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CUBIC EDGE-TRANSITIVE GRAPHS OF ORDER $8p^2$

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Abstract

A simple undirected graph is said to be semisymmetric if it is regular and edge-transitive but not vertex-transitive. Let p be a prime. It was shown by Folkman [J. Folkman, 'Regular line-symmetric graphs', *J. Combin. Theory* **3** (1967), 215–232] that a regular edge-transitive graph of order $2p$ or $2p^2$ is necessarily vertex-transitive. In this paper an extension of his result in the case of cubic graphs is given. It is proved that every cubic edge-transitive graph of order $8p^2$ is vertex-transitive.

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1. Introduction

In this paper we consider an undirected finite connected graph without loops or multiple edges. For a graph Γ, we denote by $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ its vertex set, edge set and automorphism group, respectively. For $u, v \in V(\Gamma)$, denote by uv the edge incident to u and v in Γ, and by $N_\Gamma(u)$ the *neighbourhood* of u in Γ, that is, the set of vertices adjacent to u in Γ. A graph $\widetilde{\Gamma}$ is called a *covering* of a graph Γ with projection $p : \widetilde{\Gamma} \to \Gamma$ if there is a surjection $p : V(\widetilde{\Gamma}) \to V(\Gamma)$ such that $p|_{N_{\widetilde{\Gamma}}(\widetilde{v})} : N_{\widetilde{\Gamma}}(\widetilde{v}) \to N_\Gamma(v)$ is a bijection for any vertex $v \in V(\Gamma)$ and $\widetilde{v} \in p^{-1}(v)$. Let N be a subgroup of $\mathrm{Aut}(\Gamma)$ such that N is intransitive on $V(\Gamma)$. The quotient graph Γ/N induced by N is defined as the graph such that the set Σ of N-orbits in $V(\Gamma)$ is the vertex set of Γ/N and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $\{u, v\} \in E(\Gamma)$. A covering $\widetilde{\Gamma}$ of Γ with a projection p is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{\Gamma})$ such that graph Γ is isomorphic to the quotient graph $\widetilde{\Gamma}/K$, say by h, and the quotient map $\widetilde{\Gamma} \to \widetilde{\Gamma}/K$ is the composition ph of p and h (for the purpose of this paper, all functions are composed from left to right). If K is cyclic or elementary Abelian then $\widetilde{\Gamma}$ is called a *cyclic* or an *elementary Abelian covering* of Γ, and if $\widetilde{\Gamma}$ is connected K becomes the covering transformation group. The *fibre* of an edge or a vertex is its preimage under p. An automorphism of $\widetilde{\Gamma}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while every covering transformation maps a fibre onto itself. The set of all fibre-preserving automorphisms forms a group called the *fibre-preserving group*.

An s-*arc* in a graph Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of Γ such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$; in other words, a directed walk of length s which never includes backtracking. A graph Γ is said to be s-*arc-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the set of s-arcs in Γ. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph Γ is said to be *edge-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on $E(\Gamma)$. A subgroup of the automorphism group of a graph Γ is said to be s-*regular* if it acts regularly on the set of s-arcs of Γ. It can be shown that a edge- but not vertex-transitive graph Γ is necessarily bipartite, where the two parts of the bipartition are orbits of $\mathrm{A} = \mathrm{Aut}(\Gamma)$. Moreover, if Γ is regular these two parts have equal cardinality. A regular edge- but not vertex-transitive graph will be referred to as a *semisymmetric* graph.

Covering techniques have long been known as a powerful tool in topology and graph theory. Regular covering of a graph is currently an active topic in algebraic graph theory. The class of semisymmetric graphs was introduced by Folkman [8]. He constructed several infinite families of such graphs and posed eight open problems. Subsequently, Bouwer [1, 2], Titov [19], Klin [13], Iofinova and Ivanov [11], Ivanov [12], Du and Xu [5] and others did much work on semisymmetric graphs. They gave new constructions of such graphs by combinatorial and group-theoretical methods. The answers to most of Folkman's open problems are now known. By using the covering technique, cubic semisymmetric graphs of order $6p^2$ and $2p^3$ were classified in [14, 17]. Some general methods of elementary Abelian coverings were developed in [4, 16]. The s-regular cyclic coverings and elementary Abelian coverings of the three-dimensional hypercube Q_3 were classified in [6, 7]. In this paper, by using the same covering technique and group-theoretical construction, we investigate cubic semisymmetric graphs of order $8p^2$. The following is the main result of this paper.

THEOREM 1.1. *Let p be a prime. Then every cubic edge-transitive graph of order $8p^2$ is vertex-transitive.*

2. Primary analysis

Let Γ be a graph and K be a finite group. By a^{-1} we mean the reverse arc to an arc a. A *voltage assignment* (or K-*voltage assignment*) of Γ is a function $\phi : A(\Gamma) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(\Gamma)$. The values of ϕ are called *voltages*, and K is the *voltage group*. The graph $\Gamma \times_\phi K$ derived from a voltage assignment $\phi : A(\Gamma) \to K$ has vertex set $V(\Gamma) \times K$ and edge set $E(\Gamma) \times K$, so that an edge (e, g) of $\Gamma \times K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(\Gamma)$ and $g \in K$, where $e = uv$.

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Clearly, the derived graph $\Gamma \times_\phi K$ is a covering of Γ with the first coordinate projection $p : \Gamma \times_\phi K \to \Gamma$, which is called the *natural projection*. By defining $(u, g')^g = (u, g'g)$ for any $g \in K$ and $(u, g') \in V(\Gamma \times_\phi K)$, K becomes a subgroup of $\mathrm{Aut}(\Gamma \times_\phi K)$ which acts semiregularly on $V(\Gamma \times_\phi K)$. Therefore, $\Gamma \times_\phi K$ can be viewed as a *K-covering*. For each $u \in V(\Gamma)$ and $uv \in E(\Gamma)$, the vertex set $\{(u, g) \mid g \in K\}$ is the fibre of u and the edge set $\{(u, g)\,(v, \phi(a)g) \mid g \in K\}$ is the fibre of uv, where $a = (u, v)$. Conversely, each regular covering $\widetilde{\Gamma}$ of Γ with a covering transformation group K can be derived from a K-voltage assignment. Given a spanning tree T of the graph Γ, a voltage assignment ϕ is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [10] showed that every regular covering $\widetilde{\Gamma}$ of a graph Γ can be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of Γ. It is clear that if ϕ is reduced, the derived graph $\Gamma \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group K.

Let $\widetilde{\Gamma}$ be a K-covering of Γ with a projection p. If $\alpha \in \mathrm{Aut}(\Gamma)$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{\Gamma})$ satisfy $\widetilde{\alpha}p = p\alpha$, we call $\widetilde{\alpha}$ a *lift* of α, and α the *projection* of $\widetilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(\Gamma)$ and the projection of a subgroup of $\widetilde{\Gamma}$ are self-explanatory. The lifts and projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{\Gamma})$ and $\mathrm{Aut}(\Gamma)$, respectively. In particular, if the covering graph $\widetilde{\Gamma}$ is connected, then the covering transformation group K is the lift of the trivial group, that is,

$$K = \{\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{\Gamma}) : p = \widetilde{\alpha}p\}.$$

Clearly, if $\widetilde{\alpha}$ is a lift of α, then $K\widetilde{\alpha}$ are all the lifts of α.

Let T be a spanning tree of a graph Γ. A closed walk W that contains only one cotree arc is called a fundamental closed walk. Similarly, a cycle W that contains only one cotree arc is called a fundamental cycle.

Let $\Gamma \times_\phi K \to \Gamma$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. The problem of whether an automorphism α of Γ lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \mathrm{Aut}(\Gamma)$, we define a function $\overline{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(\Gamma)$ to the voltage group K by

$$(\phi(C))^{\overline{\alpha}} = \phi(C^\alpha),$$

where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^\alpha)$ are the voltages on C and C^α, respectively. Note that if K is Abelian, $\overline{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of Γ.

The following proposition is a special case of [15, Theorem 4.2].

Proposition 2.1. *Let $\Gamma \times_\alpha K \to \Gamma$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. Then an automorphism α of Γ lifts if and only if $\overline{\alpha}$ extends to an automorphism of K.*

The next proposition is a special case of [14, Lemma 3.2].

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PROPOSITION 2.2. *Let Γ be a connected semisymmetric cubic graph with bipartition sets $U(\Gamma)$ and $W(\Gamma)$. Moreover, suppose that N is a normal subgroup of $A := \mathrm{Aut}(\Gamma)$. If N is intransitive on bipartition sets, then N acts semiregularly on both $U(\Gamma)$ and $W(\Gamma)$, and Γ is an N-regular covering of an A/N-semisymmetric graph.*

Two coverings $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ of Γ with projection p_1 and p_2, respectively, are said to be equivalent if there exists a graph isomorphism $\widetilde{\alpha} : \widetilde{\Gamma}_1 \to \widetilde{\Gamma}_2$ such that $\widetilde{\alpha} p_2 = p_1$. We quote the following propositions.

PROPOSITION 2.3 [18]. *Two connected regular coverings $\Gamma \times_\phi K$ and $\Gamma \times_\psi K$, where ϕ and ψ are T-reduced, are equivalent if and only if there exists an automorphism $\sigma \in \mathrm{Aut}(K)$ such that $\phi(u, v)^\sigma = \psi(u, v)$ for any cotree arc (u, v) of Γ.*

PROPOSITION 2.4 [17, Proposition 2.4]. *The vertex stabilizers of a connected G-edge-transitive cubic graph Γ have order $2^r \cdot 3$, $r \geq 0$. Moreover, if u and v are two adjacent vertices, then $|G : \langle G_u, G_v\rangle| \leq 2$, and the edge stabilizer $G_u \cap G_v$ is a common Sylow 2-subgroup of G_u and G_v.*

3. Proof of Theorem 1.1

We denote by Q_3 the three-dimensional hypercube which is bipartite with partite sets $\{a, b, c, d\}$ and $\{w, x, y, z\}$. Let T be a spanning tree of Q_3, as shown by dark lines in Figure 1. Let ϕ be such a voltage assignment defined by $\phi = 0$ on T and $\phi = z_1, z_2, z_3, z_4$ and z_5 on the cotree arcs (b, y), (c, w), (c, x), (d, w) and (d, x) respectively, where 0 is the identity element of K and $z_i \in K$ $(1 \leq i \leq 5)$. It is well known that $\mathrm{Aut}(Q_3) \cong S_4 \times \mathbb{Z}_2$. Let $\alpha = (bcd)\,(xyz)$, $\beta = (ab)\,(cd)\,(wx)\,(yz)$ and $\gamma = (aw)\,(bx)\,(cy)\,(dz)$. Then α, β and γ are automorphisms of Q_3.

Note that, by [3], throughout this paper we may assume that $p \geq 11$.

LEMMA 3.1. *Suppose that Γ is a connected semisymmetric cubic graph of order $8p^2$. Then Γ is a connected N-regular covering of Q_3 such that the subgroup of $\mathrm{Aut}(Q_3)$ generated by α and β lifts, where $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$.*

PROOF. Let Γ be a cubic graph satisfying the assumptions and let $\mathrm{A} = \mathrm{Aut}(\Gamma)$. Therefore Γ is bipartite graph. Denote by $U(\Gamma)$ and $W(\Gamma)$ the bipartition sets of Γ. By Proposition 2.4, $|A| = 2^r \cdot 3 \cdot p^2$, where $r \geq 2$ as A is transitive on a set of size $4p^2$. A is solvable, for if it were not, then by [9] its composition factors would have to be $PSL(3, 3)$ or $PSL(2, 17)$, which is a contradiction. Let $Q = O_p(A)$ be the maximal normal p-subgroup of A. We show that $|Q| = p^2$.

Suppose first that $Q = 1$. Let N be a minimal normal subgroup of A. So N is solvable. N is not transitive on bipartition sets $U(\Gamma)$ and $W(\Gamma)$, and hence by Proposition 2.2, N acts semiregularly on bipartition sets $U(\Gamma)$ and $W(\Gamma)$. Therefore, N is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. First, assume that $N \cong \mathbb{Z}_2$. By Proposition 2.2, N acts semiregularly on $U(\Gamma)$ and $W(\Gamma)$. Now we consider the quotient graph

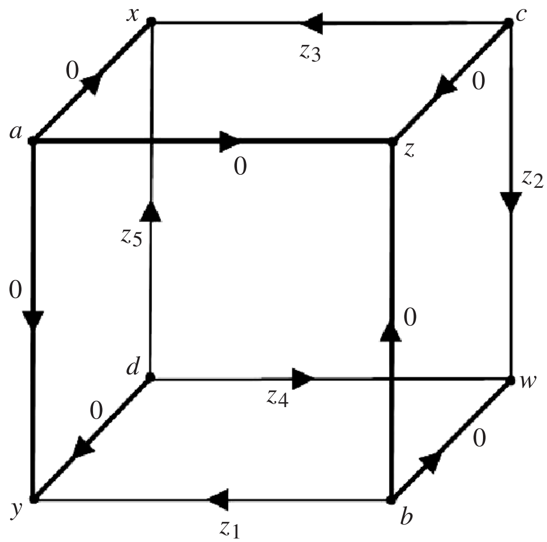

FIGURE 1. A spanning tree and a voltage assignment on Q_3.

$\Gamma_N = \Gamma/N$ of Γ relative to N, where A/N is semisymmetric on bipartition sets of Γ_N. We claim that $O_p(A/N) \neq 1$.

Suppose to the contrary that $O_p(A/N) = 1$. Now let $O_{p'}(A/N) \neq 1$ and $T/N = O_{p'}(A/N)$. T/N is not transitive on bipartition sets of $\Gamma_N = \Gamma/N$. Therefore, by Proposition 2.2, T/N acts semiregularly on bipartition sets of $\Gamma_N = \Gamma/N$, and $|O_{p'}(A/N)| = 2$. Now consider the quotient graph $\Gamma_T = \Gamma/T$ of Γ relative to T. Let H/T be a minimal normal subgroup of A/T. So H/T is solvable, and $|H/T| = p$ or $|H/T| = p^2$. Therefore H has a normal subgroup of order divisible by p, which is characteristic in H, and hence is normal in A. It contradicts our assumption that $O_p(A) = 1$. Hence $O_{p'}(A/N) = 1$.

Now suppose that T/N is a minimal normal subgroup of A/N. Since $O_p(A/N) = 1$ and $O_{p'}(A/N) = 1$, therefore T/N is nonsolvable. This is a contradiction.

Suppose now that $N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $\Gamma_N = \Gamma/N$ be the quotient graph of Γ relative to N. Let T/N be a minimal normal subgroup of A/N. Then T/N is solvable, and by Proposition 2.2, $T/N \cong \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$. Therefore T has a normal subgroup of order divisible by p, which is characteristic in T, and hence is normal in A. It contradicts our assumption that $O_p(A) = 1$. Therefore $|Q| \neq 1$.

Suppose, finally, that $|Q| = p$; we show that this leads to a contradiction. Let $\Gamma_Q = \Gamma/Q$ be the quotient graph of Γ relative to Q. Let N/Q be a minimal normal subgroup of A/Q. Hence N/Q is solvable, and by Proposition 2.2, $N/Q \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_p$. By our assumption, N/Q is not isomorphic to $\mathbb{Z}_p$. If $N/Q \cong \mathbb{Z}_2$, we consider the quotient graph Γ/N. Now let K/N be a minimal normal

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TABLE 1. Fundamental cycles and their images with corresponding voltages.

C	$\phi(C)$	C^α	$\phi(C^\alpha)$	C^β	$\phi(C^\beta)$	C^γ	$\phi(C^\gamma)$
$azby$	z_1	$axcz$	$-z_3$	$byaz$	z_1	$wdxc$	$z_2+z_5-z_3-z_4$
$bzcw$	z_2	$cxdw$	$z_3+z_4-z_2-z_5$	$aydx$	z_5	$xdya$	$-z_5$
$azcx$	z_3	$axdy$	$-z_5$	$bydw$	z_1+z_4	$wdyb$	$-z_1-z_4$
$aydwbz$	z_4	$azbwcx$	z_3-z_2	$bzcxay$	z_3-z_1	$wczaxd$	$z_4-z_2-z_5$
$aydx$	z_5	$azby$	z_1	$bzcw$	z_2	$wczb$	$-z_2$

subgroup of A/N. Therefore K/N is solvable, and $K/N \cong \mathbb{Z}_2$ or $\mathbb{Z}_p$. By our assumption K/N cannot be isomorphic to $\mathbb{Z}_p$, so $K/N \cong \mathbb{Z}_2$. Consider the quotient graph $\Gamma_K = \Gamma/K$, where A/K is semisymmetric on bipartition sets of Γ_K. Let L/K be a minimal normal subgroup of A/K. Thus L/K is solvable, and since $Q = |O_p(A)| = p$, therefore $L/K \cong \mathbb{Z}_2$. Again we consider the quotient Γ_L, and let M/L be a minimal subgroup of A/L. Hence M/L is solvable and $M/L \cong \mathbb{Z}_p$, which contradicts our assumption that $Q = |O_p(A)| = p$. If $N/Q \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, by considering the quotient graph Γ_N with the same reasoning as before, a contradiction can be obtained.

Therefore, $Q = O_p(A)$ is normal in A. The only graph of valency 3 on eight vertices is Q_3, so by Proposition 2.2, Γ is a connected Q-regular covering of Q_3, where $|Q| = p^2$. In addition, since Γ is semisymmetric and Q is normal in A, so fibre-preserving automorphism group is edge-transitive, and hence the projection of the fibre-preserving automorphism group is edge-transitive on the base graph Q_3. Therefore, the subgroup generated by α and β lifts. □

Denote by $i_1 i_2 \ldots i_s$ a directed cycle which has vertices $i_1, i_2, \ldots, i_s$ in consecutive order. There are five fundamental cycles $azby$, $bzcw$, $azcx$, $aydwbz$, and $aydx$ in Q_3, which are generated by the five cotree arcs (b, y), (c, w), (c, x), (d, w), and (d, x), respectively. Each cycle is mapped to a cycle of the same length under the actions of α, β, and γ. We list all these cycles and their voltages in Table 1, in which C denotes a fundamental cycle of Q_3 and $\phi(C)$ denotes the voltage of C.

LEMMA 3.2. *Let $N \cong \mathbb{Z}_{p^2}$ and suppose that $\Gamma = Q_3 \times_\phi \mathbb{Z}_{p^2}$ is a connected $\mathbb{Z}_{p^2}$-regular covering of Q_3. If the subgroup of* Aut(Q_3) *generated by α and β can be lifted then Γ is symmetric.*

PROOF. Since α and β can be lifted, by Proposition 2.1, $\overline{\alpha}$ and $\overline{\beta}$ can be extended to automorphisms of $\mathbb{Z}_{p^2}$. We denote these automorphisms by α^* and β^*, respectively. Since α^* and β^* always exist, by Table 1, $z_1^{\alpha^*} = -z_3$, $z_5^{\alpha^*} = z_1$, $z_5^{\beta^*} = z_2$ and $z_4^{\alpha^*} = z_3 - z_2$. The first three equations imply that z_1, z_2, z_3 and z_5 have the same order and the last equation implies that the order of z_1 is divisible by the order of z_4. Since $\langle z_1, z_2, z_3, z_4, z_5 \rangle = \mathbb{Z}_{p^2}$, each of z_1, z_2, z_3 and z_5 generates the group $\mathbb{Z}_{p^2}$. By Proposition 2.3, we may assume that $z_1 = 1$ and $z_2 = k$ such that $(k, p^2) = 1$ and $1 \le k \le p^2 - 1$. Since $z_1^{\beta^*} = z_1$, β^* is the identity automorphism

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of $\mathbb{Z}_{p^2}$. Thus, $z_2^{\beta^*} = z_5$ and $z_3^{\beta^*} = z_1 + z_4$ imply that $z_5 = k$ and $z_3 - z_4 = 1$. As $z_5^{\alpha^*} = z_1$, α^* is the automorphism of $\mathbb{Z}_{p^2}$ induced by $1 \to k^{-1}$, where k^{-1} is the inverse of k in $\mathbb{Z}_{p^2}^*$. Now, it follows that $z_3 = -k^{-1}$ and $z_4 = -k^{-1} - 1$ because $z_1^{\alpha^*} = -z_3$ and $z_3 - z_4 = 1$. Consequently, we have the equations

$$z_1 = 1, \quad z_2 = k, \quad z_3 = -k^{-1}, \quad z_4 = -k^{-1} - 1, \quad z_5 = k, \tag{3.1}$$

where $1 \le k \le p^2 - 1$ and $(k, p^2) = 1$.

Using the equations in (3.1) and $1^{\alpha^*} = k^{-1}$, we have $2(k^2 + k + 1) = 0$ and $k^3 = 1$ because $z_2^{\alpha^*} = z_3 + z_4 - z_2 - z_5$ and $z_4^{\alpha^*} = z_3 - z_2$. By Table 1, $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ can be extended to the automorphisms of $\mathbb{Z}_{p^2}$ induced by $1 \to k^{-1}$, $1 \to 1$ and $1 \to -1$, respectively, so by Proposition 2.1, the subgroup $\langle \alpha, \beta, \gamma \rangle$ of $\mathrm{Aut}(Q_3)$ lifts. Therefore Γ is symmetric. □

LEMMA 3.3. *Let $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and suppose that $\Gamma = Q_3 \times_\phi \mathbb{Z}_p^2$ is a connected N-regular covering of Q_3 such that the subgroup of* $\mathrm{Aut}(Q_3)$ *generated by α and β can be lifted. Then Γ is symmetric.*

PROOF. Since α^* and β^* always exist, with the same reasoning as in the proof of Lemma 3.2, z_1, z_2, z_3 and z_5 have the same order and the order of z_1 is divisible by the order of z_4. If $z_1 = 0$ then $z_1^{\alpha^*} = 0$. By Table 1, $z_3 = -z_1{}^{\alpha^*} = 0$. Also, $z_5 = 0$ and $z_2 = 0$ because $z_3^{\alpha^*} = -z_5$ and $z_5^{\beta^*} = z_2$. Thus, $\langle z_1, z_2, z_3, z_4, z_5 \rangle = \langle z_4 \rangle$ $= \mathbb{Z}_p^2$, which is a contradiction. Similarly, if $z_2 = 0$ the same contradiction can be obtained. Consequently, $z_1 \neq 0$ and $z_2 \neq 0$. Now, we claim that z_1 and z_2 are linearly independent. Suppose to the contrary that z_2 is a scalar multiple of z_1, say $z_2 = kz_1$. Then, $k \neq 0$. Since $z_2^{\beta^*} = kz_1^{\beta^*}$, we have $z_5 = kz_1$. As $z_5^{\alpha^*} = kz_1^{\alpha^*}$, z_3 is a scalar multiple of z_1, say $z_3 = lz_1$. And, z_4 is also a scalar multiple of z_1 because $z_3^{\beta^*} = lz_1^{\beta^*}$. It follows that $\langle z_1, z_2, z_3, z_4, z_5 \rangle = \langle z_1 \rangle = \mathbb{Z}_p^2$, which is a contradiction. Thus, z_1 and z_2 are linearly independent. Similarly, z_1 and z_3 are also linearly independent.

Since z_1 and z_2 are linearly independent and $\langle z_1, z_2, z_3, z_4, z_5 \rangle = N$, z_3, z_4 and z_5 can be expressed as a combination of z_1 and z_2.

Let $z_3 = iz_1 + jz_2$, $z_4 = i'z_1 + j'z_2$ and $z_5 = i''z_1 + j''z_2$. By Proposition 2.3, with the linear independence of z_1 and z_2 we may assume that $z_1 = (1, 0)$ and $z_2 = (0, 1)$ are the standard basis of the vector space $\mathbb{Z}_p \times \mathbb{Z}_p$.

By $z_3^{\alpha^*} = iz_1^{\alpha^*} + jz_2^{\alpha^*}$ and $z_3^{\beta^*} = iz_1^{\beta^*} + jz_2^{\beta^*}$, $-z_5 = i(-z_3) + j(z_3 + z_4 - z_2 - z_5)$ and $z_1 + z_4 = iz_1 + jz_5$, so $-i''z_1 - j''z_2 = -i^2 z_1 - ijz_2 + ijz_1 + j^2 z_2 + i'jz_1 + jj'z_2 - jz_2 - i''jz_1 - jj''z_2$ and $z_1 + i'z_1 + j'z_2 = iz_1 + j(i''z_1 + j''z_2)$. By the linear independence of z_1 and z_2, we have the following formulae:

(1) $i'' - i^2 + ij + i'j - i''j = 0$;
(2) $j'' - ij + j^2 + jj' - j - jj'' = 0$;

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(3) $1 + i' - i - i''j = 0$;
(4) $j' - jj'' = 0$.

Now by considering the image of $z_4 = i'z_1 + j'z_2$ and $z_5 = i''z_1 + j''z_2$, under α^* and β^*, we have:

(5) $i + ii' - ij' - i'j' + i''j' = 0$;
(6) $j - 1 + i'j - jj' - j'^2 + j' + j'j'' = 0$;
(7) $1 + ii'' - ij'' - i'j'' + i''j'' = 0$;
(8) $i''j - jj'' - j'j'' + j'' + j''^2 = 0$;
(9) $i - 1 - i' - i''j' = 0$;
(10) $j - j'j'' = 0$;
(11) $i'' + i''j'' = 0$;
(12) $j''^2 - 1 = 0$.

By (11), $i'' = 0$ or $j'' + 1 = 0$. First assume that $i'' = 0$, by (4) and (10) $(j + j')(1 - j'') = 0$. If $1 - j'' = 0$, by (7) and (3), $i = 1$ and $i' = 0$. Now by (1) and (5), $j = 1$ and $j' = 1$. Therefore,

$$z_1 = (1, 0), \quad z_2 = z_4 = z_5 = (0, 1) \quad z_3 = (1, 1).$$

From Table 1, it is easy to check that $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ can be extended to automorphisms of $\mathbb{Z}_{p^2}$. By Proposition 2.1, α, β and γ lift, so Γ is symmetric.

Now if $j' + j = 0$, we show that it leads to a contradiction. By (4), $j(1 + j'') = 0$. If $j = 0$ then $j' = 0$. By (2), $j'' = 0$, so by (7), $1 = 0$, which is a contradiction. Hence $1 + j'' = 0$. By (7) and (3), $i' = -1$, so by (9) $i = 0$. Now by (1) and (2), $j'' = 0$, which is a contradiction.

Now assume that $1 + j'' = 0$; again we show that this leads to a contradiction. By (10), $j = -j'$. Now by (8), $i''j = 0$. So $i'' = 0$ or $j = 0$. If $j = 0$ by (2), $-1 = 0$, which is a contradiction. If $i'' = 0$, by (7) and (3) $i = 0$ and $i' = -1$, so by (2) $-1 = 0$, which is a contradiction. □

Proof of Theorem 1.1. This follows by Lemmas 3.1, 3.2 and 3.3. □

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