis s is written as $s = c_2 X_0 + c_1 X_1 + \psi_2 X_2$ ($c_0, c_1 \in \mathbb{C},\ \psi_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(2))$). Let $q_1, q_2 \in \mathbb{P}^1$ be points with $(\widetilde{\psi}_2) = q_1 + q_2$. Put $\widehat{T}_3 := (s)$ and $B_{3,0} = \widehat{T}_3 \cap T_{3,0}$.

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If we further put $F_i := \pi_3^{-1}(q_i)$ and $\ell_i := F_i \cap \widehat{T}_3$ $(i = 1, 2)$, we have

$$\mathrm{Bs}\widehat{\mathcal{P}} = B_{3,0} \cup \ell_1 \cup \ell_2.$$

We have $\widehat{T}_3 \cong \Sigma_2$, and if we consider that $B_{3,0}$ is a divisor of $\widehat{T}_3$, then $B_{3,0} = \Delta_\infty$ holds. On the other hand, if we denote by $\widehat{Y}$ the pull back of the hypercubic Y $(\subset \mathbb{P}^3)$ defining our surface S', then we have $\widehat{Y}|_{\widehat{T}_3} \sim 3\Delta_0$. Hence, if we denote by S_1 the pull back of S' $(= Q \cap Y)$ to W_3, one of $B_{3,0} \subset S_1$ and $B_{3,0} \cap S_1 = \emptyset$ holds. The former case is excluded because $\mathcal{P}$ does not have a fixed component.

From now on, we consider the case (3-ii) by dividing it into two cases $q_1 \neq q_2$ and $q_1 = q_2$.

(3-ii-a) The case $q_1 \neq q_2$

Assume $q_1 \neq q_2$. We transform W_3 birationally as follows:

Step 1 Note that $B_{3,0}$ can be written as $B_{3,0} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}/\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$. Let $\rho_1 : \widetilde{W}_3 \to W_3$ be the blow-up along $B_{3,0}$. We have $\widetilde{W}_3 \cong \mathbb{P}(\mathcal{O}_{\Sigma_2}(\Delta_0) \oplus \mathcal{O}_{\Sigma_2})$ and the following commutative diagram:

$$\begin{array}{ccc} \widetilde{W}_3 & \xrightarrow{\rho_2} & W_3 \\ \big\downarrow & & \big\downarrow \\ \Sigma_2 & \longrightarrow & \mathbb{P}^1. \end{array}$$

Put $\zeta : \widetilde{W}_3 \to \Sigma_2$. Then ζ is a $\mathbb{P}^1$-bundle. If we let $\widetilde{T}_3$ be the proper transform of $\widehat{T}_3$ by ρ_1, then $\widetilde{T}_3$ can be written as $\widetilde{T}_3 = \zeta^*\Delta_0'$ for some $\Delta_0' \in |\Delta_0|$. Namely, there exists a subpencile $\mathcal{P}_0 \subset |\Delta_0|$ such that the proper transform of any member of $\mathcal{P}$ is the pull back of some member of $\mathcal{P}_0$. Note that there exist two base points of $\mathcal{P}_0$, say $\widetilde{q}_1$ and $\widetilde{q}_2$. For the ruling $\mu : \Sigma_2 \to \mathbb{P}^1$, we may assume $\mu(\widetilde{q}_i) = q_i$ $(i = 1, 2)$. If we consider $\ell_i \subset \widetilde{W}_3$, then $\ell_i = \zeta^{-1}(\widetilde{q}_i)$ holds. Furthermore, if we put $\Gamma_i = \mu^{-1}(q_i)$, then we have $\Delta_\infty + \Gamma_1 + \Gamma_2 \in \mathcal{P}_0$ and its pull back by ζ is the divisor defined by $\widetilde{\psi}_2 X_2$.

Step 2 Let $\rho_2 : \widehat{W}_3 \to \widetilde{W}_3$ be the blow-up along $\ell_1 \cup \ell_2$. If we let $\widetilde{\rho}_2 : \widetilde{\Sigma} \to \Sigma_2$ be the blow-up at $\widetilde{q}_1$ and $\widetilde{q}_2$, then we obtain the following commutative diagram:

$$\begin{array}{ccc} \widehat{W}_3 & \xrightarrow{\rho_2} & \widetilde{W}_3 \\ {\scriptstyle\widetilde{\zeta}}\big\downarrow & & \big\downarrow{\scriptstyle\zeta} \\ \widetilde{\Sigma} & \xrightarrow[\widetilde{\rho}_2]{} & \Sigma_2. \end{array}$$

Furthermore, we obtain the morphism $\widetilde{\mu} : \widetilde{\Sigma} \to \mathbb{P}^1$ whose fibre is a proper transform of a member of $\mathcal{P}_0$ by $\widetilde{\rho}_2$. Put $\mathcal{E}_i := \widetilde{\rho}_2^{-1}(\widetilde{q}_i)$ $(i = 1, 2)$, and denote by $\widetilde{\Gamma}_i$ the proper transform of Γ_i by $\widetilde{\rho}_2$ $(i = 1, 2)$. Let $\gamma_1 : \widetilde{\Sigma} \to \Sigma'$ be the blow-down of Γ_1, and $\gamma_2 : \Sigma' \to \Sigma_1$ the

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blow-down of the image Δ'_∞ of Δ_∞ by γ_1. If Δ'_0 is the section of $\Sigma_1 \to \mathbb{P}^1$ with ${\Delta'}_0^2 = 1$, we obtain

$$\widetilde{\rho}_2^* \Delta_0 \sim \gamma_1^*(\gamma_2^*(2\Delta'_0) - \Delta'_\infty) - \Gamma_1,$$

and hence, $\widehat{W}_3$ coincides with Equation (3.4). The only singular fibre of the morphism $\widetilde{\mu} \circ \widetilde{\zeta}$ is the inverse image of $\Delta_\infty + \Gamma_1 + \Gamma_2$ by ζ, where $\widetilde{\zeta}$ is as in the above commutative diagram. The preimage $S_3 \subset \widetilde{\widetilde{W}}_3$ of $S' \subset \mathfrak{Q}$ has a singular fibre with H-index 6/7, which is the case (II-ii-b). Then it has been proved that Λ has a subpencil of type R3-2 of Theorem 1.

(3-ii-b) The case $q_1 = q_2$

Assume $q_1 = q_2$. Let $\rho_1 : \widetilde{W}_3 \to W_3$ be as in (3-ii-a). Furthermore, let $\rho_2 : \widehat{W}_3 \to \widetilde{W}_3$ be the blow-up along ℓ_1 $(= \ell_2)$. If $\widetilde{T}_3$ is as in Step 1 of (3-ii-a), then the base locus of $|\widetilde{T}_3|$ is a rational curve $\widetilde{\ell}$, and any two general member of $|\widetilde{T}_3|$ intersect along $\widetilde{\ell}$ transversally. Let $\rho_3 : \overline{W}_3 \to \widehat{W}_3$ be a blow-up along $\widetilde{\ell}$.

If $\widetilde{q}_1 \in \Sigma_2$ is the point with $\zeta^{-1}(\widetilde{q}_1) = \ell_1$, then the subpencil $\mathcal{P}_0$ of $|\Delta_0|$ corresponding to $\widehat{\mathcal{P}}$ satisfies $\mathrm{Bs}\mathcal{P}_0 = \{\widetilde{q}_1\}$. Any two general member of $\mathcal{P}_0$ contact at $\widetilde{q}_1$ with intersection multiplicity 2. Let $\widetilde{\rho}_2 : \widehat{\Sigma} \to \Sigma_2$ be a blow-up at $\widetilde{q}_1$ and $\widetilde{\mathcal{P}}_0$ the pencil consisting with the proper transforms of the members of $\mathcal{P}_0$. Then we have $\mathrm{Bs}\widetilde{\mathcal{P}}_0 = \{q'_1\}$ for some point q'_1 on the exceptional curve of $\widetilde{\rho}_2$. Any two general members of $\widetilde{\mathcal{P}}_0$ intersect at q'_1 transversally. If $\widetilde{\rho}_3 : \overline{\Sigma} \to \widehat{\Sigma}$ is a blow-up at q'_1, then we obtain the following commutative diagram:

$$\begin{array}{ccccc} \overline{W}_3 & \xrightarrow{\rho_3} & \widehat{W}_3 & \xrightarrow{\rho_2} & \widetilde{W}_3 \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\Sigma} & \xrightarrow[\widetilde{\rho}_3]{} & \widehat{\Sigma} & \xrightarrow[\widetilde{\rho}_2]{} & \Sigma_2. \end{array}$$

Note that $\overline{W}_3 \cong \mathbb{P}(\mathcal{O}_{\overline{\Sigma}}(\widetilde{\rho}_3^* \widetilde{\rho}_2^* \Delta_0) \oplus \mathcal{O}_{\overline{\Sigma}})$ holds. Put $E_1 := \widetilde{\rho}_2^{-1}(\widetilde{q}_1)$ and $\widetilde{E}_2 := \widetilde{\rho}_3^{-1}(q'_1)$, and let Γ_1 be the fibre of μ containing $\widetilde{q}_1$. Furthermore, let $\Delta_0 \in |\Delta_0|$ be a member containing $\widetilde{q}_1$, and $\widetilde{\Delta}_0, \widetilde{\Gamma}_1$ the proper transform of Δ_0 and Γ_1, respectively. Moreover, put $E_2 := \widetilde{\rho}_3^{-1}(q'_1)$, and let Δ'_0 and $\widetilde{E}_1$ be the proper transform of $\widetilde{\Delta}_0$ and E_1, respectively.

If we consider $\widetilde{\Gamma}_0 \subset \overline{\Sigma}$, then $\widetilde{\Gamma}_0$ is a (-1)-curve, and we obtain the blow-down $\rho'_3 : \overline{\Sigma} \to \Sigma'$. If we consider that Δ_∞ and $\widetilde{E}_1$ are the curves of Σ', then they are (-1)-curves. By blowing down $\widetilde{E}_1$, we obtain the birational morphism $\rho'_2 : \Sigma' \to \mathbb{P}^1 \times \mathbb{P}^1$.

Under the above consideration, we obtain

$$\widetilde{\rho}_3^* \widetilde{\rho}_2^* \Delta_0 \sim {\rho'}_3^*({\rho'}_2^*(2E_2 + \Delta'_0) - \widetilde{E}_1) - \widetilde{\Gamma}_0,$$

which leads us to the fact that $\overline{W}_3$ coincides with Equation (3.5). Hence, if we put $p_1 = \mu(\widetilde{q}_1)$, then $\widetilde{S}$ has a singular fibre of type (II-ii-c) over p_1 and no other degenerate fibre with positive H-index. It has been proved that Λ has a subpencil of type R3-3 of Theorem 1.

Denote by $\mathcal{F} = 2C + \sum_{i=1}^{6} E_i$ the degenerate fibre with H-index 6/7 as in 3.2.2.3. By checking the birational transformation of W_3 in detail, we obtain the following.

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Namely, three members of $\{E_i\}_{i=1}^6$ (say E_1, E_2 and E_3) are contracted to the rational double points on the singular locus of $\mathfrak{Q}$. In $\widetilde{S}$, E_i $(i = 4, 5, 6)$ intersects with a (-1)-curve $\mathcal{E}_i$ which intersects every fibre of f. By contracting $\mathcal{E}_i$ $(i = 4, 5, 6)$, E_i changes to a (-1)-curve, and other fibres intersect at three points transversally. By contracting E_i, we obtain the pencil whose base locus consists of three points set theoretically, and any two members contact at these points with intersection multiplicity 2. The image of C intersects with other member transversally at these three points.

4.2. The case rk$(Q) = 4$ (The proof for (2) of Theorem 1)

Let $Q \subset \mathbb{P}^4$ be a hyperquadric of rank 4. Let the notation be as in (4-2).

Consider the following two short exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to E_4 \to \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to 0,$$

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to E_4 \to \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \to 0.$$

We have two types of the tautological divisor. The first one is isomorphic to $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and the second one is isomorphic to Σ_2. These are obtained as follows:

Let X_0, $X_1 \in H^0(\mathcal{O}_{W_4}(T_4 - F))$ and $X_2 \in H^0(\mathcal{O}_{W_4}(T_4))$ be global sections defining the homogeneous coordinates of each fibre of π_4. Any $\Psi \in H^0(\mathcal{O}_{W_4}(T_4))$ can be written as

$$\Psi = \psi_0 X_0 + \psi_1 X_1 + c_2 X_2 \quad (\psi_0, \psi_1 \in H^0(\mathcal{O}_{\mathbb{P}^1}(1)),\ c_2 \in \mathbb{C}.)$$

The following lemma is trivial:

Lemma 4. *Let the notation be as above. Then* (Ψ) *is isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$ *if and only if* $c_2 \neq 0$. (Ψ) *is isomorphic to* Σ_2 *if and only if* $c_2 = 0$ *and* ψ_0 *and* ψ_1 *have no common zero.* (Ψ) *is reducible if and only if* $c_2 = 0$ *and* ψ_0 *and* ψ_1 *have a common zero.*

Definition 1. *In the above notation, let us call* $T_4 \in |T_4|$ *the tautological divisor of type* (t_0) *if* $T_4 \cong \mathbb{P}^1 \times \mathbb{P}^1$ *and the tautological divisor of type* (t_2) *if* $T_4 \cong \Sigma_2$. *Note that any member of* $|T_4 - F|$ *is irreducible and isomorphic to* Σ_1. *For any member* $T' \in |T_4 - F|$, *let us call the divisor* $T' + F$ *the tautological divisor of type* (t_1).

Lemma 5. *Let the notation be as above. The classification of subpencils* $\widehat{\mathcal{P}}$ *of* $|T_4|$ *is as follows:*

(4-i) $\widehat{\mathcal{P}}$ *is generated by the tautological divisors of type* (t_0) *and* (t_2).
(4-ii) $\widehat{\mathcal{P}}$ *is generated by the tautological divisors of type* (t_0) *and* (t_1).
(4-iii) $\widehat{\mathcal{P}}$ *is generated by two tautological divisors of type* (t_1).
(4-iv) $\widehat{\mathcal{P}}$ *is generated by the tautological divisors of type* (t_2) *and* (t_1).

Proof. Let Ψ and Ψ' be the global sections of $\mathcal{O}_{W_4}(T_4)$ generating the two-dimensional subspace $V \subset H^0(\mathcal{O}_{W_4}(T_4))$ corresponding to $\widehat{\mathcal{P}}$. Assume Ψ and Ψ' are written as $\Psi = \psi_0 X_0 + \psi_1 X_1 + c_2 X_2$ and $\Psi' = \psi_0' X_0 + \psi_1' X_1 + c_2' X_2$.

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First, assume $c_2 \neq 0$. Put $\Psi'' := \Psi' - (c_2'/c_2)\Psi$. Assume it is written as $\Psi'' = \psi_0'' X_0 + \psi_1'' X_1$. If ψ_0'' and ψ_1'' are linearly independent, then $\widehat{\mathcal{P}}$ is of type (4-i). If ψ_0'' and ψ_1'' are linearly dependent, then $\widehat{\mathcal{P}}$ is of type (4-ii).

Next, assume $c_2 = c_2' = 0$. If at least one of Ψ and Ψ' is of type (t_1), then there is nothing to prove. So we may assume ψ_0 and ψ_1 do not have a common zero and neither do ψ_0' and ψ_1'. If ψ_0 and ψ_0' are linearly dependent, then $\widehat{\mathcal{P}}$ is of type (4-iv). Assume that the pairs (ψ_0, ψ_0') and (ψ_1, ψ_1') are both linearly independent. Furthermore, assume that it is written as $\psi_0 = A\psi_1 + B\psi_1'$ and $\psi_0' = C\psi_1 + D\psi_1'$ for $A, B, C, D \in \mathbb{C}$. Let m be the solution of the quadric equation

$$\begin{vmatrix} m - A & -C \\ B & -m + D \end{vmatrix} = 0, \tag{4.2}$$

and $(x, y) = (k, l)$ the nonzero solution of

$$\begin{pmatrix} m - A & -C \\ B & -m + D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then we have $k\psi_0 + l\psi_0' = m(k\psi_1 + l\psi_1')$ and the tautological divisor defined by $k\Psi + l\Psi'$ is of type (t_1). If Equation (4.2) has two distinct solutions, then $\widehat{\mathcal{P}}$ is of type (4-iii). If Equation (4.2) has a multiple solution, then $\widehat{\mathcal{P}}$ is of type (4-iv). □

4.2.1. Type (4-i)

We may assume S is contained in W_4 since we assume that S does not go through the vertex of Q. Let $\widehat{\mathcal{P}}$ be the pencil of type (4-i) defining $\mathcal{P}$. Then there exists a tautological divisor $T_{4,0} \in \widehat{\mathcal{P}}$ of type (t_2), and all the other members are the tautological divisor of type (t_0). Since we assume that a general member of $\mathcal{P}$ is smooth, it is an EH-general curve of genus 4. Moreover, there exists a member that is an EH-special curve of genus 4. If $\mathcal{P}$ is generic in $\mathrm{Gr}(3,1)$, then the base locus of $\mathcal{P}$ consists of six points and any two distinct member intersects at these points transversally. Hence, $\mathcal{P}$ is of type R4-1 of Theorem 1.

Remark 5. Let $T_4, T_4' \in \widehat{\mathcal{P}}$ be of type (t_0). For the image $Q \subset \mathbb{P}^4$ of W_4, we may consider as $T_4, T_4' \subset Q$. There exist hyperplanes $H, H' \subset \mathbb{P}^4$ with $H|_Q = T_4$ and $H'|_Q = T_4'$. Put $P := H \cap H'$. Then we have $P \cong \mathbb{P}^2$. Let $\alpha : \mathfrak{X} \to \mathbb{P}^4$ be the blow-up along P. We have $\mathfrak{X} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3})$. Let $\mathfrak{T}$ be the tautological divisor of $\mathfrak{X}$, $\mathfrak{F}$ a fibre of the $\mathbb{P}^3$-bundle $\beta : \mathfrak{X} \to \mathbb{P}^1$ and $\mathfrak{Q}$ the proper transform of Q by α. Then $\mathfrak{Q} \sim 2\mathfrak{T} - 2\mathfrak{F}$ holds. Namely, $\mathfrak{X}$ and $\mathfrak{Q}$ are the latter of the case of [16, Remark 3].

4.2.2. Type (4-ii)

Similarly to 4.2.1, if $\widehat{\mathcal{P}}$ is of type (4-ii) and generic in $\mathrm{Gr}(3,1)$, then the corresponding pencil $\mathcal{P}$ on S is of type R4-2 of Theorem 1.

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4.2.3. Type (4-iii)

Let Ψ_0 and Ψ_1 be the basis of two-dimensional subspace of $H^0(\mathcal{O}_{W_4}(T_4))$ corresponding to $\widehat{\mathcal{P}}$. Assume that Ψ_i is written as $\Psi_i = \psi_i X_i$ ($\psi_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ for $i = 0, 1$. Let $p_i \in \mathbb{P}^1$ be the point such that $\psi_i(p_i) = 0$, and put $F_i = \pi_4^{-1}(p_i)$. Assume $p_0 \neq p_1$. Denote by B_0 the section of π_4 defined by $X_0 = X_1 = 0$. For $i = 0, 1$, denote by ℓ_i the intersection of F_i and $X_{\sigma(i)}$, where σ is the permutation of 0 and 1. Then we have $\mathrm{Bs}\widehat{\mathcal{P}} = B_0 \cup \ell_0 \cup \ell_1$.

Let $\rho_1 : \widetilde{W}_4 \to W_4$ be the blow-up along B_0. Consider that ℓ_i $(i = 0, 1)$ is the curve of $\widetilde{W}_4$, and let $\rho_2 : \widehat{W}_4 \to \widetilde{W}_4$ be the blow-up along $\ell_0 \cup \ell_1$. Then by the same argument as in case (3-ii-a) of § 4.1.2, we obtain that $\widehat{W}_4$ coincides with Equation (3.3) and that the subpencil of type R4-3 of Theorem 1 exists.

4.2.4. Type (4-iv)

We may assume that the basis Ψ and Ψ' of the subspace corresponding $\widehat{\mathcal{P}}$ are written as $\Psi = \psi_0 X_0 + \psi_1 X_1$ and $\Psi' = \psi_1' X_1$. Hence, in the proof of Lemma 5, we have $C = D = 0$. If the quadric equation (4.2) has the multiple solution, we have $A = 0$. Hence, we may assume $\psi_1' = \psi_0$.

Let B_0 be as in the previous subsection, and ℓ_0 be the curve defined by $\psi_0 = X_1 = 0$. Then we have $\mathrm{Bs}\widehat{\mathcal{P}} = B_0 \cup \ell_0$. Let $\rho_1 : \widetilde{W}_4 \to W_4$ be as in the previous subsection and $\rho_2 : \widehat{W}_4 \to \widetilde{W}_4$ the blow-up along ℓ_0. Then $\widetilde{W}_4$ has the structure of the $\mathbb{P}^1$-bundle $\zeta_4 : \widetilde{W}_4 \to \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, if we put $q_0 := \zeta_4(\ell_0)$, and if we denote by $\nu_1 : \widetilde{\Sigma} \to \mathbb{P}^1 \times \mathbb{P}^1$ the blow-up at q_0, then we have $\widehat{W}_4 \cong \widetilde{W}_4 \times_{\mathbb{P}^1 \times \mathbb{P}^1} \widetilde{\Sigma}$. Let $\widetilde{\zeta}_4 : \widehat{W}_4 \to \widetilde{\Sigma}$ be the obtained $\mathbb{P}^1$-bundle. For the proper transform $\widehat{T}_4$ of T_4 by $\rho_1 \circ \rho_2$, there exists an infinitely near point $\widetilde{q}_0 \in \widetilde{\Sigma}$ with $\mathrm{Bs}\left|\widehat{T}_4\right| = \widetilde{\zeta}_4^{-1}(\widetilde{q}_0)$. Put $\ell_1 := \widetilde{\zeta}_4^{-1}(\widetilde{q}_0)$.

Let $\overline{\rho}_4 : \overline{W}_4 \to \widehat{W}_4$ be the blow-up along ℓ_1. By the same argument as before, we obtain that $\overline{W}_4$ coincides with Equation (3.4).

In this case, the only degenerate fibre of f with positive H-index is of the form $C_1 + C_2 + \sum_{i=1}^3 E_i$, where C_1 and C_2 are elliptic curves, and E_i is a (-2)-curve. f has three sections $\mathbb{E}_i$ $(i = 1, 2, 3)$ as (-1)-curves. We may assume $\mathbb{E}_i E_i = 1$ and $\mathbb{E}_i E_j = 0$ if and only if $i \neq j$. Furthermore, we have $\mathbb{E}_i C_j = 0$. Let $\nu_1 : \widetilde{S} \to S_1$ be the blow-down of $\mathbb{E}_1, \mathbb{E}_2$ and $\mathbb{E}_3$. Then the images of general fibres of f by ν_1 intersect at three points. Let $\nu_2 : S_1 \to S$ be the blow-down of the image of $E_1 + E_2 + E_3$ by ν_1. Then the images of general fibres of f by $\nu_2 \circ \nu_1$ contact at three points with intersection multiplicity 2, while the image of C_j intersects the images at these points transversally. Hence, the obtained subpencil is of type R4-4 of Theorem 1.

4.3. The case $\mathrm{rk}(Q) = 5$ (The proof for (3) of Theorem 1)

Let Q be a hyperquadric of rank 5. By the similar argument to Lemma 2, we obtain the following:

Lemma 6. *Let the notation be as in Lemma 2. For any subpencil $\mathcal{P}_Q \subset |\widetilde{H}_Q|$, there exists a double cover $\gamma : Q \to \mathbb{P}^3$ such that any member of $\mathcal{P}_Q$ is mapped onto a hyperplane by γ as a branched double cover.*

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Let $S \subset Q$ be our surface and $\mathcal{P}$ the subpencil of the complete linear system of the hyperplane sections. Let $\widehat{\mathcal{P}}$ be the subpencil of the complete linear system of the hyperplane sections of Q whose restriction to S is $\mathcal{P}$. Let $\gamma : Q \to \mathbb{P}^3$ be the double cover of Lemma 6. If $\widetilde{\mathcal{P}}$ is the subpencil of the hyperplanes of $\mathbb{P}^3$ corresponding to $\mathcal{P}$, then $\ell := \mathrm{Bs}\widetilde{\mathcal{P}}$ is a line of $\mathbb{P}^3$. Let $Q_0 \subset \mathbb{P}^3$ be the branch locus of γ. We have $Q_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$.

There are following three cases:

(5-i) ℓ and Q_0 intersect at two distinct points.
(5-ii) ℓ and Q_0 contact at a point.
(5-iii) $\ell \subset Q_0$ holds. In this case, ℓ is a fibre of one of the natural projection for the direct product.

4.3.1. The case (5-i)

Let us consider the case (5-i).

Let $\mathcal{P}_0$ be the restriction of $\widetilde{\mathcal{P}}$ to Q_0. If $q_1, q_2 \in Q_0$ are the points with $\ell \cap Q_0 = \{q_1, q_2\}$, then we have $\mathrm{Bs}\mathcal{P}_0 = \{q_1, q_2\}$. Let $\eta_i : Q_0 \to \mathbb{P}^1$ be the natural projection for the ith element ($i = 1, 2$), Δ_{1j} the fibre of η_1 with $q_j \in \Delta_{1j}$ and Γ_{2j} the fibre of η_2 with $q_j \in \Gamma_{2j}$. We have $\Delta_{11}+\Gamma_{21}, \Delta_{12}+\Gamma_{22} \in \mathcal{P}_0$. Furthermore, any member of $\mathcal{P}_0 \setminus \{\Delta_{11}+\Gamma_{21}, \Delta_{12}+\Gamma_{22}\}$ is irreducible and non-singular. Hence, $\widehat{\mathcal{P}}$ has two quadric cones and any other member is a smooth hyperquadric. If we consider $\mathrm{Bs}\widehat{\mathcal{P}}$ as a divisor of a general member, then it is a smooth diagonal divisor, while if we consider it as a divisor of the quadric cone, then it is smooth section not going through the vertex. If we put $\widetilde{\ell} := \mathrm{Bs}\widehat{\mathcal{P}}$, then $\widetilde{\ell}$ is an irreducible and non-singular rational curve. Let $\overline{\mathcal{P}}$ be the pencil of the complete linear system of hyperplanes of $\mathbb{P}^4$ corresponding to $\widehat{\mathcal{P}}$. We have $\mathbb{H} := \mathrm{Bs}\overline{\mathcal{P}}$ is a two-dimensional subspace with $\mathbb{H}|_Q = \widetilde{\ell}$.

Let $\alpha : \mathfrak{X} \to \mathbb{P}^4$ be the blow-up along $\mathbb{H}$. Then $\mathfrak{X} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3})$. Let $\beta : \mathfrak{X} \to \mathbb{P}^1$ be the natural projection, $\mathfrak{T}$ the tautological divisor of $\mathfrak{X}$ and $\mathfrak{F}$ a fibre of β. If we put $\mathbb{E} := \alpha^{-1}(\mathbb{H})$, then we have $\mathbb{E} \cong \mathbb{P}^1 \times \mathbb{P}^2$ and $\mathbb{E} \sim \mathfrak{T} - 2\mathfrak{F}$. If $\mathfrak{Q}$ is the proper transform of Q by α, then we have $\mathfrak{Q} \sim 2\mathfrak{T} - 2\mathfrak{F}$. This situation is similar to that of 4.2.1. The differences from 4.2.1 are that $\mathfrak{Q}$ is smooth and that the fibration $\mathfrak{Q} \to \mathbb{P}^1$ has two degenerate fibres of rank 3. Namely, the discriminant locus of $\mathfrak{Q}$ is a reduced divisor of degree 2, and hence, f is of type (I-i). In a general case, $\mathrm{Bs}\mathcal{P}$ consists of six points over $\widetilde{\ell}$, and general members intersect at these points transversally. We obtain the pencil of type R5-1 of Theorem 1.

4.3.2. The case (5-ii)

The similar argument to the case (5-i) is applied. We use the same notation. The base locus $\mathrm{Bs}\mathcal{P}_0$ consists of one point, say $q \in Q_0$. Let Δ_3 and Γ_3 be fibres of the natural projection η_1 and η_2, respectively, such that $\Delta_3 \cap \Gamma_3 = \{q\}$. $\mathcal{P}_0$ contains $\Delta_3 + \Gamma_3$ and all the other members of $\mathcal{P}_0$ are smooth diagonal divisors contacting at q with intersection multiplicity 2. $\widehat{\mathcal{P}}$ contains a quadric cone, and any other member is a smooth hyperquadric. Since $\mathrm{Bs}\widetilde{\mathcal{P}}$ is a line in $\mathbb{P}^3$ contacting with Q_0 at q, the base locus $\mathrm{Bs}\widehat{\mathcal{P}}$ is a union of two rational curves intersecting at a point. Put $\widetilde{\ell}_1 + \widetilde{\ell}_2 = \mathrm{Bs}\widehat{\mathcal{P}}$.

Let $\alpha : \mathfrak{X} \to \mathbb{P}^4$ be as in the case (iii-a). Moreover, we use the same other notation related to $\mathfrak{X}$. Then $\mathfrak{Q}$ is a relative hyperquadric and a general member of $\mathfrak{Q} \to \mathbb{P}^1$ is

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isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. There exists only one degenerate fibre that is isomorphic to the quadric cone. Namely, the discriminant locus of $\mathfrak{Q}$ is a non-reduced divisor of degree 2 and we obtain the subpencil of type R5-2 of Theorem 1. If $\mathcal{P}$ is generic, then Bs$\mathcal{P}$ consists of six points. Three of them are on ℓ_1, and the rest three points are on ℓ_2. General members of $\mathcal{P}$ intersect at these points transversally one another.

4.3.3. The case (5-iii)

Let us consider the case (5-iii). All the members of $\widehat{\mathcal{P}}$ are quadric cones, and they contact along a generating line to one another. On the other hand, the vertices of any two distinct cones are the distinct points on the generating line. Let ℓ be the generating line, and $\alpha : \mathfrak{M} \to \mathbb{P}^4$ the blow-up along ℓ. We have $\mathfrak{M} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2})$, which is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. Let $\eta : \mathfrak{M} \to \mathbb{P}^2$ be the projection. Denote by $T_{\mathfrak{M}}$ the tautological divisor of $\mathfrak{M}$. Let $L \in \mathbb{P}^2$ be a line, and put $\mathbb{E} := \alpha^{-1}(\ell)$. We have $T_{\mathfrak{M}} \sim \mathbb{E} + \eta^* L$ and $\mathbb{E} \cong \mathbb{P}^1 \times \mathbb{P}^2$. If $\widetilde{Q}$ is the proper transform of Q by α, then we have $\widetilde{Q} \sim T_{\mathfrak{M}} + \eta^* L$. Denote by $\eta_{\mathbb{E}}$ the natural projection $\mathbb{E} \to \mathbb{P}^2$ and by β the natural projection $\mathbb{E} \to \mathbb{P}^1$. If we consider the restriction $\widetilde{Q}|_{\mathbb{E}}$ as a divisor of $\mathbb{E}$, then we have

$$\widetilde{Q}|_{\mathbb{E}} \sim T_{\mathbb{E}} + \eta_{\mathbb{E}}^* L,$$

where $T_{\mathbb{E}}$ is a fibre of β. The restricted morphism $\eta_{\widetilde{Q}} : \widetilde{Q}|_{\mathbb{E}} \to \mathbb{P}^2$ coincides with the blow-up of $\mathbb{P}^2$ at a point, namely, we obtain $\widetilde{Q}|_{\mathbb{E}} \cong \Sigma_1$. Put $\widetilde{\mathbb{E}} := \mathbb{E}|_{\widetilde{Q}}$. Then $\widetilde{\mathbb{E}}$ is the exceptional divisor of the blow-up $\alpha_{\widetilde{Q}} : \widetilde{Q} \to Q$. If $\widehat{H}_Q$ is the proper transform of $\widetilde{H}_Q \in \widehat{\mathcal{P}}(\subset |\widetilde{H}_Q|)$, then we have $\widehat{H}_Q \cong \Sigma_2$ and all the members of $|\widehat{H}_Q|$ intersect along some fibre (say Γ_0) transversally. Moreover, $\widehat{H}_Q$ and $\widetilde{\mathbb{E}}$ intersect along Γ_0 transversally. The restriction of $\widehat{H}_Q$ to $\widetilde{\mathbb{E}}$ is the (-1)-curve, and hence, it is the section of the ruling $\widetilde{\mathbb{E}}(\cong \Sigma_1) \to \mathbb{P}^1$. If $\widetilde{\alpha} : \widehat{Q} \to \widetilde{Q}$ is the blow-up along Γ_0, we obtain the following commutative diagram:

$$\begin{array}{ccc} \widehat{Q} & \xrightarrow{\ \widetilde{\alpha}\ } & \widetilde{Q} \\ {\scriptstyle \widehat{\eta}}\downarrow & & \downarrow{\scriptstyle \eta_{\widetilde{Q}}} \\ \Sigma_1 & \xrightarrow[\ \mu_1\ ]{} & \mathbb{P}^2. \end{array}$$

If we consider $\widehat{H}_Q$ as a divisor of $\widehat{Q}$, then $\widehat{H}_Q$ is the pull back of a fibre of μ_1, and the restricted morphism $\widehat{\eta}|_{\widetilde{\mathbb{E}}} : \widetilde{\mathbb{E}} \to \Sigma_1$ is an isomorphism. $\widehat{Q}$ has the structure of Σ_2-bundle over $\mathbb{P}^1$, and the discriminant locus of $\widehat{Q}$ is zero.

For our K3 surface S, if we assume $\ell \subset S$, then $\mathcal{P}$ has a fixed component, and this case is excluded by the assumption that a general member of $\mathcal{P}$ is smooth. Hence, we may assume $\ell \not\subset S$. Then S and ℓ intersect at three points scheme theoretically. Put $\{p_1, p_2, p_3\} = S \cap \ell$. For $i = 1, 2, 3$, there exists a member $\widetilde{H}_{Q,i} \in |\widetilde{H}_Q|$ such that p_i is the vertex of $\widetilde{H}_{Q,i}$. The curve $\widetilde{\alpha}^{-1}(p_i)$ is mapped onto a fibre of μ_1 isomorphically. Namely,

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the set of degenerate members of $\mathcal{P}$ is one of (II-i-a), (II-i-b) and (II-i-c), and $\mathcal{P}$ is the subpencil of type R5-3 of Theorem 1.

From now on, we assume that $p_i \neq p_j$ if and only if $i \neq j$. Let $\mathcal{F}_i$ be the member of $\mathcal{P}$ contained in $\widetilde{H}_{Q,i}$ $(i = 1, 2, 3)$. General members of $\mathcal{P}$ are irreducible and non-singular and contact at p_1, p_2 and p_3 with intersection multiplicity 2. On the other hand, $\mathcal{F}_i$ intersects with a general member at $p_{\sigma(i)}$ and $p_{\sigma^2(i)}$ transversally and contacts at p_1 with intersection multiplicity 2, where σ is one of the nontrivial cyclic permutation of $1, 2$ and 3 of order 3.

Let $f : \widetilde{S} \to S$ be as before. Then f has three sections C_1, C_2 and C_3 that are (-1)-curves. Any degenerate fibre of f with positive H-index is of the form $\mathcal{E}_i + D_i$, where $\mathcal{E}_i$ is a (-2)-curve that is the preimage of $\widetilde{\alpha}^{-1}(p_i)$, and D_i is a hyperelliptic curve of genus 3. We may assume $C_i\mathcal{E}_i = 1$ and that $C_i\mathcal{E}_j = 0$ if and only if $i \neq j$. If C_1, C_2 and C_3 are contracted to smooth points, then the images of $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are (-1)-curves. If these three (-1)-curves are contracted to smooth points, we obtain S and $\mathcal{P}$. The images of any two distinct general members contact at three points with intersection multiplicity 2, while the image of D_i has an ordinary node at p_i and contact at $p_{\sigma(i)}$ and $p_{\sigma^2(i)}$ with intersection multiplicity 2.

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