RESIDUALLY FINITE RINGS

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1. Introduction. Throughout this paper a ring will always be an associative, not necessarily commutative ring with an identity. It is tacitly assumed that the identity of a subring coincides with that of the whole ring. A ring R is said to be *residually finite* if it satisfies one of the following equivalent conditions:

(1) Every non-zero ideal of R is of finite index in R;

(2) For each non-zero ideal A of R, the residue class ring R/A is finite;

(3) Every proper homomorphic image of R is finite.

The class of residually finite rings is large enough to merit our investigation. All finite rings and all simple rings are trivially residually finite. Other residually finite rings are said to be *proper*. Examples of such rings are (i) the ring of all integers; (ii) the polynomial ring $F[X]$ over a finite field F; (iii) the formal power series ring $F[[X]]$ over a finite field F; and (iv) those rings considered by Cohen and Kaplansky [**5**].

It is known that the group of integers is the only infinite abelian group in which every non-zero subgroup is of finite index. The situation in ring theory is not so simple. The present paper contains a certain number of results which originated in an attempt to determine the possible structure of a residually finite commutative ring, a problem communicated to us by Professor Carlton J. Maxson, whom we owe a debt of gratitude.

2. Basic properties. Let R be a residually finite ring. Then R satisfies the ascending chain condition on ideals. Moreover, since every finite prime ring is a simple ring, a non-zero proper prime ideal of R is maximal.

LEMMA 2.1. *Let A and B be ideals of a ring R of finite index. If the ideal $A \cap B$ is finitely generated, then AB is of finite index in R.*

Proof. Since $(A \cap B)/AB$ is a finitely generated left R/A-module, right R/B-module and since R/A and R/B are finite, AB is of finite index in $A \cap B$. However, $R/(A \cap B)$, being isomorphic to a subring of the direct sum of the finite rings R/A and R/B, is finite. Now the index of AB in R is the product of the index of AB in $A \cap B$ and that of $A \cap B$ in R. Hence AB is of finite index in R. This completes the proof.

COROLLARY 2.2. *Every infinite residually finite ring is a prime ring.*

The above corollary implies that the intersection of a finite number of non-zero ideals in an infinite residually finite ring is not zero. However, the

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intersection of an infinite number of distinct ideals in a residually finite ring is always zero. Accordingly, the descending chain condition on ideals is satisfied by a residually finite ring R if and only if R has only a finite number of ideals.

Corollary 2.2, together with the Wedderburn-Artin theorem [**6**, p. 40], implies also that the only residually finite rings with the descending chain condition on right (or left) ideals are finite rings and full matrix rings over division rings.

THEOREM 2.3. *A ring R is residually finite if and only if R satisfies the ascending chain condition on ideals and every non-zero prime ideal of R is of finite index in R.*

Proof. The necessity of the conditions is obvious. To prove the converse, let E be the collection of all non-zero ideals of R of infinite index in R. If E is not empty, then by the ascending chain condition there is a maximal element M in E. We claim that M is prime. For otherwise there would exist ideals A and B of R containing M properly such that $AB \subseteq M$. The ideals A and B would then be of finite index in R. By Lemma 2.1, AB and hence M would be of finite index in R. This contradicts the choice of M. Hence M is a non-zero prime ideal of R. By hypothesis, M is of finite index in R. This again contradicts the choice of M. Hence E is empty. This completes the proof.

As an immediate consequence of Theorem 2.3 and [**4**, p. 29, Theorem 2] we have the following result.

COROLLARY 2.4. *A commutative ring R is residually finite if and only if every non-zero prime ideal of R is finitely generated and of finite index in R.*

Given a ring S we shall denote by S_n the ring of all $n \times n$ matrices over S. Similarly, for an ideal A of S, we shall denote by A_n the ideal of S_n consisting of all $n \times n$ matrices with entries from A.

PROPOSITION 2.5. *A ring R is residually finite if and only if R_n is residually finite.*

Proof. Suppose that R is residually finite and let B be a non-zero ideal of R_n. Then $B = A_n$ for some non-zero ideal A of R. Since $R_n/B = R_n/A_n \cong (R/A)_n$ and since R/A is finite, B is of finite index in R_n and thus R_n is residually finite.

Conversely, if R_n is residually finite and A is a non-zero ideal of R, then A_n is of finite index in R_n. Again $(R/A)_n \cong R_n/A_n$, a finite ring. Hence R/A is finite and R is residually finite. This completes the proof.

3. Ideals in a residually finite ring.† Two ideals A and B of a ring R are said to be *comaximal* if $A + B = R$. If any two of the ideals $A_1, A_2, \ldots, A_m$

†For notation and terminology the reader is referred to [**2**, §§ **1**, 3]. The results of this section remain valid if we replace residually finite rings by rings with restricted minimum condition [**9**].

are comaximal, then we say that the ideals are *pairwise comaximal*. The proof of the following proposition is easy and hence will be omitted.

PROPOSITION 3.1. *Let A, B, and C be ideals of the ring R such that A and B, A and C are comaximal. Then A and $B \cap C$, A and BC are comaximal. Moreover, $A \cap B = AB + BA$ and the factor ring $R/(A \cap B)$ is isomorphic to the direct sum $R/A \oplus R/B$ under the correspondence $x \to (x + A, x + B)$.*

PROPOSITION 3.2. *Let R be a ring satisfying the ascending chain condition on ideals. Then an ideal Q of R contained in a maximal ideal P is right P-primary if and only if Q contains a power of P.*

Proof. Suppose that Q is right P-primary. By a result of Murdoch [**8**, p. 50, Theorem 10], Q contains a power of P. Conversely, if Q contains a power of P, say P^n, then the radical of Q is P. If X is an ideal of R not contained in P, then $X + P = R$ and thus $X + P^n = R$. Now

$$Q::X = (Q::X) \cap R = (Q::X) \cap (Q::P^n) = Q::(X + P^n) = Q::R = Q.$$

Hence Q is right P-primary. This completes the proof.

COROLLARY 3.3. *Let Q be a non-zero ideal of a residually finite ring R. Then the following statements are equivalent:*
(i) *Q is right primary;*
(ii) *Q is left primary;*
(iii) *Q contains a power of some non-zero prime ideal of R.*

In view of the above corollary we shall henceforth call a right (or left) primary ideal of a residually finite ring simply a primary ideal.

A ring R is called *primary* if $R/J(R)$ is a simple ring satisfying the descending chain condition on right ideals. Here $J(R)$ denotes the Jacobson radical of R.

PROPOSITION 3.4. *Let R be a residually finite ring and let Q_i $(i = 1, 2, \ldots, n)$ be non-zero primary ideals belonging to distinct maximal ideals of R. Then*
(1) *Q_i $(i = 1, 2, \ldots, n)$ are pairwise comaximal and $Q_1 \cap Q_2 \cap \ldots \cap Q_n = \sum_\pi Q_{\pi(1)}Q_{\pi(2)} \ldots Q_{\pi(n)}$, where π ranges through all permutations of $1, 2, \ldots, n$;*
(2) *$R/(Q_1 \cap Q_2 \cap \ldots \cap Q_n)$ is isomorphic to the direct sum of the (finite) primary rings R/Q_i $(i = 1, 2, \ldots, n)$.*

If, in addition, R has a Noetherian ideal theory, then
(3) *$Q_1 \cap Q_2 \cap \ldots \cap Q_n = Q_1Q_2 \ldots Q_n$.*

Proof. (1) and (2) follow immediately from Proposition 3.1 by induction on n and from the fact that $J(R/Q_i)$ is the unique maximal ideal $(\text{rad } Q_i)/Q_i$.

To prove (3), it suffices to consider the case when $n = 2$. Let $Q_1Q_2 = Q_1' \cap Q_2' \cap \ldots \cap Q_m'$ be a normal primary decomposition of Q_1Q_2. Since Q_1Q_2 is not primary, $m \geqq 2$. However, since for each i, rad Q_i' contains and,

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hence, equals rad Q_1 or rad Q_2, we have $m = 2$ and we may suppose that rad $Q_i' = \text{rad } Q_i$ ($i = 1, 2$). Now

$$Q_1 \subseteq Q_1':Q_2 = Q_1' \quad \text{and} \quad Q_1' \subseteq (Q_1' \cap Q_2'):Q_2' = (Q_1Q_2):Q_2' \subseteq Q_1:Q_2' = Q_1.$$

Hence $Q_1' = Q_1$. Similarly, $Q_2' = Q_2$. This completes the proof.

We see from Proposition 3.4 (3) that if a residually finite ring has a Noetherian ideal theory, then any two primary ideals belonging to different prime ideals commute. The converse is true as we shall see in the next theorem.

Given ideals A and B of R, we shall denote by A^*B the intersection of all $A + B^k$ ($k = 1, 2, \ldots$). Note that for a non-zero ideal A of the residually finite ring R, $A^*B = A + B^n$ for sufficiently large n. If, in addition, B is a maximal ideal P of R containing A, then A^*P is the unique minimal P-primary divisor of A.

THEOREM 3.5. *Let A be an ideal of the residually finite ring and let P_i ($i = 1, 2, \ldots, n$) be all the minimal prime divisors of A. If $P_iP_j = P_jP_i$ for all i and j, then*

$$A = (A^*P_1) \cap (A^*P_2) \cap \ldots \cap (A^*P_n)$$

is the unique normal primary decomposition of A.

Proof. Since A contains a power of its radical $P_1 \cap P_2 \cap \ldots \cap P_n = P_1P_2 \ldots P_n$, we have for sufficiently large k and for any permutation π of $1, 2, \ldots, n$,

$$\begin{aligned}(A^*P_{\pi(1)})(A^*P_{\pi(2)}) \ldots (A^*P_{\pi(n)}) &= (A + P_{\pi(1)}^k)(A + P_{\pi(2)}^k) \ldots (A + P_{\pi(n)}^k) \\ &\subseteq A + (P_1P_2 \ldots P_n)^k = A.\end{aligned}$$

By Proposition 3.4 (1),

$$A = (A^*P_1) \cap (A^*P_2) \cap \ldots \cap (A^*P_n)$$

which is clearly a normal primary decomposition of A. The uniqueness of this normal decomposition follows from a theorem of Murdoch [**8**, p. 54, Theorem 17].††

4. Commutative residually finite rings.

By Corollary 2.2, a proper residually finite commutative ring R is necessarily an integral domain. Such domains satisfy the ascending, but not the descending, chain condition on ideals. For brevity we shall call a commutative residually finite integral domain a residually finite domain.

THEOREM 4.1. *Let R be a residually finite domain with quotient field F. Then every subring S of F containing R is residually finite. More specifically, if a is a non-zero element of R such that R/aR has n elements, then S/aS has at most n^n elements.*

††We are indebted to the referee for pointing out this reference.

Proof. Since R/aR has n elements, we have

$$(a^nS \cap R) + aR = (a^{n+1}S \cap R) + aR = \dots . \tag{1}$$

Let c/d be an element of S. Then since dR is of finite index in R, there exists a positive integer k such that

$$a^kR + dR = a^{k+1}R + dR = \dots . \tag{2}$$

Now by (2), $a^k = a^{k+1}x + dr$ for some x and r in R. Thus

$$c/d = ax(c/d) + cr/a^k \equiv cr/a^k \pmod{aS}. \tag{3}$$

We shall show that $c/d \equiv u/a^{n-1} \pmod{aS}$ for some u in R. In view of (3), we may suppose that $n \leqq k$. Since $cr = a^k(cr/a^k)$ which belongs to $a^kS \cap R$, we have by (1), $cr = a^{k+1}s + at$ for some s in S and t in R. It follows that

$$c/d \equiv cr/a^k = as + t/a^{k-1} \equiv t/a^{k-1} \pmod{aS}.$$

Continuing this process we obtain

$$c/d \equiv t/a^{k-1} \equiv \dots \equiv u/a^{n-1} \pmod{aS},$$

where $t, \dots, u$ are elements of R.

Let $x_1, x_2, \dots, x_n$ be representatives of the distinct cosets of aR in R. Then $u = x_{1'} + ax_{2'} + \dots + a^{n-1}x_{n'} + a^ny$, where $1', 2', \dots, n'$ belong to $\{1, 2, \dots, n\}$ and $y \in R$. Thus

$$c/d \equiv u/a^{n-1} \equiv x_{1'}/a^{n-1} + x_{2'}/a^{n-2} + \dots + x_{n'} \pmod{aS}.$$

Hence S/aS has at most n^n elements.

Now let A be a non-zero ideal of S. Then there exist non-zero elements a and b in R such that $a/b \in A$. The ideal of S generated by a/b equals aS and is contained in A. By the above results, S/aS and hence S/A is finite. This shows that S is residually finite and hence completes the proof.

THEOREM 4.2. *Let R be an integral domain satisfying the ascending chain condition on ideals and let S be an extension domain of R, integrally dependent on R. If S is residually finite, then so is R.*

Proof. Let P be a non-zero prime ideal of R. Then by [**10**, p. 257, Theorem 3], there exists a prime ideal P' of S such that $P' \cap R = P$. Now $R/P = R/(P' \cap R) \cong (R + P')/P'$ which is a subring of the finite ring S/P' and thus is finite. By Corollary 2.4, R is residually finite. This completes the proof.

COROLLARY 4.3. *An integral domain satisfying the ascending chain condition on ideals is residually finite if and only if its integral closure is residually finite.*

LEMMA 4.4. *Let S be an extension domain of the residually finite domain R. If S is a finitely generated R-module, then S is residually finite.*

Proof. It suffices to show that for an element x of S, the subring $R[x]$ of S generated by R and x is residually finite. Let a be a non-zero element of $R[x]$.

Then $aR[x] \cap R$ is a non-zero ideal of R. Since $R[x]$ is a finitely generated R-module, $R[x]/aR[x]$ is a finitely generated module over the finite ring $R/(aR[x] \cap R)$ and thus is finite. Hence $R[x]$ is residually finite.

THEOREM 4.5. *Let R be a residually finite domain with quotient field F and let S be an extension domain of R, integrally dependent on R. If the quotient field K of S is a finite algebraic extension of F, then S is residually finite.*

Proof. Suppose that $K = F(x_1, x_2, \ldots, x_n)$, where $x_i = a_i/b_i$ with a_i, b_i in S. Then $R[a_1, \ldots, a_n, b_1, \ldots, b_n]$ with quotient field K is residually finite by Lemma 4.4. By Theorem 4.1, S is residually finite.

COROLLARY 4.6. *Let R be a residually finite domain with quotient field F and let K be a finite algebraic extension of F. Then the integral closure R' of R in K is residually finite and thus is every ring between R and R'.*

The above corollary implies in particular that the ring of algebraic integers of an algebraic number field and the ring of integral functions in the field of algebraic functions of a single variable over a finite field are residually finite.

5. Residually finite semi-local rings. A commutative ring satisfying the ascending chain condition on ideals and having a finite number of maximal (prime) ideals is called a *semi-local ring*. A semi-local ring with precisely one maximal ideal is called a *local ring*.

Throughout this section R denotes a residually finite semi-local ring; $M_1, M_2, \ldots, M_n$ the maximal ideals of R; and U the intersection (product) of the maximal ideals. Let I be the intersection of U^k ($k = 0, 1, 2, \ldots$; $U^0 = R$). If R is finite, then U is nilpotent and thus $I = (0)$. When R is a field, $I = U = (0)$. In case R is a proper integral domain, there are infinitely many non-zero ideals. The intersection of all these ideals is zero. Since U is contained in the radical of any non-zero ideal of R, I is contained in every non-zero ideal of R. Hence in any case, $I = (0)$.

A metric d can then be defined on R. Let x and y be two elements of R. If $x = y$ we set $d(x, y) = 0$. If $x \neq y$, then there exists a non-negative integer k such that $x \equiv y \pmod{U^k}$ but $x \not\equiv y \pmod{U^{k+1}}$. Set $d(x, y) = 2^{-k}$.

The topology induced by the metric d on R has the family of sets $a + U^k$ ($k = 0, 1, 2, \ldots$) as a base for the neighbourhood system at a. For any subset E of R, the closure $\mathrm{Cl}(E)$ of E is the intersection of $E + U^k$ ($k = 0, 1, 2, \ldots$). It is not difficult to verify that every non-zero ideal of R is both open and closed and R is a totally disconnected topological ring.

As a metric space, the residually finite semi-local ring R has a completion R^* (unique to within an isomorphism over R) consisting of all Cauchy sequences (x_i) in R with equality, addition, multiplication, and metric defined as follows:

$$(x_i) = (y_i) \quad \text{if and only if } \lim d(x_i, y_i) = 0;$$
$$(x_i) + (y_i) = (x_i + y_i); \qquad (x_i)(y_i) = (x_i y_i);$$
$$d((x_i), (y_i)) = \lim d(x_i, y_i).$$

The mapping $x \to (x)$, where (x) denotes the sequence all of whose terms are x, is a metric-preserving isomorphism of R into R^*. On identifying x with (x), R becomes a dense subring of the complete ring R^*.

LEMMA 5.1. *Let R be a residually finite semi-local ring, let A be a non-zero ideal of R and let B be an ideal of R^* which meets R non-trivially. Then*

(i) *AR^* consists of all Cauchy sequences of elements in A;*

(ii) *$R \cap AR^* = A$;*

(iii) *$R/A \cong R^*/AR^*$;*

(iv) *$(B \cap R)R^* = B$.*

Proof. Observe that since U is contained in the radical of A, there exists m such that $U^m \subseteq A$.

(i) Let (x_i) be an element of AR^*. Then there exist $a_1, a_2, \ldots, a_k$ in A and r_{ji} $(j = 1, 2, \ldots, k; i = 0, 1, \ldots)$ in R such that $(x_i) = \sum_j a_j(r_{ji}) = (\sum_j a_j r_{ji})$, which is a Cauchy sequence of elements of A. To prove the converse, let (a_i) be a Cauchy sequence of elements in A. Without loss of generality, we may assume that $a_0 = 0$ and $a_k - a_{k-1} \in U^{m+k} \subseteq A\,U^k$ for all k. It follows that $a_k - a_{k-1} = \sum_j g_j h_{jk}$, where g_j are the generators of the ideal A and h_{jk} belongs to U^k. Now $(a_i) = (\sum_{k=1}^{i} a_k - a_{k-1}) = \sum_j g_j(f_{ji})$, where (f_{ji}) is a Cauchy sequence whose ith term is $h_{j1} + h_{j2} + \ldots + h_{ji}$. Hence (a_i) belongs to AR^*.

(ii) Let (x_i) be an element of $R \cap AR^*$. Then by (i), (x_i) may be taken to be a sequence of elements in A. Now $(x_i) = r$ for some r in R. This implies that $x_i \equiv r \pmod{U^m}$ for sufficiently large i. Hence $r \in A$.

(iii) Since $R/A = R/(R \cap AR^*) \cong (R + AR^*)/AR^*$, it suffices to show that $R^* = R + AR^*$. Let (x_i) be a Cauchy sequence in R. Then there exists j such that $x_i - x_j \in U^m \subseteq A$ for all $i \geqq j$. Now $(x_i) - x_j = (x_i - x_j)$ which is a Cauchy sequence in A and thus belongs to AR^* by (i). Hence (x_i) is in $R + AR^*$.

(iv) By (iii), $R/(B \cap R) \cong R^*/(B \cap R)R^*$ under the isomorphism

$$r + (B \cap R) \to (r) + (B \cap R)R^*.$$

However, $(B + R)/B \cong R/(B \cap R)$ under the isomorphism $r + B \to r + (B \cap R)$. Thus $r + B \to (r) + (B \cap R)R^*$ is an isomorphism. In particular, $B = (B \cap R)R^*$. This completes the proof.

It follows from Lemma 5.1 (iii) that $M_1R^*, M_2R^*, \ldots, M_nR^*$ are maximal ideals of R^*. In fact, these are the only maximal ideals of R^* since any element of R^* not in any of the M_iR^* is invertible.

Finite metric spaces are always complete. We shall therefore concern ourselves with infinite rings. As an implication of [**1**, p. 693, Proposition 2] and Corollary 2.2, an infinite residually finite complete semi-local ring is local. Hence if R is an infinite residually finite semi-local ring which is not local, then its completion R^* cannot be residually finite. However, we shall show later (Theorem 5.2) that there exists a residually finite complete local homomorphic image of R^* which contains an isomorphic image of R.

In view of Lemma 5.1, we may ask if every non-zero ideal of R^* meets R non-trivially. If the answer is affirmative, then by Lemma 5.1 (iii), (iv), R^* is residually finite. This forces R to be local. Hence, for an infinite residually finite semi-local ring R which is not local there are non-zero ideals B of R^* such that $B \cap R = (0)$.

THEOREM 5.2. *Let R be an infinite residually finite semi-local ring. Then R can be embedded in a residually finite complete local ring R' which is a homomorphic image of the completion R^* of R.*

Proof. Let K be an ideal of R^* which is maximal among the ideals C of R^* such that $C \cap R = (0)$ and let $R' = R^*/K$. Since $R \cong R/(R \cap K) \cong (R + K)/K$, R is isomorphic to a subring of R'. If B' is a non-zero ideal of R', then $B' = B/K$ for some ideal B of R^* containing K properly. Thus $B \cap R \neq (0)$. By Lemma 5.1,

$$R'/B' = (R^*/K)/(B/K) \cong R^*/B = R^*/(B \cap R)R^* \cong R/(B \cap R)$$

which is a finite ring. Hence R' is residually finite. Moreover, the maximal ideals of R' are those MR^*/K, where M is a maximal ideal of R such that $MR^* \supseteq K$. Thus R' is semi-local. It remains to show that R' is complete in its natural topology. This, however, follows immediately from [**1**, p. 693, Proposition 1].

COROLLARY 5.3. *The completion of an infinite residually finite local ring is residually finite.*

Proof. Let M be the maximal ideal of the infinite residually finite local ring R and let K and R' be as in the proof Theorem 5.2. Then MR^*/K is the maximal ideal of the residually finite complete local ring R'. Since $(MR^*/K)^n = ((MR^*)^n + K)/K = (M^nR^* + K)/K$, we have $(MR^*/K)^n \cap R \supseteq M^n$ (with equality sign when $n = 1$). Hence R' is a completion of R. This completes the proof.

It is clear that every finite ring is compact, while an infinite field is complete but not compact relative to the natural topology. Aside from these two exceptional cases, we have the following result.

PROPOSITION 5.4. *Let R be a proper residually finite semi-local ring. Then R is complete if and only if it is compact.*

Proof. Assume that R is complete and let (x_{0i}) be a sequence of elements of R. Since R/U^k is finite, there exists a subsequence (x_{ki}) of (x_{0i}) all of whose terms are in the same coset of U^k in R. By induction we may choose (x_{ki}) to be a subsequence of $(x_{k-1,i})$. Let $y_i = x_{ii}$. Then (y_i) is a Cauchy, and hence convergent, subsequence of the given sequence. Hence R is compact. The converse is obvious. This completes the proof.

An element of a commutative ring is a unit (i.e., an invertible element) if and only if it does not belong to any of the maximal ideals. It follows that the units of a residually finite semi-local ring form a subset which is both closed and open.

PROPOSITION 5.5. *Let R be a residually finite semi-local ring and let G be the group of units of R. Then the following statements are equivalent:*

(1) *R is compact;*

(2) *G is a compact topological group with inherited topology;*

(3) *R contains a compact non-zero ideal* (*i.e.*, *R is locally compact*).

Proof. (1) implies (2): If R is compact, then by [**7**, p. 155, Lemma 5], the inversion $x \rightarrow x^{-1}$ on G is continuous and thus G is a topological group. This group is compact since it is a closed subset of R.

(2) implies (3): We may suppose that R is a proper integral domain so that the intersection U of all maximal ideals of R is not zero. Since U is closed, $1 + U$ is a closed subset of the compact set G. Accordingly, $1 + U$, and hence U, is compact.

(3) implies (1): Let A be a compact non-zero ideal of R. Then each coset of A in R is compact and R is a union of a finite number of compact subsets. Hence R is compact.

THEOREM 5.6. *Let R be a ring which is not an infinite field. Then R is residually finite complete local if and only if R is a compact topological ring such that* (i) *R contains a unique non-zero proper prime ideal P; and* (ii) *P^2 is open.*

Proof. If R is compact and satisfies conditions (i) and (ii), then by [**7**, p. 169, Theorem 20], R is a local ring with the natural topology. Now R/P is a compact field and thus is finite [**7**, p. 164, Theorem 16]. Hence by Corollary 2.4, R is residually finite.

Conversely, if R is residually finite complete local, then by Proposition 5.4, R is compact. Since R is residually finite, every non-zero proper prime ideal is maximal. Hence R contains a unique non-zero proper prime ideal P. P^2 is clearly open. This completes the proof.

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