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Measure transfer and S-adic developments for subshifts

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Abstract. Based on previous work of the authors, to any S-adic development of a subshift X a 'directive sequence' of commutative diagrams is associated, which consists at every level $n \geq 0$ of the measure cone and the letter frequency cone of the level subshift X_n associated canonically to the given S-adic development. The issuing rich picture enables one to deduce results about X with unexpected directness. For instance, we exhibit a large class of minimal subshifts with entropy zero that all have infinitely many ergodic probability measures. As a side result, we also exhibit, for any integer $d \geq 2$, an S-adic development of a minimal, aperiodic, uniquely ergodic subshift X, where all level alphabets $\mathcal{A}_n$ have cardinality d, while none of the $d-2$ bottom level morphisms is recognizable in its level subshift $X_n \subseteq \mathcal{A}_n^{\mathbb{Z}}$.

Key words: S-adic development, measure transfer, ergodic measures, letter frequencies, eventually recognizable
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1. *Introduction*

A *subshift* over a finite alphabet $\mathcal{A}$ is a non-empty, closed, and shift-invariant subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$. A very efficient tool to investigate such a subshift X is given by an *S-adic development* of X: the latter is obtained by a *directive sequence* $\overleftarrow{\sigma}$ of monoid morphisms $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$ for all integers $n \geq 0$, where each $\mathcal{A}_n$ is again a finite alphabet, and $\mathcal{A}_n^*$ denotes the free monoid over $\mathcal{A}_n$. The morphisms σ_n here are all assumed to be *non-erasing*, that is, none of the letters of $\mathcal{A}_{n+1}$ is mapped to the empty word. The directive sequence $\overleftarrow{\sigma}$ *generates* the given subshift X if for some identification $\mathcal{A} = \mathcal{A}_0$, any finite factor $x_k \cdots x_\ell$ of any biinfinite word $\mathbf{x} = \cdots x_{-1}x_0x_1 \cdots \in X$ is also a factor of some

$\sigma_0 \circ \cdots \circ \sigma_{n-1}(a_i)$ with $a_i \in \mathcal{A}_n$, and conversely, any such $\mathbf{x}$ belongs to X. One usually also assumes that $\overleftarrow{\sigma}$ is *everywhere growing*, which means that $\liminf_{n\to\infty}(\min\{|\sigma_0 \circ \cdots \circ \sigma_{n-1}(a_i)| \mid a_i \in \mathcal{A}_n\}) = \infty$. It is well known that any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is generated by some everywhere growing directive sequence $\overleftarrow{\sigma}$.

A directive sequence $\overleftarrow{\sigma}$ as above determines at every level $n \geq 0$ a *level subshift* $X_n \subseteq \mathcal{A}_n^{\mathbb{Z}}$, which is the subshift generated by the truncated sequence $\overleftarrow{\sigma}\dagger_n$, obtained from $\overleftarrow{\sigma}$ through forgetting all levels $k < n$ and the corresponding level morphisms. It is a straight forward observation that every level morphism σ_n induces a map $X_{n+1} \to X_n$ which is surjective on shift-orbits.

More generally, any non-erasing morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ between free monoids over finite alphabets $\mathcal{A}$ and $\mathcal{B}$, defines for any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ an image subshift $\sigma(X)$, and it is natural to ask which properties of X are inherited (under suitable hypotheses) by the image subshift $\sigma(X)$. In our cousin paper [5], we have formally introduced and studied, for any such morphism σ, a *measure transfer map* $\sigma_X^{\mathcal{M}} : \mathcal{M}(X) \to \mathcal{M}(\sigma(X))$, where $\mathcal{M}(X)$ denotes the *measure cone* on X, that is, the set of all shift-invariant Borel measures on the subshift X. The map $\sigma_X^{\mathcal{M}}$ is the restriction/co-restriction of a map $\sigma^{\mathcal{M}} : \mathcal{M}(\mathcal{A}^{\mathbb{Z}}) \to \mathcal{M}(\mathcal{B}^{\mathbb{Z}})$ which is $\mathbb{R}_{\geq 0}$-linear, functorial, and commutes with the support map on subshifts (see §3.1).

We thus obtain canonically, for any everywhere growing directive sequence $\overleftarrow{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ as above, an induced sequence $\mathcal{M}(\overleftarrow{\sigma}) = (\sigma_n^{\mathcal{M}} : \mathcal{M}(X_{n+1}) \to \mathcal{M}(X_n))_{n\geq 0}$ of $\mathbb{R}_{\geq 0}$-linear maps $\sigma_n^{\mathcal{M}} := \sigma_{X_{n+1}}^{\mathcal{M}}$ on the measure cones $\mathcal{M}(X_{n+1})$.

Furthermore, any invariant measure μ on a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ defines canonically a *letter frequency vector* $\vec{v}(\mu)$ in the non-negative cone $\mathbb{R}_{\geq 0}^{\mathcal{A}}$ of the vector space $\mathbb{R}^{\mathcal{A}}$, where for each letter $a_i \in \mathcal{A}$, the coordinate of $\vec{v}(\mu)$ is given by the measure $\mu([a_i])$ of the cylinder $[a_i]$. The latter consists of all biinfinite words $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$ as above for which the letter with index 1 satisfies $x_1 = a_i$. The cone of all such letter frequency vectors is denoted by $\mathcal{C}(X) \subseteq \mathbb{R}_{\geq 0}^{\mathcal{A}}$; it gives rise to a canonical $\mathbb{R}_{\geq 0}$-linear *evaluation map* $\zeta_X : \mathcal{M}(X) \to \mathcal{C}(X)$ which by definition is surjective.

It has been shown in [5] that the linear map $\mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{B}}$, defined by the incidence matrix $M(\sigma)$ of any non-erasing free monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$, commutes via the evaluation maps $\zeta_{\mathcal{A}^{\mathbb{Z}}}$ and $\zeta_{\mathcal{B}^{\mathbb{Z}}}$ with the measure transfer map $\sigma^{\mathcal{M}}$. We thus obtain, for any directive sequence $\overleftarrow{\sigma}$ as above, a rather useful commutative diagram:

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\sigma_{n+1}^{\mathcal{M}}} & \mathcal{M}(X_{n+1}) & \xrightarrow{\sigma_n^{\mathcal{M}}} & \mathcal{M}(X_n) & \xrightarrow{\sigma_{n-1}^{\mathcal{M}}} & \cdots & \xrightarrow{\sigma_2^{\mathcal{M}}} & \mathcal{M}(X_1) & \xrightarrow{\sigma_1^{\mathcal{M}}} & \mathcal{M}(X) \\
 & & \downarrow \zeta_{X_{n+1}} & & \downarrow \zeta_{X_n} & & & & \downarrow \zeta_{X_1} & & \downarrow \zeta_X \\
\cdots & \xrightarrow{M(\sigma_{n+1})} & \mathcal{C}(X_{n+1}) & \xrightarrow{M(\sigma_n)} & \mathcal{C}(X_n) & \xrightarrow{M(\sigma_{n-1})} & \cdots & \xrightarrow{M(\sigma_2)} & \mathcal{C}(X_1) & \xrightarrow{M(\sigma_1)} & \mathcal{C}(X)
\end{array}$$

A *measure tower* $\overleftarrow{\mu} = (\mu_n)_{n\geq 0}$ on a directive sequence $\overleftarrow{\sigma}$ as above, defined by postulating $\mu_n \in \mathcal{M}(X_n)$ and $\sigma_n^{\mathcal{M}}(\mu_{n+1}) = \mu_n$, defines a tower of letter frequency vectors $\vec{v}(\overleftarrow{\mu}) = (\vec{v}(\mu_n))_{n\geq 0}$ which satisfy $M(\sigma_{n+1}) \cdot \vec{v}(\mu_{n+1}) = \vec{v}(\mu_n)$. This last equality had been used in [3] as defining equality for what was called there a *vector tower* over the directive sequence $\overleftarrow{\sigma}$. A $\mathbb{R}_{\geq 0}$-linear evaluation map $\mathfrak{m} : \mathcal{V}(\overleftarrow{\sigma}) \to \mathcal{M}(X)$, from the set

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$\mathcal{V}(\overleftarrow{\sigma})$ of all such vector towers to the measure cone $\mathcal{M}(X)$ of the subshift X generated by $\overleftarrow{\sigma}$, has been established in [3], and the map $\mathfrak{m}$ is shown in [3] to be always surjective, as long as $\overleftarrow{\sigma}$ is everywhere growing (but no other hypotheses are needed). We obtain the following propositon (see Proposition 5.3).

PROPOSITION 1.1. *For any everywhere growing directive sequence $\overleftarrow{\sigma}$, there is a canonical $\mathbb{R}_{\geq 0}$-linear bijection between the cone $\mathcal{V}(\overleftarrow{\sigma})$ of vector towers and the cone $\mathcal{M}(\overleftarrow{\sigma})$ of measure towers on $\overleftarrow{\sigma}$, given by the letter frequency map*

$$\overleftarrow{\mu} = (\mu_n)_{n\geq 0} \mapsto \overleftarrow{v} = (\vec{v}_n)_{n\geq 0},$$

with $\vec{v}_n = \vec{v}(\mu_n) = (\mu_n([a_k]))_{a_k \in \mathcal{A}_n}$ for all levels $n \geq 0$.

From this set-up, we derive (see Proposition 4.4) the following result. A crucial ingredient in its proof is the main result of our previous paper [3], quoted below as Theorem 2.10.

THEOREM 1.2. *For any non-erasing monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ and any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, the induced measure transfer map $\sigma^{\mathcal{M}}$ maps the measure cone $\mathcal{M}(X)$ of X surjectively to the measure cone $\mathcal{M}(\sigma(X))$ of the image subshift $\sigma(X)$:*

$$\sigma^{\mathcal{M}}(\mathcal{M}(X)) = \mathcal{M}(\sigma(X)).$$

This general surjectivity result for the measure transfer map $\sigma^{\mathcal{M}}$ is mirrored in the special case where σ is recognizable in X (see Definition 3.5) by the the following fact, proved below in Corollary 3.9.

PROPOSITION 1.3. *If a non-erasing morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is recognizable in a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, then the measure transfer map $\sigma_X^{\mathcal{M}} : \mathcal{M}(X) \to \mathcal{M}(\sigma(X))$ is injective.*

We apply this injectivity result to any directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$, where each level map σ_n is assumed to be recognizable in the corresponding level subshift X_{n+1}. Such *totally recognizable* directive sequences (or slight variations of it) have recently received a lot of attention (see for instance [1, 9, 13, 17]) and they are shown to play a central role in the S-adic approach to symbolic dynamics. We obtain the following theorem (see Theorem 5.6).

THEOREM 1.4. *For any totally recognizable everywhere growing directive sequence $\overleftarrow{\sigma}$, with generated subshift $X = X_{\overleftarrow{\sigma}}$, the $\mathbb{R}_{\geq 0}$-linear surjective map of cones*

$$\mathfrak{m} : \mathcal{V}(\overleftarrow{\sigma}) \to \mathcal{M}(X)$$

is a bijection.

We combine this result with a construction from our earlier paper [4], where for any integer $d \geq 2$, a subshift X with d distinct invariant ergodic probability measures has been shown to exist, while X is defined by an everywhere growing directive sequence with level alphabets $\mathcal{A}_n$ that all have cardinality $\text{card}(\mathcal{A}_n) = d$. This construction is used in §7 below to define a large 'diagonal' family $\mathfrak{X}$ of directive sequences $\overleftarrow{\sigma}$ and to give a quick proof (see Theorem 7.4) that they all generate subshifts $X_{\overleftarrow{\sigma}}$ which have a remarkable property, exhibited first by a quite different and more elaborate method for very particular subshifts in a recent paper by Cyr and Kra (see [**11**]).

COROLLARY 1.5. *For any directive sequence $\overleftarrow{\sigma} \in \mathfrak{X}$, the subshift $X_{\overleftarrow{\sigma}}$ is minimal, has topological entropy $h_{X_{\overleftarrow{\sigma}}} = 0$, and admits infinitely many distinct ergodic probability measures in $\mathcal{M}(X_{\overleftarrow{\sigma}})$.*

The directive sequences considered in the last corollary are all totally recognizable and they are 'large', in that their *alphabet rank*, that is, the limit inferior of the cardinality of the level alphabets, is infinite. For finite alphabet rank, however, the condition 'totally recognizable' can be replaced by a distinctly weaker condition: in this case, the linear map defined by the incidence matrix $M(\sigma_n)$ is for any sufficiently high level $n \geq 0$, *a forteriori* (from the surjectivity result in Theorem 1.2) injective on the subspace spanned by the cone $\mathcal{M}(X_n)$. In the special—but rather frequent—case that this injectivity property of the $M(\sigma_n)$ is also true for all low levels, the bijectivity of the map $\mathfrak{m}$ as in Theorem 1.4 above is a direct consequence of our set-up. We thus obtain the following corollary (see Corollary 6.5).

COROLLARY 1.6. *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift generated by an everywhere growing directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ of finite alphabet rank. Assume that for any $n \geq 0$, the incidence matrix $M(\sigma_n)$ is invertible over $\mathbb{R}$. Then any invariant measure μ on the subshift X is determined by the letter frequency vector associated to μ, that is, by the values $\mu([a_k])$ for all $a_k \in \mathcal{A}$.*

This generalizes a result of [**6**], obtained under additional hypotheses by very different methods.

A slightly more general situation than considered in Theorem 1.4, which deserves some particular interest, occurs if the given directive sequence is only *eventually recognizable*, that is, only for sufficiently high levels, one assumes that the level morphisms are recognizable in the corresponding level subshift. In §8, we investigate non-recognizable morphisms and, in particular, we show in Corollary 8.5 the following result, which is somewhat surprising in view of the claims in [**13, 17**] (see Remark 8.7).

PROPOSITION 1.7. *For any integer $n_0 \geq 0$, there exists an everywhere growing directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ with the following properties.*

(1) *For any $n \geq n_0$, the level alphabets satisfy $\mathcal{A}_n = \mathcal{A}_{n_0}$ and the level morphisms are stationary: $\sigma_n = \sigma_{n_0}$. Furthermore, each level morphisms σ_n is recognizable in the level subshift X_{n+1}.*

(2) *For any level n with $0 \leq n \leq n_0 - 1$, we have $\mathrm{card}(\mathcal{A}_n) = n + 2 = \mathrm{card}(\mathcal{A}_{n+1}) - 1$, and none of the level morphisms σ_n is recognizable in the level subshift X_{n+1}.*

(3) *All level subshifts X_n are minimal, uniquely ergodic, and aperiodic.*

(In fact, each level subshift X_n is actually an interval exchange subshift, obtained from the stable lamination of a pseudo-Anosov homeomorphism on a suitably punctured surface.)

2. *Terminology, notation, conventions, and some quotes*
In this section, we first recall some standard terminology from symbolic dynamics (see §2.1), then summarize the notation introduced in [5] and some of its results (see §3.1), and in §2.2, we recall some classical S-adic terminology and quote the main result from [3], which plays a key role later in this paper.

2.1. *Standard terminology from symbolic dynamics.* Throughout this paper, we denote by $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$ non-empty finite sets, called *alphabets*, and by $\mathcal{A}^*$, $\mathcal{B}^*$, or $\mathcal{C}^*$ the free monoid over those alphabets. Every element $w \in \mathcal{A}^*$ is a word in the *letters* $a_1, a_2, \ldots, a_d$ of $\mathcal{A}$, that is,

$$w = x_1 x_2 \ldots x_n \quad \text{with } x_i \in \{a_1, a_2, \ldots, a_d\} = \mathcal{A}$$

for any $i = 1, \ldots, n$, and the empty word is denoted by ε. Here, n is the *length* of w, denoted by $|w|$, and one sets $|\varepsilon| = 0$. We immediately verify the formula $|w| = \sum_{a_j \in \mathcal{A}} |w|_{a_j}$, where $|w|_{a_j}$ denotes the number of occurrences of the letter a_j in w. More generally, for any second word $u \in \mathcal{A}^*$, we denote by $|w|_u$ the number of (possibly overlapping) occurrences of u as subword $x_k \ldots x_\ell$ (also called a *factor*) of w.

Any monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is determined by the family of letter images $\sigma(a_i) \in \mathcal{B}^*$ for all $a_i \in \mathcal{A}$, and this family can be chosen freely. Such a morphism σ is *non-erasing* if $|\sigma(a_i)| \geq 1$ for all $a_i \in \mathcal{A}$. Note that any composition of non-erasing morphisms is non-erasing. Morphisms which are 'erasing' (by which we mean 'not non-erasing') can occasionally create unexpected and undesired phenomena (see [2, §5]), which is why we decided to exclude them. It turns out (see Remark 2.5) that in the context considered here, this assumption is almost immaterial.

Assumption 2.1. All morphisms considered in this paper are assumed to be non-erasing.

Every monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ induces canonically a $\mathbb{R}_{\geq 0}$-linear map $\mathbb{R}_{\geq 0}^{\mathcal{A}} \to \mathbb{R}_{\geq 0}^{\mathcal{B}}$, given by the *incidence matrix*

$$M(\sigma) = (|\sigma(a_j)|_{b_i})_{b_i \in \mathcal{B},\, a_j \in \mathcal{A}}. \tag{2.1}$$

To any alphabet $\mathcal{A}$, there is also associated the *full shift* $\mathcal{A}^{\mathbb{Z}}$; its elements are written as biinfinite words

$$\mathbf{x} = \cdots x_{i-1} x_i x_{i+1} \cdots \tag{2.2}$$

with $x_i \in \mathcal{A}$ for any index $i \in \mathbb{Z}$. The set $\mathcal{A}^{\mathbb{Z}}$ is naturally equipped with the product topology (with respect to the discrete topology on $\mathcal{A}$), and $\mathcal{A}^{\mathbb{Z}}$ is a Cantor set unless $\operatorname{card}(\mathcal{A}) = 1$. Furthermore, the space $\mathcal{A}^{\mathbb{Z}}$ comes naturally with a shift-operator T, defined for any $\mathbf{x}$ as in equation (2.2) by $T(\mathbf{x}) = \cdots y_{i-1} y_i y_{i+1} \cdots$ with $y_i = x_{i+1}$ for any $i \in \mathbb{Z}$. The shift-operator acts as homeomorphism on the space $\mathcal{A}^{\mathbb{Z}}$; for convenience, it will always be denoted by the symbol T, independently of the choice of the given alphabet $\mathcal{A}$.

For any integers $k \leq l$, we denote by $\mathbf{x}_{[k,\ell]}$ the subword (again also called *factor*) $x_k \cdots x_\ell$ of the biinfinite word $\mathbf{x}$ as in equation (2.2). We also consider the one-sided infinite *positive half-word* $\mathbf{x}_{[1,\infty)} = x_1 x_2 \cdots$ of $\mathbf{x}$.

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To any word $w \in \mathcal{A}^*$, there is associated the *cylinder* $[w] \subseteq \mathcal{A}^{\mathbb{Z}}$, which consists of all words $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$ which satisfy $\mathbf{x}_{[1,|w|]} = w$. If w is the empty word, then $[w] = \mathcal{A}^{\mathbb{Z}}$. The set of all cylinders $[w]$ together with their shift translates $T^m([w])$ for any $m \in \mathbb{Z}$ constitute a basis for the above specified topology of the space $\mathcal{A}^{\mathbb{Z}}$.

A non-empty subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a *subshift* if X is closed and if $T(X) = X$. A subshift X is *minimal* if none of its subsets is a subshift except X itself. This is equivalent to the statement that for any $\mathbf{x} \in X$, the *shift-orbit* $\mathcal{O}(\mathbf{x}) = \{T^m(\mathbf{x}) \mid m \in \mathbb{Z}\}$ is dense in X. A minimal subshift X is either uncountably infinite or else it is finite: in this case, X consists of the single shift-orbit $X = \mathcal{O}(w^{\pm\infty})$ of some periodic word $w^{\pm\infty} = \cdots www \cdots$, which is well defined for any non-empty $w \in \mathcal{A}^*$ by the convention $w^{\pm\infty}_{[1,\infty)} = www \ldots$ (that is, the letter with index 1 in $w^{\pm\infty}$ is the first letter of w). It follows that any infinite minimal subshift is in particular *aperiodic*, which means that X does not contain any periodic word $w^{\pm\infty}$.

Any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ defines a *language* $\mathcal{L}(X)$ which consists of all words $w \in \mathcal{A}^*$ that occur as a factor in some $\mathbf{x} \in X$. Conversely, every infinite subset $\mathcal{L} \subseteq \mathcal{A}^*$ *generates* a subshift $X(\mathcal{L}) \subseteq \mathcal{A}^{\mathbb{Z}}$, defined by the property that any word from $\mathcal{L}(X)$ must occur as a factor in some $w' \in \mathcal{L}$.

For any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and any $n \in \mathbb{N}$, one denotes by $p_X(n)$ the number of words in $\mathcal{L}(X)$ of length n. The following limit is well defined and is known as *topological entropy* h_X of the subshift X:

$$h_X = \lim_{n\to\infty} \frac{\log p_X(n)}{n}. \tag{2.3}$$

Any non-erasing monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ defines canonically a map

$$\sigma^{\mathbb{Z}} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}} \tag{2.4}$$

where for any $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$, the image $\mathbf{y} = \sigma^{\mathbb{Z}}(\mathbf{x}) \in \mathcal{B}^{\mathbb{Z}}$ is defined by extending σ first to the positive half-word $\mathbf{x}_{[1,\infty)}$ to define $\mathbf{y}_{[1,\infty)}$, and subsequently extending σ to all of $\mathbf{x}$.

For almost all subshifts $X \subseteq \mathcal{A}^{Z}$, the image set $\sigma^{\mathbb{Z}}(X)$ will not be shift-invariant and hence not be a subshift. However, there is a canonical *image subshift* $\sigma(X)$ of X, which admits several naturally equivalent definitions.

Remark 2.2. The following three definitions of the image subshift $Y := \sigma(X)$ are equivalent for any non-erasing monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$.

(1) Y is the intersection of all subshifts that contain the set $\sigma^{\mathbb{Z}}(X)$.

(2) Y is the the union of all shift-orbits $\mathcal{O}(\sigma(\mathbf{x}))$, for any $\mathbf{x} \in X$. (Note here (see [5, Lemma 2.4]) that this union is always closed, a fact that *a priori* can not be taken for granted.)

(3) Y is the subshift generated by the language $\sigma(\mathcal{L}(X))$. Thus, Y consists of all biinfinite words $\mathbf{y} \in \mathcal{B}^{\mathbb{Z}}$ with the property that every factor of $\mathbf{y}$ is also a factor of some word in $\sigma(\mathcal{L}(X))$.

We observe directly the following consequence which will be used later (see Remark 4.2).

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LEMMA 2.3. *Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a non-erasing monoid morphism and let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be any subshift. If $\mathcal{L} \subseteq \mathcal{A}^*$ is a language that generates X, then $\sigma(\mathcal{L})$ generates $\sigma(X)$.*

An *invariant measure* on $\mathcal{A}^{\mathbb{Z}}$ is a finite Borel measure μ on $\mathcal{A}^{\mathbb{Z}}$ which is invariant under the homeomorphism T (= the shift operator). The set of all such invariant measures is denoted by $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$. For any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, we denote by $\mathcal{M}(X) \subseteq \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ the set of those invariant measures μ for which their support satisfies $\text{Supp}(\mu) \subseteq X$. For notational convenience, we identify any such μ with its restriction to X.

Any invariant measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ defines a function

$$\mathcal{A}^* \to \mathbb{R}_{\geq 0}, \quad w \mapsto \mu([w])$$

which, for convenience, is also denoted by μ, yielding $\mu(w) = \mu([w])$ for any $w \in \mathcal{A}^*$. This function is a *weight function* in that it satisfies the *Kirchhoff equalities*:

$$\mu(w) = \sum_{a_i \in \mathcal{A}} \mu(a_i w) = \sum_{a_i \in \mathcal{A}} \mu(w a_i) \tag{2.5}$$

for any $w \in \mathcal{A}^*$. Conversely, it is well known that any weight function $\mu : \mathcal{A}^* \to \mathbb{R}_{\geq 0}$ defines an invariant measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ which satisfies $\mu([w]) = \mu(w)$. The set $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ can hence be understood as a subset of the infinite dimensional non-negative cone $\mathbb{R}_{\geq 0}^{\mathcal{A}^*} = \{\sum_{w \in \mathcal{A}^*} x_w \vec{e}_w \mid x_w \geq 0\}$, where $\vec{e}_w$ denotes the unit vector of $\mathbb{R}^{\mathcal{A}^*}$ in the direction defined by $w \in \mathcal{A}^*$. From this embedding, the set $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ inherits the product topology; the latter coincides with the more generally known *weak*-topology* on the measure cone $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$.

A measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is a *probability measure* if its total mass satisfies $\mu(\mathcal{A}^{\mathbb{Z}}) = 1$. A measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ is *ergodic* if μ cannot be written as a linear combination with positive coefficients of two distinct probability measures. For any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, the number $e(X)$ of ergodic probability measures in $\mathcal{M}(X)$ can be finite or infinite; it is equal to the dimension of the linear convex cone $\mathcal{M}(X) \subseteq \mathbb{R}_{\geq 0}^{\mathcal{A}^*}$. For any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, we have $e(X) \geq 1$; if $e(X) = 1$, the subshift X is called *uniquely ergodic*.

The support $\text{Supp}(\mu)$ of any $\mu \in \mathcal{A}^{\mathbb{Z}}$ is always a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$; if μ is ergodic, then $X = \text{Supp}(\mu)$ is a minimal subshift. The converse conclusion does not hold (see §7 below).

A word $w \in \mathcal{A}^* \setminus \{\varepsilon\}$ is called a *proper power* if

$$w = u^m \quad \text{for some } u \in \mathcal{A}^* \text{ and some integer } m \geq 2. \tag{2.6}$$

(Elements in $\mathcal{A}^*$ which are not a proper power are sometimes called 'primitive'. However, since $\mathcal{A}^*$ is canonically embedded into the free group $F(\mathcal{A})$, where the notion of 'primitive elements' is classical, but has a different meaning, we believe it is better not to use this terminology for a different purpose.)

Any non-empty word $w \in \mathcal{A}^*$ defines a *characteristic measure* $\mu_w \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$: if w is not a proper power, then μ_w is given by

$$\mu_w(B) := \text{card}(B \cap \mathcal{O}(w^{\pm\infty})) \tag{2.7}$$

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for any measurable set $B \subseteq \mathcal{A}^{\mathbb{Z}}$. If however $w = u^m$ for some $u \in \mathcal{A}^*$ and some integer $m \geq 2$, where u is assumed not to be a proper power, then one has

$$\mu_w := m \cdot \mu_u.$$

In either case, it follows that $(1/|w|)\mu_w$ is a probability measure. The set of *weighted characteristic measures* $\lambda\,\mu_w$ (for any $\lambda > 0$) is known to be dense in $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$. The support of any characteristic measure is given by

$$\mathrm{Supp}(\mu_w) = \mathcal{O}(w^{\pm\infty}). \tag{2.8}$$

To any alphabet $\mathcal{A}$, one associates canonically the non-negative *alphabet cone* $\mathbb{R}^{\mathcal{A}}_{\geq 0} = \{\sum_{a_k \in \mathcal{A}} x_k \vec{e}_{a_k} \mid x_k \in \mathbb{R}, x_k \geq 0\}$. For any invariant measure μ on $\mathcal{A}^{\mathbb{Z}}$, the evaluation on the letter cylinders $[a_k]$ for all $a_k \in \mathcal{A}$ defines a *letter frequency vector*

$$\vec{v}(\mu) := \sum_{a_k \in \mathcal{A}} \mu([a_k])\, \vec{e}_{a_k}, \tag{2.9}$$

so that one has a canonical $\mathbb{R}_{\geq 0}$-linear map of cones, denoted by

$$\zeta_{\mathcal{A}^{\mathbb{Z}}} : \mathcal{M}(\mathcal{A}^{\mathbb{Z}}) \to \mathbb{R}^{\mathcal{A}}_{\geq 0}, \quad \mu \mapsto \vec{v}(\mu). \tag{2.10}$$

For any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, the restriction of this map to $\mathcal{M}(X)$ will be denoted by ζ_X. The image of this map is a cone, denoted by

$$\mathcal{C}(X) := \zeta_X(\mathcal{M}(X)) \subseteq \mathbb{R}^{\mathcal{A}}_{\geq 0}, \tag{2.11}$$

and called the *letter frequency cone* of the subshift X. For simplicity, we will below, for any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ and any morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$, use the symbol $M(\sigma)$ to denote all three linear maps

$$\mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{B}}, \mathbb{R}^{\mathcal{A}}_{\geq 0} \to \mathbb{R}^{\mathcal{B}}_{\geq 0} \quad \text{and} \quad \mathcal{C}(X) \to \mathcal{C}(\sigma(X)) \tag{2.12}$$

defined by the incidence matrix of the morphism σ.

More details about these basic facts and some references can be found in [5, §2].

2.2. *Measures on subshifts via vector towers on directive sequences.* To state Theorem 2.10 below, which is the main purpose of this subsection, we first recall some standard notation that is also used later.

A *directive sequence* $\overleftarrow{\sigma} = (\sigma_n)_{n \geq 0}$ consists of *level morphisms*

$$\sigma_n : \mathcal{A}^*_{n+1} \to \mathcal{A}^*_n \tag{2.13}$$

for any *level* $n \geq 0$, where each $\mathcal{A}_n$ is a finite non-empty set, called the *level n alphabet*. We sometimes use the less formal but more suggestive notation

$$\overleftarrow{\sigma} = \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \cdots$$

to denote a directive sequence.

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For any integers $m > n \geq 0$, we define the *telescoped level morphism*

$$\sigma_{[n,m)} := \sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{m-1}$$

as well as the level n *truncated* directive sequence

$$\overleftarrow{\sigma}\dagger_n = (\sigma_k)_{k \geq n}. \tag{2.14}$$

Any directive sequence $\overleftarrow{\sigma}$ as in equation (2.13) above *generates* a subshift $X = X_{\overleftarrow{\sigma}}$ over the *base alphabet* $\mathcal{A}_0$, defined by the convention that $\mathbf{x} \in \mathcal{A}_0^{\mathbb{Z}}$ belongs to X if and only if for any finite factor w of $\mathbf{x}$, there exists some level $n \geq 1$ and some letter $a_j \in \mathcal{A}_n$ such that w is also a factor of $\sigma_{[0,n-1)}(a_j)$.

For any level $n \geq 0$, a directive sequence $\overleftarrow{\sigma}$ as above defines an *intermediate level subshift* $X_n \subseteq \mathcal{A}_n^{\mathbb{Z}}$ which is generated by the truncated sequence $\overleftarrow{\sigma}\dagger_n$:

$$X_n := X_{\overleftarrow{\sigma}\dagger_n}. \tag{2.15}$$

The subshift X_n is the image subshift of the analogously defined level $n+1$ intermediate subshift X_{n+1} under the morphism σ_n, that is:

$$X_n = \sigma_n(X_{n+1}) \quad \text{for any level } n \geq 0. \tag{2.16}$$

In this paper, we will almost exclusively consider directive sequences that are *everywhere growing*, by which we mean that the sequence of *minimal level letter image lengths*

$$\beta_-(n) := \min \{ |\sigma_{[0,n-1)}(a_j)| \mid a_j \in \mathcal{A}_n \} \tag{2.17}$$

tends to ∞ for $n \to \infty$. We have the following fact (see for instance [3, Proposition 5.10]).

Fact 2.4. For every subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, there exists an everywhere growing directive sequence $\overleftarrow{\sigma}$ that generates X. More precisely, using the notation from equation (2.13), one has

$$\mathcal{A}_0 = \mathcal{A} \quad \text{and} \quad X_{\overleftarrow{\sigma}} = X. \tag{2.18}$$

Remark 2.5. Going back to Assumption 2.1, we would like to note that the above evoked directive sequence $\overleftarrow{\sigma}$ from [3, Proposition 5.10] has indeed the property that every level morphism is non-erasing.

In this context, we observe that, *a priori*, if a directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n \geq 0}$ is everywhere growing, it could well be that some of the level maps $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$ map a generator $a_i \in \mathcal{A}_{n+1}$ to the empty word in $\mathcal{A}_n^*$. However, it follows directly from the assumption 'everywhere growing' that this can only occur for a finite number of level maps σ_n, so that one could easily bypass these levels by suitable telescoping and thus obtain an everywhere growing directive sequence which generates the same subshift and has only level maps which are all non-erasing.

Remark 2.6.

(1) A directive sequence $\overleftarrow{\sigma}$ that generates a subshift X is also called an *S-adic development* (or an *S-adic expansion*) of X, where S stands sometimes for an (often assumed to be finite) set of substitutions that contains all level morphisms. This concept and in particular the terminology 'S-adic' has been introduced by Ferenczi in

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[19]. In this context, one often assumes that the sequence $\overleftarrow{\sigma}$ has *finite alphabet rank*. By this, we mean that there is a uniform upper bound to the cardinality of any level alphabet, so that we can identify all level alphabets with a single finite alphabet $\mathcal{A}$.

(2) If the set S consists of a single endomorphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$, then the S-adic subshift X, which is generated by the stationary directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ with $\sigma_n = \sigma$ for all $n \geq 0$, is called *substitutive*. It is important to note that we require here the substitution σ (or rather, the above stationary directive sequence $\overleftarrow{\sigma}$) to be everywhere growing. The term 'substitution' itself is often used synonymous to 'endomorphism of a free monoid', but sometimes (varying) additional conditions are imposed (see for instance [16]).

Definition 2.7. A directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ is called *weakly primitive* if for every level $n \geq 0$, there exists a level $m > n$ such that the telescoped incidence matrix $M(\sigma_{[n,m)})$ is *positive* (that is, it has all coefficients > 0).

Remark 2.8.

(1) One verifies easily that any directive sequence $\overleftarrow{\sigma}$ which is weakly primitive is in particular everywhere growing, unless all level alphabets have cardinality 1.

(2) Weakly primitive directive sequences $\overleftarrow{\sigma}$ have another important property, namely that the subshift $X_{\overleftarrow{\sigma}}$ generated by $\overleftarrow{\sigma}$ is minimal. For this conclusion, we cite [7], proved originally in [14].

In [3], to any directive sequence $\overleftarrow{\sigma}$ as in equation (2.13), there has been associated the set $\mathcal{V}(\overleftarrow{\sigma})$ of *vector towers* $\overleftarrow{v} = (\vec{v}_n)_{n\geq 0}$ *over* $\overleftarrow{\sigma}$. (The terminological specification $\overleftarrow{\sigma}$*-compatible vector tower* used in [4] has been dropped here, as all 'vector towers' occurring in the present paper satisfy the compatibility condition in equation (2.20) for any $n \geq 0$.) Such a vector tower consists of non-negative vectors

$$\vec{v}_n = \sum_{a_j \in \mathcal{A}} \vec{v}_n(a_j)\, \vec{e}_{a_j} \in \mathbb{R}_{\geq 0}^{\mathcal{A}_n} \tag{2.19}$$

that are subject to the *compatibility condition*

$$\vec{v}_n = M(\sigma_n) \cdot \vec{v}_{n+1} \tag{2.20}$$

for all $n \geq 0$. It has been shown (see [3, Remark 9.5]) that for any word $w \in \mathcal{A}_0^*$ and any such vector tower $\overleftarrow{v}$, the sequence of sums

$$\sum_{a_j \in \mathcal{A}_n} \vec{v}_n(a_j)\, |\sigma_{[0,n)}(a_j)|_w$$

is bounded above and increasing, as long as $\overleftarrow{\sigma}$ is everywhere growing (but no other condition is needed). This shows that the value

$$\mu^{\overleftarrow{v}}(w) := \lim_{n\to\infty} \sum_{a_j \in \mathcal{A}_n} \vec{v}_n(a_j)\, |\sigma_{[0,n)}(a_j)|_w \tag{2.21}$$

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is well defined for any $w \in \mathcal{A}_0^*$. Furthermore, it is shown in [3, Propositions 7.4 and 9.4] that the issuing function $\mu^{\overleftarrow{v}} : \mathcal{A}_0^* \to \mathbb{R}_{\geq 0}$ satisfies the Kirchhoff equalities in equation (2.5), so that we can summarize to get the following proposition.

PROPOSITION 2.9. [3] *Any vector tower $\overleftarrow{v}$ on an everywhere growing directive sequence $\overleftarrow{\sigma}$ defines via equation (2.21) an invariant measure on the subshift X generated by $\overleftarrow{\sigma}$, denoted by $\mu^{\overleftarrow{v}} \in \mathcal{M}(X)$.*

In terms of S-adic language, the main result of [3] translates directly into the following theorem (see also [4, §3]).

THEOREM 2.10. [3] *Let $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ be an everywhere growing directive sequence which generates the subshift $X := X_{\overleftarrow{\sigma}}$. Then the map*

$$\mathfrak{m}_{\overleftarrow{\sigma}} : \mathcal{V}(\overleftarrow{\sigma}) \to \mathcal{M}(X), \quad \overleftarrow{v} \mapsto \mu^{\overleftarrow{v}}$$

is $\mathbb{R}_{\geq 0}$-linear and surjective.

For any of the level alphabets $\mathcal{A}_n$ of a directive sequence $\overleftarrow{\sigma}$ as above, we consider the projection map of the set of vector towers to the corresponding non-negative alphabet cone:

$$pr_n : \mathcal{V}(\overleftarrow{\sigma}) \to \mathbb{R}_{\geq 0}^{\mathcal{A}_n}, \quad \overleftarrow{v} = (\vec{v}_n)_{n\geq 0} \mapsto \vec{v}_n.$$

(The map pr_n was denoted in [3, 4] by $\mathfrak{m}_n$, but we decided to reserve this notation here for the more telling maps introduced below in §5.) On the base level $n = 0$, this projection splits over the evaluation map $\zeta_{\mathcal{A}_0^{\mathbb{Z}}}$ from equation (2.10) via the map $\mathfrak{m}_{\overleftarrow{\sigma}}$ from the last theorem. More precisely, this gives (see [3, Proposition 10.2(1) and (2)]).

PROPOSITION 2.11. *For any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ generated by an everywhere growing directive sequence $\overleftarrow{\sigma}$ as in equations (2.13) and (2.18), one has the following items.*

(1) *The map $\zeta_X : \mathcal{M}(X) \to \mathbb{R}_{\geq 0}^{\mathcal{A}}$, $\mu \mapsto (\mu([a_k])_{a_k\in\mathcal{A}}$ satisfies*

$$pr_0 = \zeta_X \circ \mathfrak{m}_{\overleftarrow{\sigma}}.$$

(2) *In particular, for the letter frequency cone $\mathcal{C}(X) = \mathrm{im}(\zeta_X)$ (see equation (2.11)), this gives*

$$\mathcal{C}(X) = \zeta_X(\mathfrak{m}_{\overleftarrow{\sigma}}(\mathcal{V}(\overleftarrow{\sigma}))).$$

(3) *Alternatively, the letter frequency cone is obtained as a nested intersection as*

$$\mathcal{C}(X) := \bigcap_{n\geq 1} M(\sigma_{[0,n)})(\mathbb{R}_{\geq 0}^{\mathcal{A}_n}).$$

(4) *In particular, $\dim \mathcal{C}(X)$ is a lower bound to the number $e(X)$ of distinct ergodic probability measures on X.*

The following statement is the translation of [3, Remark 9.2(3)] into the terminology used here.

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LEMMA 2.12. *For any vector tower* $\overleftarrow{v} = (\vec{v}_n)_{n\geq 0}$ *over an everywhere growing directive sequence* $\overleftarrow{\sigma}$ *as in equation (2.13), one has*

$$\lim_{n\to\infty} \sum_{a_j \in \mathcal{A}_n} \vec{v}_n(a_j) = 0,$$

where the coefficient $\vec{v}_n(a_j) \in \mathbb{R}_{\geq 0}$ *is defined in equation (2.19).*

3. *The measure transfer and its injectivity for recognizable morphisms*

In this section, we will first recall the definition of the measure transfer map and quote some basic properties derived in [5] (see §3.1 below), then recall the definition and some related properties of recognizable morphisms (see §3.2 below). In §3.3, we will derive the injectivity result from the title of this section.

3.1. *The measure transfer and some results from* [5].

For any non-erasing monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$, we define the *subdivision alphabet* $\mathcal{A}_\sigma = \{a_i(k) \mid a_i \in \mathcal{A}$ and $1 \leq k \leq |\sigma(a_i)|\}$. The morphism σ now defines a *subdivision morphism* $\pi_\sigma : \mathcal{A}^* \to \mathcal{A}_\sigma^*$ and a *letter-to-letter morphism* $\alpha_\sigma : \mathcal{A}_\sigma^* \to \mathcal{B}^*$, given for any $a_i \in \mathcal{A}$ and any $a_i(k) \in \mathcal{A}_\sigma$ by

$$\pi_\sigma(a_i) = a_i(1)\, a_i(2) \ldots a_i(|\sigma(a_i)|) \quad \text{and} \quad \alpha_\sigma(a_i(k)) = [\sigma(a_i)]_k.$$

Here, by $[\sigma(a_i)]_k$, we mean the kth letter of the word $\sigma(a_i) \in \mathcal{B}^*$. We obtain directly the following fact.

Fact 3.1. For any non-erasing monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$, one has

$$\sigma \;=\; \alpha_\sigma \circ \pi_\sigma.$$

For any word $w \in \mathcal{A}_\sigma^*$, we denote by $\widehat{w} \in \mathcal{A}^*$ the shortest word such that $\pi_\sigma(\widehat{w})$ contains w as a factor. If such $\widehat{w}$ exists, it is unique; otherwise (for notational convenience only), we treat $\widehat{w}$ as formal symbol for which we set

$$\mu(\widehat{w}) = 0 \tag{3.1}$$

for any $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$.

For any measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$, a measure $\mu^{\pi_\sigma} \in \mathcal{M}(\mathcal{A}_\sigma^{\mathbb{Z}})$ is defined in [5, §3.1] by setting $\mu^{\pi_\sigma}([w]) := \mu([\widehat{w}])$, where $[\widehat{w}]$ is the cylinder associated to the word $\widehat{w}$ (see §2.1). However, for any measure $\mu' \in \mathcal{M}(\mathcal{A}_\sigma^{\mathbb{Z}})$, the classical push-forward measure $(\alpha_\sigma)_*(\mu')$ is an invariant measure on $\mathcal{B}^{\mathbb{Z}}$, since α_σ is letter-to-letter. We thus obtain the following theorem (see [5, §3]).

THEOREM 3.2. *Let* $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ *be a non-erasing morphism of free monoids.*

(1) *For any invariant measure* μ *on* $\mathcal{A}^{\mathbb{Z}}$*, an invariant measure* μ^σ *on* $\mathcal{B}^{\mathbb{Z}}$ *is given by*

$$\mu^\sigma = (\alpha_\sigma)_*(\mu^{\pi_\sigma}).$$

(2) *For any word* $w' \in \mathcal{B}^*$*, the 'transferred measure'* μ^σ *takes on the cylinder* $[w']$ *the value*

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$$\mu^\sigma([w']) = \sum_{w_i \in \alpha_\sigma^{-1}(w')} \mu([\widehat{w}_i]).$$

(3) *The issuing measure transfer map*

$$\sigma^{\mathcal{M}} : \mathcal{M}(\mathcal{A}^{\mathbb{Z}}) \to \mathcal{M}(\mathcal{B}^{\mathbb{Z}}), \quad \mu \mapsto \mu^\sigma$$

induced by the morphism σ has the following properties.

(3a) *The map $\sigma^{\mathcal{M}}$ is linear (over $\mathbb{R}_{\geq 0}$) and continuous (with respect to the weak*-topology).*

(3b) *The map $\sigma^{\mathcal{M}}$ is functorial.*

(3c) *If X is the support of μ, then $\sigma(X)$ is the support of μ^σ. Hence, $\sigma^{\mathcal{M}}$ induces in particular on any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ a restriction/co-restriction map*

$$\sigma_X^{\mathcal{M}} : \mathcal{M}(X) \to \mathcal{M}(\sigma(X)).$$

We also list the following more technical properties derived in [5].

PROPOSITION 3.3. *Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a non-erasing free monoid morphism and let $\sigma^{\mathcal{M}}$ be the induced transfer map on the measure cones. Let $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$ be an invariant measure on the full shift $\mathcal{A}^{\mathbb{Z}}$, and denote as before by $\mu^\sigma = \sigma^{\mathcal{M}}(\mu)$ the transferred measure on $\mathcal{B}^{\mathbb{Z}}$. Then one has the following properties.*

(a) *The total mass of the transferred measure μ^σ is given by the formula*

$$\mu^\sigma(\mathcal{B}^{\mathbb{Z}}) = \sum_{a_k \in \mathcal{A}} \sum_{b_j \in \mathcal{B}} |\sigma(a_k)|_{b_j} \cdot \mu(a_k).$$

In particular, if μ is a probability measure, then, in general, μ^σ will not be probability.

(b) *For any generator $b_j \in \mathcal{B}$, we have*

$$\mu^\sigma([b_j]) = \sum_{a_k \in \mathcal{A}} |\sigma(a_k)|_{b_j} \cdot \mu(a_k).$$

In particular, for the letter frequency vectors from equation (2.9), we obtain

$$\vec{v}(\mu^\sigma) = M(\sigma) \cdot \vec{v}(\mu). \tag{3.2}$$

In other words (see [5, Proposition 4.5]), the measure transfer map $\sigma^{\mathcal{M}}$ commutes via the evaluation maps $\zeta_{\mathcal{A}^{\mathbb{Z}}}$ and $\zeta_{\mathcal{B}^{\mathbb{Z}}}$ from equation (2.10) with the linear map induced by σ on the non-negative cone $\mathbb{R}_{\geq 0}^{\mathcal{A}}$:

$$\zeta_{\mathcal{B}^{\mathbb{Z}}} \circ \sigma^{\mathcal{M}} = M(\sigma) \circ \zeta_{\mathcal{A}^{\mathbb{Z}}}.$$

(c) *For any $w \in \mathcal{A}^*$, the cylinder measures satisfy*

$$\mu^\sigma([\sigma(w)]) \geq \mu([w]).$$

(d) *For any word $w \in \mathcal{A}^*$, the characteristic measure μ_w satisfies*

$$\sigma^{\mathcal{M}}(\mu_w) = \mu_{\sigma(w)}.$$

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It remains to quote a useful evaluation technique for the transferred measure, derived in [5, §4] from what is stated above as part (2) of Theorem 3.2. For this purpose, we define for any non-erasing morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ and any $w \in \mathcal{A}^*$, $u \in \mathcal{B}^*$, the number $\lfloor \sigma(w) \rfloor_u$ of *essential occurrences* of u in $\sigma(w)$, by which we mean that the first letter of u occurs in the σ-image of the first letter of w, and the last letter of u occurs in the σ-image of the last letter of w. By $\langle \sigma \rangle$, we denote the smallest length of any of the letter images $\sigma(a_i)$.

PROPOSITION 3.4. [5, Proposition 4.2] *Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be any non-erasing monoid morphism and let $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$. Then for any $w' \in \mathcal{B}^*$ with $|w'| \geq 2$, the transferred measure $\mu^\sigma = \sigma^{\mathcal{M}}(\mu)$ takes on the cylinder $[w']$ the value*

$$\mu^\sigma([w']) = \sum_{\{w_j \in \mathcal{A}^* \mid |w_j| \leq (|w'|-2)/\langle \sigma \rangle + 2\}} \lfloor \sigma(w_j) \rfloor_{w'} \cdot \mu([w_j]).$$

3.2. *Recognizable morphisms and some related properties.* The following notion has become more and more central to symbolic dynamics (see for instance [9, 13, 15] or [16]).

Definition 3.5. Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a non-erasing morphism and let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift over $\mathcal{A}$. Then σ is said to be *recognizable in X* if the following conclusion is true.

Consider biinfinite words $\mathbf{x}, \mathbf{x}' \in X \subseteq \mathcal{A}^{\mathbb{Z}}$, and $\mathbf{y} \in \mathcal{B}^{\mathbb{Z}}$ which satisfy:

(*) $\mathbf{y} = T^k(\sigma^{\mathbb{Z}}(\mathbf{x}))$ and $\mathbf{y} = T^\ell(\sigma^{\mathbb{Z}}(\mathbf{x}'))$ for some integers k, ℓ which satisfy $0 \leq k \leq |\sigma(x_1)| - 1$ and $0 \leq \ell \leq |\sigma(x_1')| - 1$, where x_1 and x_1' are the first letters of the positive half-words $\mathbf{x}_{[1,\infty)} = x_1 x_2 \ldots$ of $\mathbf{x}$ and $\mathbf{x}'_{[1,\infty)} = x_1' x_2' \ldots$ of $\mathbf{x}'$, respectively.

Then one has $\mathbf{x} = \mathbf{x}'$ and $k = \ell$.

As we will see in the next subsection, recognizability in a subshift is much related to the following.

Definition 3.6. [5, §5] For any non-erasing monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ and any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, we define the following two properties.

(1) σ is *shift-orbit injective in X*: any $\mathbf{x}$ and $\mathbf{y}$ in X have images $\sigma(\mathbf{x})$ and $\sigma(\mathbf{y})$ in the same shift-orbit if and only $\mathbf{x}$ and $\mathbf{y}$ lie in a common shift-orbit.

(2) σ is *shift-period preserving in X*: for any periodic biinfinite word $w^{\pm\infty} = \cdots www \cdots \in X$, the word w can be written as a proper power (see equation (2.6)) if and only if $\sigma(w)$ can be written as a proper power.

The following useful property is a direct consequence of the previous definition (see [5]).

LEMMA 3.7. *Let $\sigma_1 : \mathcal{A}^* \to \mathcal{B}^*$ and $\sigma_2 : \mathcal{B}^* \to \mathcal{C}^*$ be two non-erasing morphisms, and consider a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ as well as its image subshift $Y = \sigma_1(X) \subseteq \mathcal{B}^{\mathbb{Z}}$. Then we have the following properties.*

(1) *The composed morphism $\sigma_2 \circ \sigma_1 : \mathcal{A}^* \to \mathcal{C}^*$ is shift-orbit injective in X if and only if σ_1 is shift-orbit injective in X and σ_2 is shift-orbit injective in Y.*

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(2) *The composed morphism $\sigma_2 \circ \sigma_1 : \mathcal{A}^* \to \mathcal{C}^*$ is shift-period preserving in X if and only if σ_1 is shift-period preserving in X and σ_2 is shift-period preserving in Y.*

3.3. *Injectivity of the measure transfer for recognizable morphisms.* Let $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a non-erasing morphism of free monoids, and let $\pi_\sigma : \mathcal{A}^* \to \mathcal{A}_\sigma^*$ and $\alpha_\sigma : \mathcal{A}_\sigma^* \to \mathcal{B}^*$ be the canonical subdivision morphism and the induced letter-to-letter morphism associated to σ which satisfy $\sigma = \alpha_\sigma \circ \pi_\sigma$ (see Fact 3.1). For any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, we consider the image subshift $\pi_\sigma(X) \subseteq \mathcal{A}_\sigma^{\mathbb{Z}}$ and the induced restriction/co-restriction

$$\alpha_\sigma^X : \pi_\sigma(X) \to \sigma(X)$$

of the map $\alpha_\sigma^{\mathbb{Z}} : \mathcal{A}_\sigma^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ to $\pi_\sigma(X)$ and $\sigma(X)$, respectively.

PROPOSITION 3.8. *For any non-erasing morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ and any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, the following statements are equivalent:*

(1) *σ is recognizable in X;*
(2) *α_σ^X is an isomorphism of subshifts;*
(3) *α_σ is shift-orbit injective and shift-period preserving in $\pi_\sigma(X)$;*
(4) *σ is shift-orbit injective and shift-period preserving in X.*

Proof. We first note that by definition, α_σ^X is continuous and surjective, so that claim (2) is equivalent to stating that α_σ^X is injective.

Next we observe that claim (1) is equivalent to stating that α_σ^X is recognizable in $\pi_\sigma(X)$. This is a direct consequence of the product decomposition $\sigma = \alpha_\sigma \circ \pi_\sigma$ from Fact 3.1 and [**9**, Lemma 3.5], since every subdivision morphism π_σ is recognizable in the full shift, as follows directly from the definition of π_σ.

To show the equivalence (1) $\Longleftrightarrow$ (2), we apply Definition 3.5 to the morphism α_σ and the subshift $\pi_\sigma(X)$: we observe that, since $|\alpha_\sigma(x)| = 1$ for any letter $x \in \mathcal{A}_\sigma$, in the hypothesis (*) of Definition 3.5, the integers k ℓ are necessarily equal to 0. However, in this case, the conclusion $\mathbf{x} = \mathbf{x}'$ stated there amounts precisely to assuring that the map $\alpha_\sigma^{\mathbb{Z}}$ is injective on $\pi_\sigma(X)$, or in other words, that α_σ^X is injective.

The equivalence (2) $\Longleftrightarrow$ (3) is immediate, since any subshift-isomorphism preserves orbits and shift-periods, while conversely, any shift-orbit injective letter-to-letter morphism could only fail to be injective if on some periodic orbit, the shift-period is not preserved.

Finally, the equivalence (3) $\Longleftrightarrow$ (4) is a direct consequence of Lemma 3.7, since every subdivision morphism π_σ is shift-orbit injective and shift-period preserving in the full shift (see [**5**, Lemma 5.3]). □

Note that the equivalence of the statements (1) and (2) from Proposition 3.8 has already been observed in [**16**, Proposition 2.4.24]. Indeed, Fabien Durand has suggested to us to use this equivalence to derive the following corollary. In the meantime, we have obtained a result which is actually a bit stronger: it turns out (see [**5**, Theorem 5.5]) that the hypothesis ‘shift-orbit injective’ suffices to obtain the same conclusion as stated in Corollary 3.9 below, but the proof is much less direct.

We can now derive Proposition 1.3 from §1, restated here for the convenience of the reader.

COROLLARY 3.9. *For any non-erasing morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ and any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, the measure transfer map $\sigma_X^{\mathcal{M}} : \mu \to \mu^{\sigma}$ is injective if σ is recognizable in X.*

Proof. We decompose $\sigma = \alpha_\sigma \circ \pi_\sigma$ as in Fact 3.1, so that from the functoriality of the measure transfer (see property (3b) of Theorem 3.2), we have $\sigma_X^{\mathcal{M}} = (\alpha_\sigma^X)^{\mathcal{M}} \circ \pi_\sigma^{\mathcal{M}}$. The injectivity of $\pi_\sigma^{\mathcal{M}}$ is immediate from the definition of a subdivision morphism (see [**5**, Lemma 5.4]), and the injectivity of $(\alpha_\sigma^X)^{\mathcal{M}}$ is a direct consequence of Proposition 3.8(2). □

Remark 3.10. Consider any non-erasing morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ and any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ with image subshift $Y = \sigma(X) \subseteq \mathcal{B}^{\mathbb{Z}}$.

(1) Assume that the subshift Y contains a periodic word $w^{\pm\infty}$ for some $w \in \mathcal{B}^* \setminus \{\varepsilon\}$, and that the morphism σ is shift-orbit injective. Then, for σ to be shift-period preserving in X, a necessary condition is that at least one of the letters $a_i \in \mathcal{A}$ satisfies $|\sigma(a_i)| \leq |w|$.

As a consequence, unless a given subshift Y is aperiodic, in any everywhere growing S-adic development of Y, there will always be infinitely many level morphisms which are not recognizable in their corresponding level subshift.

(2) This has sparked the following weakening of the notion of 'recognizability' which has become recently very popular (see for instance [**2**]).

The morphism σ is said to be *recognizable for aperiodic points in X* if the conclusion in Definition 3.5 holds under the strengthened assumption that $\mathbf{y}$ is not a periodic word.

(3) From the above proof of Proposition 3.8, we observe that the property 'shift-orbit injective in X' implies the property 'recognizable for aperiodic points in X'.

Indeed, since the subdivision morphism π_σ is always shift-orbit injective and shift-periodic preserving (and thus recognizable) in the full shift, the property 'σ is recognizable for aperiodic points in X' is equivalent to 'α_σ^X is recognizable for aperiodic points in $\pi_\sigma(X)$'. This in turn is equivalent to stating that every non-periodic word in $\sigma(X)$ has precisely one preimage under the letter-to-letter map α_σ. However, since we assume that σ and hence α_σ^X is shift-orbit injective, two distinct such preimages must lie in the same shift-orbit, which implies that their image in $\sigma(X)$ must be periodic.

4. *The measure transfer via vector towers*

In this section, we will consider a subshift X given by means of a directive sequence, an invariant measure μ on X given by means of a vector tower on this directive sequence, and a morphism $\tau : X \to Y = \tau(X)$ which we use to build a new directive sequence for Y by simply adding τ at the bottom to the given sequence. Then the given vector tower is naturally transferred to a new vector tower on the new directive sequence, and, as do all vector towers, it defines an invariant measure μ' on the subshift Y generated by this new sequence. The main goal of this section is to show that the new measure μ' is precisely the image of given measure μ under the transfer map $\tau^{\mathcal{M}}$ induced by the morphism τ (see Theorem 3.2(1)).

For convenience, we summarize the running hypotheses for this section as follows.

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Assumption 4.1. Let $\tau : \mathcal{A}^* \to \mathcal{B}^*$ be a non-erasing morphism of free monoids over finite alphabets $\mathcal{A}$ and $\mathcal{B}$, and let $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ be an everywhere growing directive sequence with base level alphabet $\mathcal{A}_0 = \mathcal{A}$. Let $X := X_{\overleftarrow{\sigma}} \subseteq \mathcal{A}^{\mathbb{Z}}$ be the subshift generated by $\overleftarrow{\sigma}$ and denote by $Y := \tau(X)$ the image subshift of X given by the morphism τ (see Remark 2.2).

Remark 4.2.

(1) For any morphism τ and any directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ as in Assumption 4.1, with subshifts X and Y as defined there, there is a second 'prolonged' directive sequence $\overleftarrow{\sigma}^{\tau} = (\sigma'_n)_{n\geq 0}$, given by setting $\sigma'_n := \sigma_{n-1}$ for any level $n \geq 1$ and $\sigma'_0 := \tau$. We observe from Lemma 2.3 that the subshift $X_{\overleftarrow{\sigma}^{\tau}}$ generated by $\overleftarrow{\sigma}^{\tau}$ agrees precisely with the τ-image subshift $Y = \tau(X) \in \mathcal{B}^{\mathbb{Z}}$.

(2) Consider now any vector tower $\overleftarrow{v} = (\vec{v}_n)_{n\geq 0}$ over $\overleftarrow{\sigma}$, and let $\mu = \mathfrak{m}_{\overleftarrow{\sigma}}(\overleftarrow{v})$ be the invariant measure on X associated to $\overleftarrow{v}$ via Theorem 2.10. There is a second 'prolonged' vector tower $\overleftarrow{v}^{\tau} = (\vec{v}'_n)_{n\geq 0}$ over $\overleftarrow{\sigma}^{\tau}$, given by setting $\vec{v}'_n = \vec{v}_{n-1}$ for any level $n \geq 1$ and by setting $\vec{v}'_0 := M(\tau) \cdot \vec{v}_0$. We denote by μ' be the measure on Y associated to $\overleftarrow{v}^{\tau} = (\vec{v}'_n)_{n\geq 0}$, that is,

$$\mu' = \mathfrak{m}_{\overleftarrow{\sigma}^{\tau}}(\overleftarrow{v}^{\tau}). \tag{4.1}$$

We can now link up the measure transfer map defined and studied in [**5**] with the technology of vector towers from our previous papers [**3**, **4**]. The following will be the basis for all results presented in this paper.

PROPOSITION 4.3. *Let τ, $\overleftarrow{\sigma}$, and X be as in Assumption 4.1, and let $\overleftarrow{v} = (\vec{v}_n)_{n\geq 0}$ be a vector tower over $\overleftarrow{\sigma}$, with associated invariant measure $\mu = \mathfrak{m}_{\overleftarrow{\sigma}}(\overleftarrow{v})$ on X. Let $\overleftarrow{\sigma}^{\tau} = (\sigma'_n)_{n\geq 0}$, $\overleftarrow{v}^{\tau} = (\vec{v}'_n)_{n\geq 0}$, and $\mu' = \mathfrak{m}_{\overleftarrow{\sigma}^{\tau}}(\overleftarrow{v}^{\tau})$ be as in Remark 4.2.*

Then the measure transfer map $\tau^{\mathcal{M}} : \mathcal{M}(\mathcal{A}^{\mathbb{Z}}) \to \mathcal{M}(\mathcal{B}^{\mathbb{Z}})$ induced by the morphism τ satisfies

$$\mu' = \tau^{\mathcal{M}}(\mu) \quad [= \mu^{\tau}].$$

Proof. In this proof, we will freely use the terminology from [**5**] as recalled in §3.1.

For any word $w' \in \mathcal{B}^*$, we consider in the subdivision monoid $\mathcal{A}^*_{\tau}$ the subset $W(w')$ of preimages w_i of w' under the induced letter-to-letter morphism $\alpha_{\tau} : \mathcal{A}^*_{\tau} \to \mathcal{B}^*$. For each $w_i \in W(w')$, consider (as in the paragraph subsequent to Fact 3.1) the word $\widehat{w}_i \in \mathcal{A}^*$ defined by the conditions that (a) its canonically subdivided image $\pi_{\tau}(\widehat{w}_i)$ contains w_i, and that (b) the word $\widehat{w}_i$ is shortest among all words in $\mathcal{A}^*$ which satisfy condition (a). Recall from equation (3.1) that either $\widehat{w}_i$ exists and is unique, or else we formally set $\mu(\widehat{w}_i) = 0$ for any $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$. From Theorem 3.2(2), we know that $\mu^{\tau}(w') = \sum_{w_i \in W(w')} \mu(\widehat{w}_i)$ (with $\mu^{\tau} = \tau^{\mathcal{M}}(\mu)$ as before).

For the purpose of using equation (2.21), we consider now the value of the approximating sum on the right-hand side of this formula for any (large) level $n - 1$, for each of the words $\widehat{w}_i$ and the given vector tower $\overleftarrow{v}$, that is, the term (see equation (2.19) for the notation)

$$\sum_{a\, \in \mathcal{A}_{n-1}} \vec{v}_{n-1}(a)\ |\sigma_{[0,n-1)}(a)|_{\widehat{w}_i}. \tag{4.2}$$

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We sum up the results of equation (4.2) over all $w_i \in W(w')$ to get

$$\sum_{w_i \in W(w')} \sum_{a \in \mathcal{A}_{n-1}} \vec{v}_{n-1}(a) \, |\sigma_{[0,n-1)}(a)|_{\widehat{w}_i}, \tag{4.3}$$

and compare the obtained sum to the limit on the right-hand side of equation (2.21), when this formula is applied to w' and to the vector tower $\overleftarrow{v}^{\tau}$ over the prolonged directive sequence $\overleftarrow{\sigma}^{\tau}$. The nth term of this limit gives the sum

$$\sum_{a \in \mathcal{A}_{n-1}} \vec{v}\,'_n(a) \, |\sigma'_{[0,n)}(a)|_{w'}. \tag{4.4}$$

When comparing the two sums in equations (4.3) and (4.4), we keep in mind that according to the set-up from Remark 4.2 for any $n \geq 1$, we have $\vec{v}\,'_n(a) = \vec{v}_{n-1}(a)$ for any $a \in \mathcal{A}_{n-1}$, as well as $\sigma'_{[0,n)} = \tau \circ \sigma_{[0,n-1)}$.

We now notice that each occurrence of any of the $\widehat{w}_i$ in any of the image words $\sigma_{[0,n-1)}(a)$ with $a \in \mathcal{A}_{n-1}$ defines precisely an occurrence of w_i in $\pi_\tau(\sigma_{[0,n-1)}(a))$, and thus an occurrence of w' in $\alpha_\tau(\pi_\tau(\sigma_{[0,n-1)}(a))) = \tau(\sigma_{[0,n-1)}(a)) = \sigma'_{[0,n)}(a)$. Furthermore, two distinct occurrences of $\widehat{w}_i$ in some $\sigma_{[0,n-1)}(a)$ define distinct occurrences of w_i in $\pi_\tau(\sigma_{[0,n-1)}(a))$, and thus distinct occurrences of w' in $\sigma'_{[0,n)}(a)$. The same is true for occurrences of distinct $\widehat{w}_i$ in $\sigma_{[0,n-1)}(a)$. It follows (using the above recalled equality $\vec{v}\,'_n = \vec{v}_{n-1}$) that

$$\sum_{w_i \in W(w')} \sum_{a \in \mathcal{A}_{n-1}} \vec{v}_{n-1}(a) \, |\sigma_{[0,n-1)}(a)|_{\widehat{w}_i} \leq \sum_{a \in \mathcal{A}_{n-1}} \vec{v}\,'_n(a) \, |\sigma'_{[0,n)}(a)|_{w'}. \tag{4.5}$$

However, the opposite inequality is also true, up to a constant K_n which only depends on $\overleftarrow{\sigma}$ and not on $\overleftarrow{v}$:

$$\sum_{a \in \mathcal{A}_{n-1}} \vec{v}\,'_n(a) \, |\sigma'_{[0,n)}(a)|_{w'} \leq \sum_{w_i \in W(w')} \sum_{a \in \mathcal{A}_{n-1}} \vec{v}_{n-1}(a) \, |\sigma_{[0,n-1)}(a)|_{\widehat{w}_i} + K_n. \tag{4.6}$$

Indeed, any occurrence of w' in $\sigma'_{[0,n)}(a)$ defines, in a unique manner, an occurrence of some w_i in $\pi_\tau(\sigma_{[0,n-1)}(a))$. The latter defines (again uniquely) an occurrence of $\widehat{w}_i$ in $\sigma_{[0,n-1)}(a)$, unless the corresponding occurrence of w_i in $\pi_\tau(\sigma_{[0,n-1)}(a))$ takes place in a suffix or prefix of length bounded by the maximum $m(w') \geq 0$ of all $|\widehat{w}_i|$. We hence deduce:

$$K_n \leq 2m(w') \sum_{a \in \mathcal{A}_{n-1}} \vec{v}_{n-1}(a).$$

It follows now from Lemma 2.12 that the right-hand side of the last inequality tends to 0 for $n \to \infty$, so that we obtain from equations (4.5) and (4.6) through the above definitions $\mu = \mathfrak{m}_{\overleftarrow{\sigma}}(\overleftarrow{v}) = \mu^{\overleftarrow{v}}$ and $\mu' = \mathfrak{m}_{\overleftarrow{\sigma}^{\tau}}(\overleftarrow{v}^{\tau}) = \mu^{\overleftarrow{v}^{\tau}}$ the desired result

$$\begin{aligned} \mu^{\tau}(w') &= \sum_{w_i \in W(w')} \left(\lim_{n \to \infty} \sum_{a \in \mathcal{A}_{n-1}} \vec{v}_{n-1}(a) \, |\sigma_{[0,n-1)}(a)|_{\widehat{w}_i} \right) \\ &= \lim_{n \to \infty} \sum_{a \in \mathcal{A}_{n-1}} \vec{v}\,'_n(a) \, |\sigma'_{[0,n)}(a)|_{w'} = \mu'(w') \end{aligned}$$

for any $w' \in \mathcal{B}^*$. □

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As a first application of the above shown 'basic' Proposition 4.3, we derive the following proposition.

PROPOSITION 4.4. *For any non-erasing morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ and any subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ with image subshift $\sigma(X)$, the induced measure transfer map*

$$\sigma^{\mathcal{M}} : \mathcal{M}(\mathcal{A}^{\mathbb{Z}}) \to \mathcal{M}(\mathcal{B}^{\mathbb{Z}}), \quad \mu \mapsto \mu^{\sigma}$$

maps the measure cone $\mathcal{M}(X)$ surjectively to the measure cone $\mathcal{M}(\sigma(X))$.

Proof. We consider any everywhere growing directive sequence $\overleftarrow{\sigma}$ which generates X; from Fact 2.4, we know that such $\overleftarrow{\sigma}$ exists for any subshift X. By prolonging $\overleftarrow{\sigma}$ through the morphism σ, as explained above in Remark 4.2, we obtain any everywhere growing directive sequence $\overleftarrow{\sigma}' := \overleftarrow{\sigma}^{\sigma}$ which generates $\sigma(X)$. We then apply Theorem 2.10 to obtain for any measure $\mu' \in \mathcal{M}(\sigma(X))$, a vector tower $\overleftarrow{v}'$ on $\overleftarrow{\sigma}'$ with $\mathfrak{m}_{\overleftarrow{\sigma}'}(\overleftarrow{v}') = \mu$. Truncating now the last term of $\overleftarrow{v}'$ gives a vector tower $\overleftarrow{v}$ on $\overleftarrow{\sigma}$, which by Remark 2.9(2) defines a measure $\mu := \mathfrak{m}_{\overleftarrow{\sigma}}(\overleftarrow{v})$ on X. We can now apply Proposition 4.3 to obtain $\mu' = \mu^{\sigma}$ $[= \sigma^{\mathcal{M}}(\mu)]$. □

Remark 4.5. We would like to note that as a result of the material presented in this section, we have now derived an alternative way of how to understand the transferred measure $\mu^{\sigma} = \sigma^{\mathcal{M}}(\mu) \in \mathcal{M}(\mathcal{B}^{\mathbb{Z}})$, for any non-erasing morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ and any invariant measure $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$.

It turns out that in many circumstances, the use of vector towers as presented here is more convenient when dealing with μ^{σ} in practice, compared with the definition as studied in [**5**, §§3 and 4], and also compared with the approximation method via weighted characteristic measures indicated in [**5**, Remark 3.9].

5. *Measure towers and vector towers*

Throughout this section, we will assume that

$$\overleftarrow{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n \geq 0}$$

is an everywhere growing directive sequence which generates a subshift $X = X_0 \subseteq \mathcal{A}_0^{\mathbb{Z}}$ (and where all level maps σ_n are non-erasing, see Remark 2.5). As in equation (2.15), we denote, for any level $k \geq 0$, by $X_k \subseteq \mathcal{A}_k^{\mathbb{Z}}$ the intermediate subshift of level k, which is generated by the truncated sequence $\overleftarrow{\sigma} \dagger_k = (\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n \geq k}$ from equation (2.14).

Definition 5.1. A *measure tower on* $\overleftarrow{\sigma}$, denoted by $\overleftarrow{\mu} = (\mu_n)_{n \geq 0}$, is given by a sequence of measures $\mu_n \in \mathcal{M}(\mathcal{A}_n^{\mathbb{Z}})$ which satisfy

$$\mu_n = \sigma_n^{\mathcal{M}}(\mu_{n+1}).$$

The set of measure towers on $\overleftarrow{\sigma}$ will be denoted by $\mathcal{M}(\overleftarrow{\sigma})$.

We will now construct a particular type of measure towers on a given directive sequence $\overleftarrow{\sigma}$ as above, starting from a vector tower $\overleftarrow{v} = (\vec{v}_n)_{n \geq 0}$ on $\overleftarrow{\sigma}$. We first observe that for any intermediate level $k \geq 0$, we obtain from $\overleftarrow{\sigma}$ via the truncated directive sequence $\overleftarrow{\sigma} \dagger_k$ a

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'truncated evaluation map' $\mathfrak{m}_k := \mathfrak{m}_{\overleftarrow{\sigma}\dagger_k} : \mathcal{V}(\overleftarrow{\sigma}\dagger_k) \to \mathcal{M}(X_k)$. From the vector tower $\overleftarrow{v}$, we obtain similarly a 'truncated' vector tower $\overleftarrow{v}\dagger_k = (\vec{v}_n)_{n \geq k}$ on $\overleftarrow{\sigma}\dagger_k$, which defines the corresponding shift-invariant 'level k measure'

$$\mu_k := \mathfrak{m}_k(\overleftarrow{v}\dagger_k) \tag{5.1}$$

on the subshift $X_k \subseteq \mathcal{A}_k^{\mathbb{Z}}$.

LEMMA 5.2.

(1) *For any vector tower $\overleftarrow{v}$ on an everywhere growing directive sequence $\overleftarrow{\sigma}$, the family of level k measures μ_k as in equation (5.1), for all $k \geq 0$, defines a measure tower $\overleftarrow{\mathfrak{m}}(\overleftarrow{v}) := (\mu_k)_{k \geq 0}$ on $\overleftarrow{\sigma}$.*

(2) *Conversely, every measure tower $\overleftarrow{\mu} = (\mu_n)_{n \geq 0}$ on a directive sequence $\overleftarrow{\sigma}$ as above determines a vector tower $\overleftarrow{\zeta}(\overleftarrow{\mu}) = (\vec{v}_n)_{n \geq 0}$ on $\overleftarrow{\sigma}$, given by the letter frequency vectors $\vec{v}_n := \vec{v}(\mu_n) = \zeta_{X_n}(\mu_n)$ from equations (2.9) and (2.10).*

Proof. (1) It suffices to observe that Proposition 4.3 gives directly $\sigma_k^{\mathcal{M}}(\mu_{k+1}) = \mu_k$ for all $k \geq 0$.

(2) We only need to verify that $\overleftarrow{\zeta}(\overleftarrow{\mu})$ is indeed a vector tower, that is, that compatibility conditions in equation (2.20) are satisfied. This is a direct application of of [**5**, Proposition 4.5], stated above as equation (3.2). □

The above set-up of measure towers and vector towers over a given directive sequence is very natural and, indeed, it turns out that the two are essentially equivalent. More precisely, we obtain the following result, which has been quoted in a notationally adapted version in §1 as Proposition 1.1.

PROPOSITION 5.3. *For any everywhere growing directive sequence $\overleftarrow{\sigma}$, there is a canonical $\mathbb{R}_{\geq 0}$-linear bijection*

$$\overleftarrow{\zeta} : \mathcal{M}(\overleftarrow{\sigma}) \to \mathcal{V}(\overleftarrow{\sigma})$$

between the cone of measure towers on one hand and the cone of vector towers on the other, given by the map

$$\overleftarrow{\mu} \mapsto \overleftarrow{\zeta}(\overleftarrow{\mu}) \quad \text{and its inverse} \quad \overleftarrow{v} \mapsto \overleftarrow{\mathfrak{m}}(\overleftarrow{v}).$$

Proof. The fact that the composition $\overleftarrow{\zeta} \circ \overleftarrow{\mathfrak{m}}$ gives the identity on $\mathcal{V}(\overleftarrow{\sigma})$ follows directly from Proposition 2.11(1), when applied to all truncated sequences $\overleftarrow{\sigma}\dagger_k$ with $k \geq 0$. We obtain in particular that the map $\overleftarrow{\mathfrak{m}}$ is injective.

However, we can apply Theorem 2.10 to each of the truncated sequences $\overleftarrow{\sigma}\dagger_k$ to obtain the surjectivity of the map $\mathfrak{m}_k : \mathcal{V}(\overleftarrow{\sigma}\dagger_k) \to \mathcal{M}(X_k)$ for any level $k \geq 0$. It follows then directly from the definition set up in Lemma 5.2 (1) above that the map $\overleftarrow{\mathfrak{m}} : \mathcal{V}(\overleftarrow{\sigma}) \mapsto \mathcal{M}(\overleftarrow{\sigma})$ must be surjective.

Hence, $\overleftarrow{\mathfrak{m}}$ is a bijective map, which implies that $\overleftarrow{\zeta}$ must also be bijective, and that $\overleftarrow{\mathfrak{m}} \circ \overleftarrow{\zeta}$ is the identity on $\mathcal{M}(\overleftarrow{\sigma})$.

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The linearity of the maps $\overleftarrow{\zeta}$ and $\overleftarrow{\mathfrak{m}}$ is a direct consequence of the linearity (see §2.2) of the maps $\mathfrak{m}_k$ and ζ_{X_n} used in the above definitions of the measure or vector towers $\overleftarrow{\mathfrak{m}}(\overleftarrow{v}) = (\mathfrak{m}_k(\overleftarrow{v}\dagger_k))_{k\geq 0}$ and $\overleftarrow{\zeta}(\overleftarrow{\mu}) = (\zeta_{X_n}(\mu_n))_{n\geq 0}$, respectively. □

Although slightly similar in notation, the two cones $\mathcal{M}(\overleftarrow{\sigma})$ and $\mathcal{M}(X_{\overleftarrow{\sigma}})$ should not be confused. Indeed, without further assumptions on the given set-up, the structure of the cone $\mathcal{M}(\overleftarrow{\sigma})$ of measure towers will not only depend on the given subshift $X = X_{\overleftarrow{\sigma}}$ but can vary quite a bit depending on the choice of the S-adic development $\overleftarrow{\sigma}$ of X. More precisely, we have the following remark.

Remark 5.4. For any everywhere growing directive sequence $\overleftarrow{\sigma}$ which generates a subshift $X = X_{\overleftarrow{\sigma}}$, the composition

$$\mathfrak{m}_{\overleftarrow{\sigma}} \circ \overleftarrow{\zeta} : \mathcal{M}(\overleftarrow{\sigma}) \to \mathcal{M}(X) \tag{5.2}$$

is $\mathbb{R}_{\geq 0}$-linear and surjective since $\overleftarrow{\zeta}$ is $\mathbb{R}_{\geq 0}$-linear and bijective by Proposition 5.3, and $\mathfrak{m}_{\overleftarrow{\sigma}}$ is $\mathbb{R}_{\geq 0}$-linear and surjective by Theorem 2.10. However, in general, the map $\mathfrak{m}_{\overleftarrow{\sigma}} \circ \overleftarrow{\zeta}$ will be far from being injective.

We thus consider the following strengthening on the hypotheses of the given directive sequence, which has been considered already by several other authors in a related context (compare [**9**, Definition 4.1] or [**13**, §3.3]).

Definition 5.5. A directive sequence (or an S-adic development) $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ is called *totally recognizable* if every level map σ_n is recognizable in the corresponding subshift X_{n+1} (see Definition 3.5 and Proposition 3.6). If all but finitely many of the level maps σ_n are recognizable in X_{n+1}, we call $\overleftarrow{\sigma}$ *eventually recognizable*.

The following result is quoted in a slightly shortened version in §1 as Theorem 1.4.

THEOREM 5.6. *For any everywhere growing totally recognizable S-adic development $\overleftarrow{\sigma}$ of a subshift X and its associated cone $\mathcal{V}(\overleftarrow{\sigma})$ of vector towers, the canonical $\mathbb{R}_{\geq 0}$-linear map*

$$\mathfrak{m}_{\overleftarrow{\sigma}} : \mathcal{V}(\overleftarrow{\sigma}) \to \mathcal{M}(X)$$

is a bijection.

In particular, for any level $n \geq 0$, the map $\sigma^{\mathcal{M}}_{[0,n)} : \mathcal{M}(X_n) \to \mathcal{M}(X)$ is a linear bijection of cones. Similarly, the same conclusion follows for the map $\mathfrak{m}_{\overleftarrow{\sigma}} \circ \overleftarrow{\zeta}$ from equation (5.2).

Proof. From the assumption that $\overleftarrow{\sigma}$ is totally recognizable, it follows (using statement (3d) of Theorem 3.2) that the induced $\mathbb{R}_{\geq 0}$-linear map

$$(\sigma_n)^{\mathcal{M}}_{X_{n+1}} : \mathcal{M}(X_{n+1}) \to \mathcal{M}(X_n)$$

is bijective for any level $n \geq 0$. It follows that the composed map $m_{\overleftarrow{\sigma}} \circ \overleftarrow{\zeta} : \mathcal{M}(\overleftarrow{\sigma}) \to \mathcal{V}(\overleftarrow{\sigma}) \to \mathcal{M}(X)$ from equation (5.2) is bijective. Since we know from Proposition 5.3 that the map $\overleftarrow{\zeta}$ is a bijection, we deduce that $m_{\overleftarrow{\sigma}}$ must be bijective. □

6. *Directive sequences with 'small' intermediate letter frequency cones*

In this section, we will give a first application of the machinery set up in the previous two sections. However, before doing so, we want to summarize, for the convenience of the reader, the various ingredients that the rich picture issuing from this set-up offers, and to list some basic facts to avoid potential misunderstandings. As an illustration, we give at the end of this section a detailed example, where all the data listed now can be seen in practice.

We use the same terminology as previously, that is, $X \in \mathcal{A}^{\mathbb{Z}}$ is a subshift over the finite alphabet $\mathcal{A} = \mathcal{A}_0$, and $\overleftarrow{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ is an everywhere growing directive sequence which generates X. For the subsequent discussion, the following notion will be helpful.

Definition 6.1.

(1) For any integer $n \geq 0$, the *intermediate letter frequency cone* $\mathcal{C}_n = \mathcal{C}_n(\overleftarrow{\sigma}) \subseteq \mathbb{R}_{\geq 0}^{\mathcal{A}_n}$ of the directive sequence $\overleftarrow{\sigma}$ is given, through the level subshifts X_n (see equation (2.15)) and their measure cones $\mathcal{M}(X_n)$, via equation (2.11) by

$$\mathcal{C}_n := \mathcal{C}(X_n) = \zeta_{X_n}(\mathcal{M}(X_n)).$$

As set up in §2, here $\zeta_{X_n} : \mathcal{M}(X_n) \to \mathbb{R}_{\geq 0}^{\mathcal{A}_n}$ is given for $\mathcal{A}_n = \{a_{n,1}, \ldots, a_{n,d(n)}\}$ by $\mu \mapsto ([\mu(a_{n,1})], \ldots, [\mu(a_{n,d(n)})])$ for any $\mu \in \mathcal{M}(X_n)$.

(2) We denote the dimension of the cone $\mathcal{C}_n$ by c_n, that is,

$$c_n := \dim \mathcal{C}_n \leq \operatorname{card}(\mathcal{A}_n).$$

Remark 6.2. From Proposition 2.11, applied to the truncated directive sequence $\overleftarrow{\sigma}\dagger_n$, we observe directly that the cone $\mathcal{C}_n$ is the image of the set $\mathcal{V}(\overleftarrow{\sigma})$ of vector towers under the level n projection map pr_n, which amounts to stating that $\mathcal{C}_n$ is the intersection of the nested images of the non-negative alphabet cones of level $m \geq n$ under the telescoped level maps:

$$\mathcal{C}_n = \bigcap \{\mathbb{R}_{\geq 0}^{\mathcal{A}_n} \supseteq \cdots \supseteq M(\sigma_{[n,m)})(\mathbb{R}_{\geq 0}^{\mathcal{A}_{m+1}}) \supseteq \cdots\}.$$

In particular, one always has

$$\mathcal{C}_n = M(\sigma_n)(\mathcal{C}_{n+1}) \tag{6.1}$$

and thus

$$c_n \leq c_{n+1}$$

for all $n \geq 0$.

Our main focus here is to explain how this set-up and in particular the value of the c_n can be used to find out information about the number $e(X) \in \mathbb{N} \cup \{\infty\}$ of invariant ergodic probability measures on X.

Remark 6.3. Under the above stated conditions, the following conclusions are immediate.

(1) It is quite possible that $e(X) > c_n$ for some 'low' level $n \geq 0$, even if $\overleftarrow{\sigma}$ is totally recognizable.

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(2) The converse inequality, $e(X) < c_n$, is also possible, but in this case, the directive sequence $\overleftarrow{\sigma}$ is not totally recognizable. More precisely, in this case, the telescoped morphism $\sigma_{[0,n)}$ is not recognizable.

(3) In any case, we always have

$$e(X) \leq \lim c_n \leq \lim\inf(\operatorname{card} \mathcal{A}_n),$$

but in general, both inequalities may well be strict.

(4) However, if $\overleftarrow{\sigma}$ is totally recognizable, then we have

$$e(X) = \lim c_n.$$

In particular, we recover the well-known upper bound $e(X) \leq \lim\inf(\operatorname{card} \mathcal{A}_n)$, as well as the lower bounds $c_n \leq e(X)$ for all $n \geq 0$.

From Remark 6.3(3), we observe directly that for any directive sequence $\overleftarrow{\sigma}$ with finite alphabet rank (that is, $\lim\inf(\operatorname{card} \mathcal{A}_n) < \infty$), there is a *critical level* $n_0 \geq 0$ such that one has

$$c_n = c_{n_0} \quad \text{for all } n \geq n_0 \text{ and } c_n < c_{n_0} \text{ for all } n < n_0. \tag{6.2}$$

More generally, any everywhere growing directive sequence $\overleftarrow{\sigma}$ (possibly with infinite alphabet rank) which possesses such a critical level has been termed in [3] *thinning*, and in the particular case where the critical level agrees with the base level $n_0 = 0$, the sequence $\overleftarrow{\sigma}$ has been called *thin*. Of course, any thinning sequence can be made thin by simply truncating it at its critical level (or any level higher up); furthermore, we can telescope all levels below the critical level into a single 'thinning' morphism. Subshifts that are 'thin' in that they are generated by a thin (and in particular everywhere growing) directive sequence have the following useful property.

PROPOSITION 6.4. [3] *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift generated by a thin directive sequence $\overleftarrow{\sigma}$. Then the letter frequency map $\zeta_X : \mathcal{M}(X) \to \mathbb{R}_{\geq 0}^{\mathcal{A}}$ co-restricts to a $\mathbb{R}_{\geq 0}$-linear bijection*

$$\mathcal{M}(X) \to C(X), \mu \mapsto (\mu(a_k))_{a_k \in \mathcal{A}}.$$

In particular, any two invariant measures μ_1 and μ_2 on X are equal if and only if one has $\mu_1([a_k]) = \mu_2([a_k])$ for the finitely many cylinders $[a_k]$ given by all letters $a_k \in \mathcal{A}$.

This statement can be derived directly from [3, Proposition 10.2(1) and Corollary 10.4]. For convenience of the reader, we give here a proof in the terminology introduced above.

Proof of Proposition 6.4. For any two measure $\mu, \mu' \in \mathcal{M}(X)$, there exist, by Theorem 2.10, vector towers $\overleftarrow{v} = (\vec{v}_n)_{n\geq 0}$ and $\overleftarrow{v}' = (\vec{v}_n^{\,\prime})_{n\geq 0}$ on $\overleftarrow{\sigma}$ with $\mathfrak{m}_{\overleftarrow{\sigma}}(\overleftarrow{v}) = \mu$ and $\mathfrak{m}_{\overleftarrow{\sigma}}(\overleftarrow{v}') = \mu'$. Thus, $\mu \neq \mu'$ implies $\overleftarrow{v} \neq \overleftarrow{v}'$ and hence $\vec{v}_n \neq \vec{v}_n^{\,\prime}$ for some $n \geq 0$. However, then we deduce from equation (6.1) and the hypothesis that $\dim \mathcal{C}(X_n) = c_n = c_0 = \dim \mathcal{C}(X)$ that $\vec{v}_0 = M(\sigma_{[0,n)})(\vec{v}_n) \neq M(\sigma_{[0,n)})(\vec{v}_n^{\,\prime}) = \vec{v}_0^{\,\prime}$. From Proposition 2.11(1), we know that $\vec{v}_0 = pr_0(\overleftarrow{v}) = \zeta_X(\mu)$ and $\vec{v}_0\,' = pr_0(\overleftarrow{v}\,') = \zeta_X(\mu')$, which shows that the map ζ_X is injective. For the linearity of ζ_X and the equality $\zeta_X(\mathcal{M}(X)) = \mathcal{C}(X)$, see equations (2.10) and (2.11). □

Directive sequences of finite alphabet rank occur naturally in many important contexts in the symbolic dynamics literature (e.g. substitutive subshifts, IETs, etc). Furthermore, the extra invertibility condition from the following Corollary 6.5 is rather frequently satisfied. This corollary has been quoted as Corollary 1.6 in §1; it is stated here again for the convenience of the reader.

COROLLARY 6.5. *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift generated by an everywhere growing directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ of finite alphabet rank. Assume that for every $n \geq 0$, the incidence matrix $M(\sigma_n)$ is invertible over $\mathbb{R}$. Then any invariant measure μ on the subshift X is determined by the evaluation of μ on the letter cylinders, that is, by the values $\mu([a_k])$ for all $a_k \in \mathcal{A}$.*

Proof. From equation (6.1) and the hypothesis that $M(\sigma_n)$ is invertible, it follows directly that $c_{n+1} = c_n$ for all $n \geq 0$, so that the directive sequence $\overleftarrow{\sigma}$ is thin. Hence, the hypotheses of Proposition 6.4 are satisfied, which gives directly the claimed statement. □

Note that the conclusion of Corollary 6.5 has recently been proved by Berthé *et al* under somewhat more restrictive hypotheses (see [**6**, Corollary 4.2]); in particular, it is required there that every $M(\sigma_n)$ has determinant equal to 1 or to −1, and that X is minimal.

Remark 6.6.

(1) If in Proposition 6.4 the hypothesis 'thin' is replaced by 'thinning', with critical level $n_0 \geq 1$, then the conclusion that any two distinct measures $\mu \neq \mu' \in \mathcal{M}(X)$ can be distinguished by the evaluation on the letter cylinders $[a_k]$ for all $a_k \in \mathcal{A}$, may in some cases still hold, despite the fact that from the definition of the critical level, we have

$$\dim \mathcal{C}_0 = c_0 < c_{n_0} = \dim \mathcal{C}_{n_0} = \dim \mathcal{M}(X_{n_0}).$$

Here the last equality follows from Proposition 6.4, applied to the directive sequence truncated at the critical level n_0. The reason why the above strict inequality does not contradict the presumed equality $c_0 = \dim \mathcal{C}_0 = \dim \mathcal{M}(X)$ is that the measure transfer map $\sigma^{\mathcal{M}}_{[0,n_0)} : \mathcal{M}(X_{n_0}) \to \mathcal{M}(X)$ may well not be injective, in the case that the telescoped level map $\sigma_{[0,n_0)}$ is not recognizable in the level subshift X_{n_0}.

However, if $\overleftarrow{\sigma}$ is totally recognizable, or if at least $\sigma_{[0,n_0)}$ is recognizable in X_{n_0}, and if furthermore $\overleftarrow{\sigma}$ is thinning but not thin, then the conclusion of Proposition 6.4 necessarily fails: this case is treated in Example 6.7 below.

(2) In view of the fact that the measure transfer map $\sigma^{\mathcal{M}}$ induced by a non-recognizable monoid morphism σ is in general far from being injective, it seems noteworthy that in Proposition 6.4 and Corollary 6.5, no recognizability condition on the level maps σ_n is imposed. One should recall in this context that in [**9**, Theorem 5.2], it has been proved that directive sequences of bounded alphabet rank, with aperiodic

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level subshifts, are 'eventually recognizable', that is, all level maps above some 'other critical level' must be recognizable in their level subshift. However, this 'other critical level' may well be a lot bigger than the above critical level n_0 and, indeed, we give in Corollary 8.5(2) examples of thin directive sequences where this 'other critical level' can be chosen to be arbitrarily high up, while none of the level morphisms below it is recognizable in its corresponding level subshift (which is aperiodic for any level).

We now present the promised 'detailed example with all above data made visible'.

Example 6.7. The subshift X in this example consists of two periodic words and is hence all by itself not so interesting. We chose it to give a transparent presentation of a simple subshift via some not so obvious everywhere growing directive sequence, which we now describe in detail. We first describe the level $n = 1$, then pass to the base level $n = 0$, and finally build the higher levels $n \geq 2$ on top of the two lowest levels. We also include for each level n a description of the measure cone $\mathcal{M}(X_n)$ and of the associated letter frequency cone $\mathcal{C}_n$.

Set $\mathcal{A}_1 = \{a, b\}$ and let $X_1 \subseteq \mathcal{A}_1^{\mathbb{Z}}$ be the union of the two periodic subshifts $\mathcal{O}(w^{\pm\infty})$ and $\mathcal{O}(w'^{\pm\infty})$, defined by the words $w = a^2b$ and $w' = b^2a$. We consider the two characteristic measures $\mu := \mu_w$ and $\mu' := \mu_{w'}$, and observe that $\mathcal{M}(X_1)$ consists of all non-negative linear combinations of these two measures. The letter frequency map $\zeta_{X_1} : \mathcal{M}(X_1) \to \mathcal{C}(X_1) \subseteq \mathbb{R}_{\geq 0}^{\{a,b\}}$ is injective, in that $\zeta_{X_1}(\mu) = 2\vec{e}_a + \vec{e}_b$ and $\zeta_{X_1}(\mu') = \vec{e}_a + 2\vec{e}_b$. This results in $c_1 = \dim(\mathcal{C}_1) = 2$.

For $\mathcal{A}_0 = \{c, d\}$, consider now the 'Thue–Morse' morphism $\sigma_0 : \mathcal{A}_1^{\mathbb{Z}} \to \mathcal{A}_0^{\mathbb{Z}}$, $a \mapsto cd$, $b \mapsto dc$, and recall (see Proposition 3.3(d)) that $\sigma_0^{\mathcal{M}}(\mu) = \mu_{\sigma_0(w)}$ and $\sigma_0^{\mathcal{M}}(\mu') = \mu_{\sigma_0(w')}$, with $\sigma_0(w) = cdcddc$ and $\sigma_0(w') = dcdccd$. Since $cdcddc$ and $dcdccd$ cannot be obtained from each other by a cyclic permutation, we have $\mathcal{O}((cdcddc)^{\pm\infty}) \neq \mathcal{O}((dcdccd)^{\pm\infty})$, so that from equation (2.8), it follows that $\text{Supp}(\mu_{cdcddc}) \neq \text{Supp}(\mu_{dcdccd})$. We thus deduce for the image subshift $X_0 = \sigma_0(X_1)$ that the measure cone $\mathcal{M}(X_0)$, which is spanned by μ_{cdcddc} and μ_{dcdccd}, is of dimension 2.

However, using Proposition 3.4 (or more directly, equation (2.7)), we readily compute $\mu_{cdcddc}([cd]) = \mu_{cdcddc}([dc]) = 2$ as well as $\mu_{dcdccd}([cd]) = \mu_{dcdccd}([dc]) = 2$. It follows that the frequency map $\zeta = \zeta_{X_0}$ is not injective and that $\mathcal{C}_0$ has dimension $c_0 = 1$.

We now define the higher up levels of the directive sequence by setting $\mathcal{A}_n = \{x, y\}$ for any $n \geq 2$, and by defining all level morphisms $\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n$ for $n \geq 2$ to be equal to the substitution defined by $x \mapsto x^2$, $y \mapsto y^2$. It follows that for $n \geq 2$, all level subshifts X_n consist of the two biinfinite periodic words $x^{\pm\infty}$ and $y^{\pm\infty}$. Moreover, we easily see that the incidence matrix of σ_n is equal to two times the two-by-two identity matrix I_2, that is, $M(\sigma_n) = 2 \cdot I_2$, so that we have $\mathcal{M}(X_n) = \mathcal{C}_n = \mathbb{R}_{\geq 0}^{\{x,y\}}$.

It remains now to define $\sigma_1 : \mathcal{A}_2 \to \mathcal{A}_1$ via $x \mapsto w$, $y \mapsto w'$, which ensures $\sigma_1(X_2) = X_1$, to obtain a directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ over alphabets that all have cardinality 2. We have shown above that the critical level of this directive sequence is $n_0 = 1$, while the evaluation on the cylinders $[\sigma_0(a)] = [cd]$ and $[\sigma_0(b)] = [dc]$ does not suffice to distinguish the two measures $\mu_{cdcddc} \neq \mu_{dcdccd}$ that span $\mathcal{M}(X_0)$.

7. *Minimal subshifts with zero entropy and infinitely many ergodic probability measures*

A subshift X, which is 'small' in that it has topological entropy $h_X = 0$ (see equation (2.3)), and simultaneously 'large' in that the number $e(X)$ of ergodic probability measures carried by X is infinite, is a bit of a contradiction in itself (if one restricts to non-atomic measures). However, such subshifts are known to exist, but they are not easy to come by. One of the first such subshift known to us is the Pascal-adic subshift, treated in [**22**]; more recent such examples (with additional strong properties, in particular minimality) are exhibited in [**11**]. Not surprisingly, there is always a certain amount of work involved to simultaneously achieve the above two opposite properties.

In this section, we will present an alternative way to construct minimal subshifts X which satisfy both, $h_X = 0$ and $e(X) = \infty$. The main purpose of this section is to underline how directly such examples can be exhibited by means of the technology established in the previous sections.

We first recall two known results. The first appears as [**7**, Theorem 4.3] and is attributed there to Thierry Monteil; alternatively, it can be found in [**8**] as Lemma 6.7.1 of Ch. 6, written by Fabien Durand, who told us that the result can actually be traced back to the paper [**10**] by Boyle and Handelman.

PROPOSITION 7.1. *Let X be a subshift which is generated by a directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ with level alphabets $\mathcal{A}_n$. Then, for the minimal level letter image length $\beta_-(n)$ from equation (2.17), one has*

$$h_X \leq \inf_{n\geq 0} \frac{\log(\text{card } \mathcal{A}_n)}{\beta_-(n)}.$$

PROPOSITION 7.2. [**4**, §4.1] *For any integer $d \geq 2$, let X be a subshift which is generated by a directive sequence $\overleftarrow{\sigma} = (\sigma_{n,d})_{n\geq 0}$ with level alphabets that are all of uniform cardinality d (and are thus identified with $\mathcal{A}_{(d)} = \{a_1, \ldots, a_d\}$). Assume that for any level $n \geq 0$, the incidence matrix of the level map $\sigma_{n,d}$ is given by*

$$M(\sigma_{n,d}) = M_{\ell(n),d} := \ell(n) I_d + 1_{d\times d}, \tag{7.1}$$

where I_d is the identity matrix of size $d \times d$, $1_{d\times d}$ is the $d \times d$ matrix with all entries equal to 1, and $\ell(n)$ is a positive integer depending on n.

Then X is minimal, and for any sufficiently fast growing sequence $(\ell(n))_{n\in\mathbb{N}}$, the subshift X admits precisely d distinct invariant ergodic probability measures.

The use of Proposition 7.1 will be an ingredient below in our proof of Theorem 7.4. Proposition 7.2, however, will not be formally used in the following, but it may pay anyway for the reader to look it up. We use below the very same basic idea as in this earlier result, but do not carry out all calculations as had been done in [**4**, §4] (where, in particular, precise lower bounds for the integers $\ell(n)$ are computed which guarantee the 'sufficiently fast growing' in the above statement).

For the proof below, we first need to define for any integer $d \geq 2$ and alphabet $\mathcal{A}_{(d)} = \{a_1, \ldots, a_d\}$, the morphism $\tau_d : \mathcal{A}^*_{(d+1)} \to \mathcal{A}^*_{(d)}$, given by $a_i \mapsto a_i^2$ for any a_i with $1 \leq i \leq d$ and $a_{d+1} \mapsto a_1 a_2 \ldots a_d$.

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Remark 7.3.

(1) For the morphism τ_d as given above, it is easy to see that any biinfinite word $\mathbf{y} \in \mathcal{A}_{(d)}^{\mathbb{Z}} \setminus \{a_1^{\pm\infty}, a_2^{\pm\infty}, \ldots, a_d^{\pm\infty}\}$ can be 'desubstituted' in at most one way (compare [**5**, Remark 6.2(2)]) to give a biinfinite word $\mathbf{x} \in \mathcal{A}_{(d+1)}^{\mathbb{Z}}$ with $\tau_d(\mathcal{O}(\mathbf{x})) = \mathcal{O}(\mathbf{y})$. Since for any $i = 1, \ldots, n$, the periodic word $a_i^{\pm\infty}$ is the only element $\mathbf{x} \in \mathcal{A}_{(d+1)}^{\mathbb{Z}}$ with $\tau_d(\mathcal{O}(\mathbf{x})) = \mathcal{O}(a_i^{\pm\infty})$, it follows that τ_d is recognizable in every subshift which does not contain any of the periodic words $a_i^{\pm\infty}$.

(2) Again by elementary desubstitution arguments, one verifies quickly that any morphism $\sigma_{n,d}$ with incidence matrix given by equation (7.1), with $\ell(n) \geq 2$, is recognizable in the full shift $\mathcal{A}_{(d)}^{\mathbb{Z}}$.

(Indeed, it suffices to check in any biinfinite word $\mathbf{y} \in \sigma_{n,d}(\mathcal{A}_{(d)}^{\mathbb{Z}})$ for a factor $w \in \mathcal{A}_{(d)}^{*}$ which is 'distinguished' in that some letter $a_i \in \mathcal{A}_{(d)}$ occurs precisely three times in w, while all other letters $a_j \in \mathcal{A}_{(d)}$ occur at most twice. Such a distinguished word w occurs in $\sigma_{n,d}(a_i)$, and any such occurrence is contained in the image of some word from $\mathcal{A}_{(d)}^{*}$ of length at most 3. In either case, one verifies quickly that the middle occurrence of a_i in w must belong to $\sigma_{n,d}(a_i)$. For this middle occurrence y_s in the factor $w = y_r \ldots y_t$ of $\mathbf{y} = \cdots y_{n-1} y_n y_{n+1} \cdots$, one considers the factors $w_+ = y_s \ldots y_{t'}$ and $w_- = y_{r'} \ldots y_s$ of $\mathbf{y}$, with $y_s = y_{s+1} = \cdots = y_{t'-1} = a_i$ and $y_{t'} \neq a_i$, and similarly $y_s = y_{s-1} = \cdots = y_{r'+1} = a_i$ and $y_{r'} \neq a_i$. From the fact that $\sigma_{n,d}(a_i)$ contains each letter $a_j \neq a_i$ precisely once, one deduces directly that the words w_+ and w_- determine which occurrence of a_i in $\sigma_{n,d}(a_i)$ is given by the letter y_s. It follows that, starting from y_s, the biininite word $\mathbf{y}$ can be desubstituted in precisely one way.)

(3) From the conditions on the incidence matrix $M(\sigma_n)$ in equation (7.1), it follows directly that every word in $\sigma_{n,d}(\mathcal{A}_{(d)}^{*})$ must contain each of the letters of $\mathcal{A}_{(d)}$. Hence, we observe that $\sigma_{n,d}(\mathcal{A}_{(d)}^{\mathbb{Z}})$ cannot contain any of the periodic words $a_i^{\pm\infty}$.

(4) As a consequence of the above observations (1)–(3), we deduce for the following 'alternating' directive sequence

$$\overleftarrow{\sigma} = \sigma_2 \circ \tau_2 \circ \sigma_3 \circ \tau_3 \circ \cdots, \tag{7.2}$$

where we set $\sigma_d := \sigma_{d,d}$, that each level map is recognizable in its corresponding level subshift, so that the sequence $\overleftarrow{\sigma}$ is fully recognizable.

THEOREM 7.4. *For any integer $d \geq 2$, let $\mathcal{A}_{(d)} = \{a_1, \ldots, a_d\}$ and let $\sigma_d : \mathcal{A}_{(d)}^{*} \to \mathcal{A}_{(d)}^{*}$ be a morphism with incidence matrix $M(\sigma_d) = M_{\ell(d),d}$ from equation (7.1), for some integer $\ell(d) \geq 2$ depending on d. Let X be the subshift generated by the alternating directive sequence $\overleftarrow{\sigma}$ given in equation (7.2).*

If the exponent sequence $(\ell(n))_{n\in\mathbb{N}}$ is sufficiently fast growing, then the subshift X is minimal, has entropy $h_X = 0$, and admits infinitely many distinct invariant ergodic probability measures.

(We denote by $\mathfrak{X}$ the class of all subshifts $X \subseteq \mathcal{A}_{(d)}^{\mathbb{Z}}$ which satisfy all of the above conditions.)

Proof. For each integer $d \geq 2$, we identify the finite alphabet $\mathcal{A}_{(d)} = \{a_1, a_2, \ldots, a_d\}$ with the corresponding subset of an infinite alphabet, via $\mathcal{A}_{(2)} \subseteq \mathcal{A}_{(3)} \subseteq \cdots \subseteq \mathcal{A}_{(\infty)} = \{a_1, a_2, \ldots\}$. For the issuing infinite non-negative cone $\mathbb{R}_{\geq 0}^{\mathcal{A}_{(\infty)}}$, we abbreviate for notational convenience the base unit vectors to $\vec{e}_i := \vec{e}_{a_i}$.

For any level $n = 2d - 2$ or $n = 2d - 1$, we consider the subcone $\mathcal{C}^n := \mathbb{R}_{\geq 0}^{\mathcal{A}_{(d)}} \subseteq \mathbb{R}_{\geq 0}^{\mathcal{A}_{(\infty)}}$ and, in particular, the 'center vector' $\vec{c}_n = \sum \vec{e}_i$ of $\mathcal{C}^n$. We observe that both families, the morphisms σ_d as well as the morphisms τ_d, induce maps $M(\sigma_d) : \mathcal{C}^n \to \mathcal{C}^n$ and $M(\tau_d) : \mathcal{C}^{n+1} \to \mathcal{C}^n$, respectively, which each maps the center vector $\vec{c}_n$ (for σ_n) or $\vec{c}_{n+1}$ (for τ_n) to a scalar multiple of the center vector $\vec{c}_n$. Furthermore, any unit vector $\vec{e}_i$ with $1 \leq i \leq d$ is mapped by both $M(\sigma_d)$ and $M(\tau_d)$ to a non-negative linear combination $\lambda_1 \vec{e}_i + \lambda_2 \vec{c}_d$. Note here that (again for both σ_d and τ_d,)

$$\text{the quotient } \frac{\lambda_2}{\lambda_1} \text{ can be made arbitrarily small} \tag{7.3}$$

by choosing $\ell(d)$ sufficiently large.

We now fix some level $n_0 = 2d - 2 \geq 0$, and for any index i with $1 \leq i \leq d$, we look for a vector tower $\overleftarrow{v}_i = (\vec{v}_n^{\,i})_{n \geq n_0}$ on the truncated directed sequence $\overleftarrow{\sigma}\dagger_{n_0} = \sigma_{n_0} \circ \tau_{n_0} \circ \sigma_{n_0+1} \circ \tau_{n_0+1} \circ \cdots$ with the property that $\overleftarrow{v}_i$ has for any level $n \geq n_0$, a level vector $\vec{v}_n^{\,i} = \lambda_{1,n} \vec{e}_i + \lambda_{2,n} \vec{c}_n$, with coefficients

$$\lambda_{1,n} > 0 \quad \text{and} \quad \lambda_{2,n} > 0 \tag{7.4}$$

(which must both tend to 0 for $n \to \infty$). From equation (7.3), we deduce that a sufficiently large choice of the exponents $\ell(d)$ effects indeed that there exist families of such coefficients where both of the inequalities in equation (7.4) are satisfied, while the compatibility condition in equation (2.20) is maintained, for any $n \geq n_0$. It follows that on the lowest level $n = n_0$ (and thus similarly also on all levels $n \geq n_0$), the level vectors $\vec{v}_{n_0}^{\,1}, \vec{v}_{n_0}^{\,2}, \ldots, \vec{v}_{n_0}^{\,d}$ are linearly independent.

For the level subshift $X_{n_0} \subseteq \mathcal{A}_{(d)}^{\mathbb{Z}}$, generated by the truncated sequence $\overleftarrow{\sigma}\dagger_{n_0}$, the truncated evaluation map $\mathfrak{m}_{n_0} := \mathfrak{m}_{\overleftarrow{\sigma}\dagger_{n_0}} \overleftarrow{\mathcal{V}}(\overleftarrow{\sigma}\dagger_{n_0}) \to \mathcal{M}(X_{n_0})$ from equation (5.1) defines d invariant measures $\mu_1, \ldots, \mu_d$ on the level subshift X_{n_0} as images of the d vector towers $\overleftarrow{v}_1, \ldots, \overleftarrow{v}_d$, respectively:

$$\mu_i = \mathfrak{m}_{n_0}(\overleftarrow{v}_i).$$

It follows from Proposition 2.11(1) that the subcone

$$\mathcal{M}_{n_0} := \mathbb{R}_{\geq 0} \langle \mu_1, \ldots, \mu_d \rangle \subseteq \mathcal{M}(X_{n_0})$$

spanned by the μ_i has dimension d. Since we verified in Remark 7.3(4) above that each of the maps σ_j and τ_j is recognizable in its corresponding level subshift, it follows from Theorem 3.2 (3d) that $\mathcal{M}_{n_0}$ is mapped by $\sigma_2^{\mathcal{M}} \circ \tau_2^{\mathcal{M}} \circ \cdots \circ \sigma_{n_0-1}^{\mathcal{M}} \circ \tau_{n_0-1}^{\mathcal{M}}$ to a subcone of $\mathcal{M}(X)$ that also has dimension d.

We have thus proved that $\mathcal{M}(X)$ contains subcones of arbitrary large dimension, and hence must be infinite dimensional, that is, $e(X) = \infty$. The desired equality $h_X = 0$ is immediate from Proposition 7.1 for large $\ell(d)$, and the minimality of X follows directly from the positivity of the matrices $M(\sigma_d)$, see Remark 2.8(2). □

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8. *Non-recognizable directive sequences*

The purpose of this section is to show how non-recognizable morphisms appear naturally in a well-known context (IETs and pseudo-Anosov surface homeomorphisms), and how this phenomenon can be exploited to construct interesting directive sequences that are not totally recognizable or even not eventually recognizable.

Our construction will be presented in four steps, organized below as follows. In §8.1, we present our basic quotient construction in geometric language. In §8.2, we show how the canonical 'inverse quotient construction' is obtained in a natural geometric context to define a non-recognizable monoid morphism. In §8.3, the results from the previous subsections are properly 'pasted together' to give a directive sequence where every level morphism is non-recognizable (and, in addition, it is a particularly nice letter-to-letter factor map). Finally, in §8.4, we modify this sequence slightly to obtain the desired everywhere growing but not (eventually) recognizable directive sequences. Note that all intermediate level subshifts which occur in our constructions turn out to be minimal; they are furthermore both substitutive and IET.

Before starting the detailed description, we will highlight its essential features in a special case, in a language that may be more easily accessible to those of us who are less familiar with Thurston's work on surface homeomorphisms.

Remark 8.1.

(1) Let us consider the tiling of the real plane $\mathbb{R}^2$ by squares of side length 1 that have their vertices on the points with integer coordinates. We now pick a slope s, say $0 < s < 1$, and we foliate the plane by lines that have slope s. By choosing the slope s to be irrational, we make sure that on any line of the foliation, there is at most one vertex of our square tiling. To every line ℓ that avoids any such vertex, one can associate canonically a biinfinite word $w(\ell)$ in the letters h and v, which records the sequence of intersections of ℓ with a horizontal ('h') or vertical ('v') line of our square grid. To fix an indexing of the letters of $w(\ell)$, we pick a distinguished 'base square' Q and require that ℓ passes through the interior of Q. We quickly observe that the orbits in our family of lines ℓ, with respect to the canonical $\mathbb{Z} \oplus \mathbb{Z}$-action on $\mathbb{R}^2$, are in one-to-one relation with the shift-orbits of the resulting set of words $w(\ell)$. Indeed, for this one-to-one relation, it suffices to consider the positive half-words of any $w(\ell)$, so that it extends naturally to the lines ℓ that pass over any of the vertices.

We consider now more closely any of the 'troublesome' lines ℓ_P that cross over a vertex P of the square grid. To ℓ_P, we associate two words $w_{\text{above}}(\ell_P)$ and $w_{\text{below}}(\ell_P)$ in $\{h, v\}^{\mathbb{Z}}$, which are read off from ℓ_P after isotopying it slightly in the neighborhood of P so that it passes either above or below P. From the above observed one-to-one relation between the $\mathbb{Z} \oplus \mathbb{Z}$-orbits of the lines ℓ and the shift-orbits of the corresponding words $w(\ell)$, we deduce that the words $w_{\text{above}}(\ell_P)$ and $w_{\text{below}}(\ell_P)$ do not belong to the same shift-orbit.

The set $X_s \subseteq \{h, v\}^{\mathbb{Z}}$ of all biinfinite words $w(\ell)$, including the above defined $w_{\text{above}}(\ell_P)$ and $w_{\text{below}}(\ell_P)$, for any line ℓ that passes through our distinguished base square Q, is a subshift—indeed, a well-known Sturmian subshift.

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(2) We now proceed by subdividing the top and bottom side of each square into segments of equal length through introducing a new vertex at the midpoint of any horizontal segment of the square grid. Any transition of a line ℓ through the left half of the subdivided horizontal square side will now be recorded by the letter h_{left}, and any transition through the right half by h_{right}, to give a new biinfinite word $w'(\ell) \in \{h_{\text{left}}, h_{\text{right}}, v\}^{\mathbb{Z}}$. The morphism $\sigma : \{h_{\text{left}}, h_{\text{right}}, v\}^{\mathbb{Z}} \to \{h, v\}^{\mathbb{Z}}$ defined by $h_{\text{left}} \mapsto h$, $h_{\text{right}} \mapsto h$ and $v \mapsto v$ maps any $w'(\ell)$ to $w(\ell)$, and it will be one-to-one, except for the new 'troublesome' lines ℓ_R that pass through any of the new vertices R in the middle of our original horizontal square grid intervals. For such lines, we have as before two words $w'_{\text{above}}(\ell_R)$ and $w'_{\text{below}}(\ell_R)$, and both have the same image word $w(\ell_R)$. Since $w'_{\text{above}}(\ell_R)$ and $w'_{\text{below}}(\ell_R)$ belong as above to distinct shift-orbits, the morphism σ is not shift-orbit injective, and hence not recognizable (see Proposition 3.6(1)).

   Clearly, this process can be iterated arbitrarily often and, every time, the obtained morphism is shift-orbit injective except for two particular shift-orbits, which have the same image orbit.

(3) The above set-up of lines in a square grid of $\mathbb{R}^2$ admits a particularly convincing translation into an IET setting, since for any of the squares, we can use the left-hand and the bottom sides together as 'bottom intervals', and the top side together with the right-hand side as 'top intervals', and the line segments of our foliation that are contained in the chosen square give canonically a classical IET system. If the chosen square agrees with the above picked base square Q, then the interval coding associated traditionally to the IET defines a subshift that agrees precisely with the one given by the set of biinfinite words $w(\ell)$ (or similarly for $w'(\ell)$), which have been read off above from the intersections of the lines ℓ with the given square grid.

After this 'appetizer', we now give a detailed description of our construction in the subsequent four subsections. We assume a minimal familiarity with the basic terminology of Thurston's work on surfaces, such as 'pseudo-Anosov homeomorphism', 'stable lamination', or 'invariant train track'.

8.1. *The basic geometric quotient construction.* We will start by describing our basic geometric construction, using a pseudo-Anosov homeomorphism h of a compact orientable surface Σ, and its expanding invariant lamination Λ^s, which consists of uncountably many biinfinite geodesics (called 'leaves') with respect to a fixed hyperbolic structure on Σ. (The family Λ^s was called 'the stable lamination' by Thurston, as he was looking at its behavior when lifted to the universal covering of Σ, identified with the hyperbolic plane $\mathbb{H}^2$, in the neighborhood of a $\partial\tilde{h}$-fixed point on $\partial\mathbb{H}^2$ (where $\tilde{h}$ is a lift of h to $\mathbb{H}^2$ and $\partial\tilde{h}$ is the canonical extension of $\tilde{h}$ to $\partial\mathbb{H}^2$).)

It is a standard procedure to translate such laminations (for instance, by using an h-invariant train track neighborhood of Λ^s) into a classical interval exchange setting, which in turn (assuming that Λ^s is orientable and Σ has at least one boundary component) allows a direct translation of Λ^s into a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}$ is given by the intervals in the IET. Since both of these translations are well known (see for instance [12, 20, 21]), we will restrict ourselves here only to a description of the geometry of h and Λ^s.

For our purposes, it is convenient to impose the following extra conditions.

(H1) Assume that Σ has $r \geq 2$ boundary components, which are all fixed by h.

(H2) Each complementary component of Λ^s contains precisely one boundary component.

(Note that this assumption effects that there is a natural identification of $\pi_1\Sigma$ with the free group $F(\mathcal{A})$.)

(H3) Each complementary component has at least two cusps, and each cusp is fixed by h.

We now pick a particular complementary component $\Sigma_i \subseteq \Sigma$ of Λ^s, and assume that Σ_i has precisely two cusps, and thus also precisely two boundary leaves ℓ_1 and ℓ_2, which (as do all boundary leaves of complementary components) will then both belong to Λ^s. We now pass to a quotient surface Σ' by 'filling in' the boundary component of Σ that is contained in Σ_i, through identifying all points of the boundary curve in Σ_i into a single point P of Σ'. Then h induces a pseudo-Anosov homeomorphism $h' : \Sigma' \to \Sigma'$ with stable lamination $\Lambda's$, and there is a canonical quotient map $q : \Lambda^s \to \Lambda's$ that commutes with h and h', respectively. The map q is one-to-one everywhere, except at points on the leaves ℓ_1 and ℓ_2, which are identified by q to a single leaf $\ell' \in \Lambda's$. The leaf ℓ' is fixed and expanded by h', and the sole h'-fixed point on ℓ' is precisely the above point P. This can be seen for example by the canonical passage from the stable lamination Λ^s to the associated stable foliation $\mathcal{F}^s$ for h.

Remark 8.2.

(1) There is a remarkable feature here in that both Λ^s and $\Lambda's$ are minimal laminations (that is, each leaf is dense), while the map q induces on the leaf spaces of Λ^s and $\Lambda's$ a map that is surjective, but not injective.

(2) This is translated (via the associated IETs as indicated above) into a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ that is mapped by a morphism $\sigma : \mathcal{A}^* \to \mathcal{A}'*$ to a subshift $\sigma(X) =: X' \subseteq \mathcal{A}'\mathbb{Z}$ (for $\mathcal{A}'* \subseteq F(\mathcal{A}') = \pi_1\Sigma'$, in complete analogy to $\mathcal{A}$ and Σ in the above set-up). Here both X and X' are minimal, while the map induced by σ on X is not shift-orbit injective, so that σ is not recognizable in X.

(3) More precisely, since there is a natural one-to-one correspondence between the shift-orbits of X and the leaves of Λ^s (and similarly for X' and $\Lambda's$), we observe that σ maps precisely two shift-orbits of X to a common image shift-orbit of X', while everywhere else, the induced map on shift-orbits is one-to-one.

8.2. *The 'inverse' geometric quotient construction.* After having presented our basic geometric quotient construction, we will now describe the precise converse procedure. For this purpose, we assume in this subsection that σ_0, h_0, Λ_0^s, $\mathcal{A}_0$ and X_0 are as Σ, h, Λ^s, $\mathcal{A}$ and X in §8.1 above, and that in particular the conditions (H1)–(H3) are satisfied, except that in condition (H1), we lower the assumption on the number r of boundary components of Σ_0 to $r \geq 1$. We now select any non-boundary leaf ℓ_0 of Λ_0^s which is fixed by h_0:

$$h_0(\ell_0) = \ell_0. \tag{8.1}$$

Since Λ_0^s is expanded by h_0, it follows that there is precisely one fixed point $P = h_0(P) \in \ell$. We derive the surface Σ_1 from Σ_0 by puncturing a hole in Σ_0 at the point P, and observe from equation (8.1) that h_0 induces a homeomorphism $h_1 : \Sigma_1 \to \Sigma_1$. Again from considering the stable foliation $\mathcal{F}_0^s$ associated to Λ_0^s, we obtain the stable lamination $\Lambda_1^s \subseteq \Sigma_1$ for h_1 from Λ_0^s by doubling the leaf ℓ_0 into two leaves $\widehat{\ell_0}$ and $\widehat{\ell'_0}$, which are boundary leaves of a new complementary component $\Sigma'_1 \subseteq \Sigma_1$ that has no further boundary leaf. The component Σ'_1 contains a new boundary component of Σ_1 that runs around the puncture where formerly the point $P \in \Sigma_0$ was located.

From this construction, we obtain a quotient map $q_0 : \Lambda_1^s \to \Lambda_0^s$ that satisfies

$$h_0 \circ q_0 = q_0 \circ h_1, \tag{8.2}$$

and q_0 is one-to-one everywhere except on the leaves $\widehat{\ell_0}$ and $\widehat{\ell'_0}$, which are identified by q_0 to the single leaf $\ell_0 \in \Lambda_0^s$. We thus observe that the 'quotient procedure' from Σ_1, h_1 and Λ_1^s to Σ_0, h_0 and Λ_0^s is precisely the same as described in §8.1 when passing from Σ, h and Λ^s to Σ', h' and $\Lambda's$.

Remark 8.3. In the passage from Λ_0^s to Λ_1^s, when translated into the IET language as in Remark 8.2, we observe that the IET for Λ_1^s derives from the IET for Λ_0^s by subdividing one of the intervals (namely the one onto which we choose to isotope P along the leaf ℓ_0). Hence, the alphabet $\mathcal{A}_1$ for Λ_1^s derives from $\mathcal{A}_0$ by doubling one of its letters, namely the one corresponding to the subdivided interval.

For the minimal subshift $X_1 \subseteq \mathcal{A}_1^{\mathbb{Z}}$ associated to Λ_1 and the morphism $\sigma_0 : \mathcal{A}_1^* \to \mathcal{A}_0^*$ determined by the map q_0, which maps X_1 to X_0 and is non-recognizable in X_1, it follows that σ_0 is letter-to-letter, so that X_0 is actually a factor of X_1.

8.3. *Iteration of the inverse quotient construction.* We now look for a leaf $\ell_1 \in \Lambda_1^s$ with $h_1(\ell_1) = \ell_1$. As shown in the previous subsection, this is the only ingredient needed to repeat the above procedure to obtain a surface Σ_2, a pseudo-Anosov homeomorphism $h_2 : \Sigma_2 \to \Sigma_2$ with stable lamination Λ_2^s, a map $q_1 : \Lambda_2^s \to \Lambda_1^s$, and a morphism $\sigma_1 : \mathcal{A}_2^* \to \mathcal{A}_1^*$ that is non-recognizable on the minimal subshift X_2 which satisfies $\sigma_1(X_2) = X_1$.

Hence, to be able to repeat this procedure infinitely often, with the purpose to get for any $n \geq 0$, a morphism $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$ that is non-recognizable on a minimal subshift X_{n+1} with $\sigma_n(X_{n+1}) = X_n$, we just need for any Λ_n^s, a leaf $\ell_n \in \Lambda_n^s$ with $h_n(\ell_n) = \ell_n$. However, up to replacing h_n by a power $h_n^{t(n)}$ for some suitable integer $t(n) \geq 1$, this is no problem. It is well known that any pseudo-Anosov map h has infinitely many h-periodic leaves in its stable lamination. We obtain the following result, which is however only an intermediate step in our construction. In particular, the subshifts X_n are *not* the intermediate level subshifts of the given directive sequence $\overleftarrow{\sigma}$.

PROPOSITION 8.4. *There exists a directive sequence* $\overleftarrow{\sigma} = (\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n \geq 0}$ *and subshifts* $X_n \subseteq \mathcal{A}_n^{\mathbb{Z}}$*, such that for any* $n \geq 0$*, the following hold:*

(1) $\sigma_n(X_{n+1}) = X_n$ *and* σ_n *is not recognizable in* X_{n+1};
(2) $\mathrm{card}(\mathcal{A}_{n+1}) = \mathrm{card}(\mathcal{A}_n) + 1$*;*
(3) σ_n *is letter-to-letter. In particular,* σ_n *commutes with the shift operator and* X_n *is a factor of* X_{n+1};
(4) X_n *is minimal, aperiodic, and uniquely ergodic;*

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(5) *X_n is substitutive (see Remark 2.6(2)) for some primitive substitution $\tau_n : \mathcal{A}_n^* \to \mathcal{A}_n^*$;*
(6) *$\tau_n^{t(n)} \circ \sigma_n = \sigma_n \circ \tau_{n+1}$ for some integer $t(n) \geq 1$.*

Proof. Properties (1), (2), and (3) have been derived in the construction described above. The substitution τ_n from property (5) is the translation of the homeomorphism h_n into the monoid setting through the canonical embedding $\mathcal{A}_n^* \subseteq F(\mathcal{A}_n) = \pi_1 \Sigma_n$. The primitivity of τ_n is a direct consequence of the assumption 'pseudo-Anosov' for h and thus for all h_n. Property (4) is a direct consequence of property (5), and property (6) is the translation into the monoid setting of the commutativity relation $h_n^{t(n)} \circ q_n = q_n \circ h_{n+1}$, which is a consequence of equation (8.2) together with the above replacement of h_n by $h_n^{t(n)}$. □

8.4. *Everywhere growing directive sequences that are not (eventually) recognizable.* The sequence $\overleftarrow{\sigma}$ from Proposition 8.4 is not everywhere growing; in fact, for any integers $m > n \geq 0$, the telescoped level map $\sigma_{[n,m)}$ is letter-to-letter. However, by choosing suitable 'diagonal' or 'eventually horizontal' paths through the infinite commutative diagram built from the above morphisms σ_n ('vertical') and τ_n ('horizontal'), we will derive below everywhere growing directive sequences with interesting properties.

Using the terminology from Proposition 8.4, we first define for each $n \geq 0$, the morphism

$$\sigma_n' := \tau_n^{t'(n)} \circ \sigma_n \quad (= \sigma_n \circ \tau_{n+1}^{s(n)}),$$

where we set $t'(n) := s(n)\, t(n)$ for some suitably chosen integer $s(n) \geq 1$ which ensures that the incidence matrix $M(\tau_n^{t'(n)})$ is positive. Such $s(n)$ exists because of property (5) of Proposition 8.4, and since $M(\sigma_n)$ has no zero-columns, it follows furthermore that

$$\text{the incidence matrix } M(\sigma_n') \text{ is positive for any index } n \geq 0. \tag{8.3}$$

We now define a directive sequence $\overleftarrow{\sigma}' = (\sigma_n' : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_{n\geq 0}$ with intermediate level subshifts called X_n'. Since $\tau_n(X_n) = X_n$ and $\sigma_n(X_{n+1}) = X_n$, we have

$$\sigma_n'(X_{n+1}) = X_n \tag{8.4}$$

for any $n \geq 0$, so that from the minimality of X_n, we can deduce $X_n \subseteq X_n'$. In particular, we obtain from statement (1) of Proposition 8.4 together with Lemma 3.7 that σ_n' is not recognizable in X_{n+1} and thus neither in X_{n+1}'. From equation (8.3), we obtain directly (see Remark 2.8(1)) that the sequence $\overleftarrow{\sigma}'$ is everywhere growing.

Furthermore, we define for any integer $k \geq 0$, a directive sequence $\overleftarrow{\tau}_k = (\tau_n')_{n\geq 0}$ through setting $\tau_n' := \tau_k$ for all $n \geq k$ and $\tau_n' = \sigma_n'$ if $0 \leq n \leq k-1$. We also specify the starting surface Σ_0 to be a punctured torus, so that one has $|\mathcal{A}_0| = 2$, and X_0 is Sturmian. It follows that for any level $n \geq k$, the intermediate level n subshift of $\overleftarrow{\tau}_k$ is equal to the substitutive subshift X_k defined by the substitution τ_k from statement (5) of Proposition 8.4, so that for every $0 \leq n \leq k-1$, we deduce from equation (8.4) that the level n subshift is equal to X_n. The primitivity of τ_k implies in particular that the directive sequence $\overleftarrow{\tau}_k$ is everywhere growing. Recall also that (as is true for all stationary sequences, see [1] and the references given there) the truncated stationary sequence $\overleftarrow{\tau}_k = (\tau_n')_{n\geq k}$ is totally recognizable.

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We obtain hence as immediate consequence of Proposition 8.4 the following result; we observe that its parts (2) and (3) give directly the statements that have been rephrased in §1 and stated there as Proposition 1.7.

COROLLARY 8.5.

(1) *The directive sequence* $\overleftarrow{\sigma}'$ *is everywhere growing and satisfies the properties (1), (2), (4), (5), and (6) from Proposition 8.4, with* σ_n *replaced by* σ'_n.

(2) *For any integer* $k \geq 0$, *there exists a directive sequence* $\overleftarrow{\tau}_k$, *with level alphabets* $\mathcal{A}_n$ *of size* $\text{card}(\mathcal{A}_n) = k + 2$ *for any level* $n \geq k$, *and* $\text{card}(\mathcal{A}_n) = n + 2$ *if* $n \leq k$.

The sequence $\overleftarrow{\tau}_k$ *is everywhere growing and eventually recognizable: each of the first* k *level morphisms on the bottom of* $\overleftarrow{\tau}_k$ *is not recognizable in its corresponding level subshift, while all level morphisms of level* $n \geq k$ *are recognizable in their corresponding level subshift. Indeed, the sequence* $\overleftarrow{\tau}_k$ *is stationary above level* k.

(3) *All intermediate level subshifts of the above directive sequences* $\overleftarrow{\tau}_k$ *are minimal, uniquely ergodic, and aperiodic. In particular, the properties 'recognizable', 'shift-orbit injective' (see Definition 3.6), and 'recognizable for aperiodic points' (see Remark 3.10(2)) are equivalent, for each level morphism in its corresponding intermediate level subshift.*

Remark 8.6. It turns out that property (3) of Corollary 8.5 is also true for the directive sequence $\overleftarrow{\sigma}'$. Indeed, from property (6) of Proposition 8.4 and the well-known North-South dynamics induced by any pseudo-Anosov homeomorphism of Σ on the projectivized space of all measured laminations (= the boundary of Teichmüller space for Σ), one can deduce that the inclusion $X_n \subseteq X'_n$ derived after equation (8.4) is actually an equality. However, laying out the details of these arguments would go beyond our self-imposed limits on the amount of Nielsen–Thurston theory imported into this section.

Remark 8.7. Given any eventually recognizable everywhere growing directive sequence $\overleftarrow{\sigma} = (\sigma_n)_{n\geq 0}$ of finite alphabet rank, one may ask whether there is an upper bound to the number level morphisms σ_n which are not recognizable in their corresponding intermediate level subshift. This question has sparked some interest, see [**9**, **13**]. We note that the examples given in part (2) of Corollary 8.5 above contradict the bound claimed in [**13**, Theorem 3.7] as stated; to rectify that statement, additional hypotheses would need to be imposed. This error could also effect the upper bound claimed in [**17**, Corollary 1.5] on the number of successive factor maps, for a large class of subshifts.

In this context, we also want to point to the very recent paper [**2**, Example 7.5], where a family of directive sequences is presented that has the same properties as exhibited in Corollary 8.5(2) above for the sequences $\overleftarrow{\tau}_k$. The examples from [**2**] are easier to describe, but fail to have the extra properties listed in part (3) of Corollary 8.5.

Another construction of a similar kind (but closer to our Corollary 8.5 above) has been communicated to us by Espinoza [**18**] in the final stages of the revision of this paper.

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REFERENCES

[1] M.-P. Béal, D. Perrin and A. Restivo. Recognizability of morphisms. *Ergod. Th. & Dynam. Sys.* **43**(11) (2023), 3578–3602.
[2] M.-P. Béal, D. Perrin, A. Restivo and W. Steiner. Recognizability in S-adic shifts. *Preprint*, 2023, arXiv:2302.06258.
[3] N. Bédaride, A. Hilion and M. Lustig. Graph towers, laminations and their invariant measures. *J. Lond. Math. Soc. (2)* **101** (2020), 1112–1172.
[4] N. Bédaride, A. Hilion and M. Lustig. Tower power for S-adics. *Math. Z.* **297** (2021), 1853–1875.
[5] N. Bédaride, A. Hilion and M. Lustig. The measure transfer for subshifts induced by a morphism of free monoids. *Preprint*, 2023, arXiv:2211.11234.
[6] V. Berthé, P. Cecchi Bernales, F. Durand, J. Leroy, D. Perrin and S. Petite. On the dimension group of unimodular S-adic subshifts. *Monatsh. Math.* **194** (2021), 687–717.
[7] V Berthé and V. Delecroix. Beyond substitutive dynamical systems: S-adic expansions. *RIMS Kôkyûroku Bessatsu* **B46** (2014), 81–123.
[8] V. Berthé and M. Rigo (Ed.). *Combinatorics, Automata and Number Theory (Encyclopedia of Mathematics and its Applications, 135)*. Cambridge University Press, Cambridge, 2010.
[9] V. Berthé, W. Steiner, J. Thuswaldner and R. Yassawi. Recognizability for sequences of morphisms. *Ergod. Th. & Dynam. Sys.* **39** (2019), 2896–2931.
[10] M. Boyle and D. Handelman. Entropy versus orbit equivalence for minimal homeomorphisms. *Pacific J. Math.* **164** (1994), 1–13.
[11] V. Cyr and B. Kra. Realizing ergodic properties in zero entropy subshifts. *Israel J. Math.* **240** (2020), 119–148
[12] V. Delecroix. Interval exchange transformations. *Lecture Notes from Salta (Argentina)*, November 2016.
[13] S. Donoso, F. Durand, A. Maass and S. Petite. Interplay between finite topological rank minimal Cantor systems, S-adic subshifts and their complexity. *Trans. Amer. Math. Soc.* **374** (2021), 3453–3489.
[14] F. Durand. Linearly recurrent subshifts have a finite number of non-periodic subshift factors. *Ergod. Th. & Dynam. Sys.* **20** (2000), 1061–1078; Corrigendum and addendum, *Ergod. Th. & Dynam. Sys.* **23** (2003), 663–669.
[15] F. Durand, J. Leroy and G. Richomme. Do the properties of an S-adic representation determine factor complexity? *J. Integer Seq.* **16** (2013), Article 13-2-6
[16] F. Durand and D. Perrin. *Dimension Groups and Dynamical Systems, Bratteli Diagrams and Cantor Systems (Cambridge Studies in Advanced Mathematics, 196)*. Cambridge University Press, Cambridge, 2022.
[17] B. Espinoza. On symbolic factors of S-adic subshifts of finite topological rank. *Ergod. Th. & Dynam. Sys.* **43**(5) (2023), 1511–1547.
[18] B. Espinoza. Worst-case number of non-recognizable levels for finite alphabet rank S-adic sequences. Available at: https://sites.google.com/ug.uchile.cl/espinoza/miscellaneous.
[19] S. Ferenczi. Rank and symbolic complexity. *Ergod. Th. & Dynam. Sys.* **16** (1996), 663–682.
[20] V. H. Gadre. Dynamics of non-classical interval exchanges. *Ergod. Th. & Dynam. Sys.* **32** (2012), 1930–1971
[21] M. Kapovich. Laminations, foliations, and trees. *Hyperbolic Manifolds and Discrete Groups (Modern Birkhäuser Classics)*. Ed. M. Kapovich. Birkhäuser, Boston, 2009.
[22] X. Méla and K. Petersen. Dynamical properties of the Pascal adic transformation. *Ergod. Th. & Dynam. Sys.* **25** (2005), 227–256.