Compositio Mathematica **106:** 321–348, 1997.

Grothendieck topologies and deformation Theory II

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Received 13 February 1996; accepted in revised form 7 March 1996

Abstract. Starting from a sheaf of associative algebras over a scheme we show that its deformation theory is described by cohomologies of a canonical object, called the cotangent complex, in the derived category of sheaves of bi-modules over this sheaf of algebras. The passage from deformations to cohomology is based on considering a site which is naturally constructed out of our sheaf of algebras. It turns out that on the one hand, cohomology of certain sheaves on this site control deformations, and on the other hand, they can be rewritten in terms of the category of sheaves of bi-modules.

Mathematics Subject Classifications (1991): 14F20, 18F10.

Key words: Grothendieck topologies, sheaves, deformations, derived categories.

0. Introduction

0.1. In the present paper we continue the study of deformation theory of algebras using the approach of [Ga]. We will extend the main results of [Ga] to the global case. Namely, we pose and solve the following problem: what cohomological machinery controls deformations of a sheaf of algebras over a scheme? This question has already been studied by many authors [Ill], [GeSch], [Schl].

0.2. Let first A be an associative algebra over a ring. Consider the category of all algebras over A, let us call it $C(A)$. One can observe that every question concerning the deformation theory of A can be formulated in terms of this category.

Our first step will be to apply a linearization procedure to $C(A)$, in other words we will endow it with a Grothendieck topology and then we will consider sheaves of abelian groups on it. It will turn out that deformations of A are controlled by cohomologies of certain sheaves on this site. Cohomologies arise naturally as classes attached to torsors and gerbes. All this was done in [Ga].

When A is no longer an algebra over a ring but rather a quasi-coherent sheaf of algebras over a scheme X, the definition of $C(A)$ must be modified in order to take into account possible localization with respect to X, since the appropriate cohomology theory would incorporate algebra cohomology of A and scheme cohomology of X. In this case instead of working with the whole category of sheaves on our site, we single out a subcategory which we call the category of quasi-coherent sheaves. This category will have properties similar to those of the category of quasi-coherent sheaves of A-bimodules among all sheaves of A-bimodules and it will be more manageable.

The second step will be to find a connection between the category of sheaves on $C(A)$ and the category of quasi-coherent sheaves of A-bimodules on our scheme X. This connection will be described by two mutually adjoint functors, which would enable us to rewrite the cohomology groups that control deformations of A in terms of cohomologies of some canonical object $T^\bullet(A)$ of the derived category of quasi-coherent sheaves of A-bimodules. The object $T^\bullet(A)$ will be called the cotangent complex of A. Another approach to the construction of the cotangent complex in a slightly different situation was used by Illusie [Ill].

0.3. Let us now describe the contents of the paper.

In Section 1 we present a brief exposition of some well known facts and results from the theory of sites. For a more detailed discussion the reader is referred to [Ar], [Gr]. In the remaining sections we will freely operate with the machinery of sheaves, cohomologies, direct and inverse images; therefore the reader is advised to look through this section in order to become familiar with the notation.

In Section 2 we define the site $C_X(A)$ along with its variants for affine schemes. We introduce also the appropriate categories of sheaves and functors between them. The central results are

(1) Theorem 2.3.3 with its corollaries, that insure that the category $\mathrm{Sh}^{qc}(A)$ is well defined
(2) Theorem 2.5 that says that cohomologies of quasi-coherent sheaves computed inside the quasi-coherent category and inside the category of all sheaves give the same answers.

In Section 3 we introduce functors $\mathfrak{I}$ and $\mathcal{L}$ that establish connection between the category $\mathrm{Sh}^{qc}(X)$ and the category A_{qc}-mod. Let us remark that it would be possible to work with the category of all sheaves on $C_X(A)$ without introducing quasi-coherent sheaves explicitly. We, nevertheless, decided to do that, since to our mind, introducing this category and basic functors that are connected to it reflects the nature of the things and clarifies the exposition.

Finally, Section 4 is devoted to deformation theory. Theorem 4.2 describes how to pass from deformations to cohomology of sheaves on $C_X(A)$ via torsors and gerbes, and in 4.3 we translate the assertions of this theorem to the language of cohomology of quasi-coherent sheaves of A-bimodules.

0.4. The results of the present paper can be easily generalized to the case of algebras over an arbitrary operad (cf. [Ga]). We opted for treating the case of associative algebras only in order to simplify the exposition. One can also develop a similar theory for operad co-algebras.

0.5. In recent years there have been a lot of interest in deformation theory. We have to mention the works [Ge-Sch 1,2], [Ma], [Ma-St], [St], [Fo]. Our approach is close to that of [Ill]. Let us also point out that one of the central ideas of the present paper: to resolve an algebra A by free algebras (at least locally) goes back to Quillen and to Grothendieck [Qu].

1. Preliminaries on Grothendieck topologies

1.0. In this section we will review certain notions from the theory of sites. Proofs will be given mostly in cases when our exposition differs from the standard one.

1.1. Let C be a category possessing fiber products. A Grothendieck topology (cf. [Gr]) on it (or a structure of a site) is a collection of morphisms that are called covering maps if it satisfies the following three conditions:

(1) Any isomorphism is a covering.
(2) If $\phi: U \to V$ and $\psi: V \to W$ are coverings, then their composition $\psi \circ \phi: U \to W$ is a covering too.
(3) If $\phi: U \to V$ is a covering and if $\alpha: V_1 \to V$ is an arbitrary morphism, then the base change map $\phi_1: U \underset{V}{\times} V_1 \to V_1$ is a covering.

1.1.1. EXAMPLES

1. For any category C there exists the minimal Grothendieck topology: the only coverings are isomorphisms. This site will be denoted by (C, min).

2. Let Set be the category of sets. We introduce the structure of a site on it by declaring surjections to be the covering maps.

3. Let Set^o be the category opposite to Set. We introduce a Grothendieck topology by declaring $\phi: X \to Y$ to be a covering if the corresponding map of sets $Y \to X$ is an injection.

4. Constructions similar to the above ones can be carried out when the category Set is replaced by an abelian category, in particular, by the category Ab, the category of abelian groups.

5. Let X be a topological space. Let $C(X)$ be the category whose objects are finite disjoint unions of open subspaces of X.

$$\text{Hom}(U, V) \overset{\text{def}}{=} \text{maps from } U \text{ to } V \text{ compatible with the embedding to } X.$$

A map $\phi \in \text{Hom}(U, V)$ is a covering if it is surjective.

6. Let a cite C have a final object X_0 and let X be any other object of C. We can define a new cite C_X whose underlying category is the category of 'objects of C over X', with morphisms being compatible with projections to X. A morphism ϕ in C_X is declared to be a covering if it is a covering in C.

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1.1.2. Let C_1 and C_2 be two sites. A functor F between the underlying categories is said to be a functor between sites if the following holds:

(1) F maps coverings to coverings.

(2) If A, B, D are three objects in C_1 with A, B mapping to D, then the canonical map $F(A \underset{D}{\times} B) \to F(A) \underset{F(D)}{\times} F(B)$ is a covering in C_2.

We say that a functor F between two sites is strict if it preserves fiber products, i.e. if the map in (2) is an isomorphism.

1.2. DEFINITION. A sheaf of sets (resp. of abelian groups) on a site C is a functor S between the sites C and Set^o (resp. Ab^o), the latter considered with the topology specified in the Example 3 above.

Morphisms between sheaves are by definition natural transformations between such functors.

DEFINITION. A presheaf of sets on C (resp. of abelian groups) is sheaf on C when the latter is considered with the minimal topology.

It is an easy exercise to verify that the above definition of a sheaf coincides with the traditional one. From now on, by a sheaf we will mean a sheaf of abelian groups. It will be left to the reader to make appropriate modifications for sheaves of sets. Note, that a sheaf of abelian groups is the same as a group-like object in the category of sheaves of sets.

The category of sheaves will be denoted by $\mathrm{Sh}(C)$. This category possesses a natural additive structure and is in fact an abelian category. If S is a sheaf, and if $X \in C, S(X)$ will be denoted by $\Gamma(X, S)$ and will be called the set of sections of S over X. The map $\Gamma(X, S) \to \Gamma(Y, S)$ for a map $Y \to X$ will be called the restriction map.

1.3. Let $F: C_1 \to C_2$ be a functor between sites. We have then the natural functor (called direct image) $F_\bullet: \mathrm{Sh}(C_2) \to \mathrm{Sh}(C_1)$. This functor is always left exact.

The functor $F_\bullet$ has a left adjoint (called the inverse image): $F^\bullet: \mathrm{Sh}(C_1) \to \mathrm{Sh}(C_2)$. The functor $F^\bullet$ is always right exact.

The following standard facts are often used in the theory of sheaves:

PROPOSITION

(1) *The functor $F_\bullet$ is right exact if for any covering $Z \to F(X)$ there exists a covering $\beta: Y \to X$, endowed with a map $\alpha: F(Y) \to Z$ such that the composition $F(Y) \to Z \to F(X)$ coincides with $F(\beta)$.*

(2) *The functor $F^\bullet$ is left exact if the functor F is strict in the sense of* 1.1.2.

(3) *Assume that F is strict. Then the functor $F^\bullet$ is faithful if for any $\alpha: Y \to X$ and $\beta: Z \to F(Y)$ such that the composition $Z \to F(Y) \to F(X)$ is a covering, α is a covering too.*

1.3.1. EXAMPLES

1. Let Forget: $(C, \min) \to C$ be the canonical functor of sites. The above constructions yield the embedding functor from sheaves to presheaves and its left adjoint, which is called the functor of associating a sheaf to a presheaf. It is a good exercise to describe the associated sheaf explicitly.

2. Let pt be the category of one object and one morphism. If C is a site, for any $X \in C$ we have a functor $pt_X: pt \to C$, that sends the unique object of pt to X. We have the canonical constant sheaf $\mathbb{Z}$ on pt. Let by definition $\mathbb{Z}_X = pt_X^\bullet(\mathbb{Z})$. This sheaf will be called the constant sheaf corresponding to X. By definition we have: $\mathrm{Hom}(\mathbb{Z}_X, S) = \Gamma(X, S)$ functorially with respect to $S \in \mathrm{Sh}(C)$.

Analogous construction produces also a sheaf Const_X in the category of sheaves of sets. A sheaf of sets is called representable if it is of the form Const_X for some X.

3. Let $F: C_1 \to C_2$ be a functor between sites and let $X \in C_1$. Then

$$F^\bullet(\mathbb{Z}_X) \simeq \mathbb{Z}_{F(X)}.$$

4. Recall the situation of 1.1.1 Example 6. We have the natural embedding functor $i: C_X \to C$ and its right adjoint Cart: $Y \to Y \underset{X_0}{\times} X$. Note, that the functors $i_\bullet$ and $\mathrm{Cart}^\bullet$ are canonically isomorphic. We denote this functor by $S \to S|C_X$ and call it the functor of restriction of a sheaf to C_X. By definition, for $Y \in C_X$ we have $\Gamma(Y, S|C_X) \simeq \Gamma(Y, S)$.

If now $X \to X_0$ is a covering, the functor $S \to S|C_X$ is exact and faithful.

1.4. *Cohomology of sheaves*. Along with the abelian category $\mathrm{Sh}(C)$ one considers also the corresponding derived categories $D(\mathrm{Sh}(C)), D^+(\mathrm{Sh}(C)), D^-(\mathrm{Sh}(C))$ and $D^b(\mathrm{Sh}(C))$. It can be shown [Ar], [Gr] that the category $\mathrm{Sh}(C)$ has enough injective objects. In particular, any left exact functor admits a right derived functor. If $X \in C$, $R^i\Gamma(X, S)$ will be denoted by $H^i(X, S)$.

1.5. *Torsors and Gerbes*

1.5.0. Let now our category possess a final object X_0 and let S be a sheaf of abelian groups. $H^i(S)$ will denote $H^i(X_0, S) \simeq R^i\Gamma(X_0, S)$.

1.5.1. Before defining torsors and gerbes in the sheaf-theoretic context we need to recall several definitions.

Let Γ be an abelian group and let Γ act on a set τ. We say that τ is a torsor over Γ if this action is simply transitive. Torsors over a given group form a rigid

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monoidal category (cf. [DM]) under $\tau_1 \otimes \tau_2 \to \tau_1 \times \tau_2/\Gamma$ with the anti-diagonal action of Γ.

Let now O be a monoidal category and let M be an arbitrary category. We say that O acts on M if we are given

(1) A functor Action: $O \times M \to M$.
(2) A natural transformation between the two functors $O \times O \times M \to M$

$$\begin{array}{ccc} O \times O \times M & \xrightarrow{\text{Action}} & O \times M \\ \downarrow & & \downarrow{\scriptstyle \text{Action}} \\ O \times M & \xrightarrow{\text{Action}} & M \end{array}$$

such that the obvious 'pentagon' identity is satisfied.

We say that M is a gerbe bound by O, if for any $X \in M$ the functor $O \to M$ given by $A \to \text{Action}(A \times X)$ is an equivalence of categories.

If O is a groupoid and if M is a gerbe bound by O, then M is also a groupoid and $\pi_0(M)$ is a torsor over $\pi_0(O)$.

1.5.2. A sheaf of sets Υ is called a torsor over S if

(1) S viewed as a group-like object in the category of sheaves of sets acts on the object Υ, i.e. if for every $X \in C, \Gamma(X, S)$ acts on $\Gamma(X, \Upsilon)$ in a way compatible with restrictions.
(2) For every $X \in C, \Gamma(X, \Upsilon)$ is a torsor over $\Gamma(X, S)$, whenever the former is nonempty.
(3) For some covering X of X_0, the set $\Gamma(X, \Upsilon)$ is nonempty.

Let $\text{Tors}_C(S)$ denote the category of torsors over S. From 1.5.1 we deduce that it is a groupoid and that it possesses a structure of a rigid monoidal category.

LEMMA. *The group* $\pi_0(\text{Tors}_C(S))$ *is canonically isomorphic to* $H^1(S)$.

Proof. In fact, we claim more: Consider the category $\text{Ext}_C(\mathbb{Z}_{X_0}, S)$, whose objects are short exact sequences $0 \to S \to E \to \mathbb{Z}_{X_0} \to 0$ and whose morphisms are maps between such sequences that induce identity maps on the ends.

We claim, that this category is canonically equivalent to $\text{Tors}_C(S)$.

Indeed, for any such extension $0 \to S \to E \to \mathbb{Z}_{X_0} \to 0$ we associate a torsor Υ by setting for every X over X_0

$$\Gamma(X, \Upsilon) = \text{splittings: } \mathbb{Z} \simeq \Gamma(X, \mathbb{Z}_{X_0}) \to \Gamma(X, E).$$

This functor is easily seen to be an equivalence of (monoidal) categories and

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$$\pi_0(\mathrm{Ext}(\mathbb{Z}_{X_0}, S)) \simeq \mathrm{Ext}^1(\mathbb{Z}_{X_0}, S) \simeq R^1 \mathrm{Hom}(\mathbb{Z}_{X_0}, S) \\ \simeq R^1\Gamma(X_0, S)) \simeq H^1(S). \qquad \square$$

1.5.3. We are heading towards the definition of gerbes, but first we need to recollect the notion of a stack.

Let C be a site with a final object X_0. Suppose that for each $X \in C$ we are given a category $G(X)$, for each map $\alpha: Y \to X$ we are given a functor $G_\alpha: G(X) \to G(Y)$ and for each composition of maps $\alpha: Y \to X$ and $\beta: Z \to Y$, we are given a natural transformation $G_\alpha \circ G_\beta \Rightarrow G_{\alpha\circ\beta}$, such that all the data are compatible with respect to two-fold compositions.

Remark. Functors $G_\alpha: G(X) \to G(Y)$ for $\alpha: Y \to X$ will be called the restriction functors and will be often denoted as $s \in G(X) \to s|Y \in G(Y)$.

Such a collection of categories and of functors is called a presheaf of categories. It is said to be a sheaf of categories (or a stack) if moreover the following two axioms are satisfied:

(1) Let $X \in C$, and let us consider the category C_X as in 1.1.1 Example 6. Let also s_1, s_2 be two objects of $G(X)$. We can consider the presheaf of sets on C_X

$$(Y, \alpha: Y \to X) \in C_X \to \mathrm{Hom}(G_\alpha(s_1), G_\alpha(s_2)).$$

We require that this presheaf is a sheaf for each $X \in C$.

(2) Let $\phi: Y \to X$ be a covering. Consider the category of descent data on Y with respect to X, whose objects are pairs $s \in G(Y)$ and an isomorphism $G_{p_1}(s) \to G_{p_2}(s)$, where p_1, p_2 are the two projections from $Y \underset{X}{\times} Y$ to Y, such that the above isomorphism satisfies the obvious cocycle condition on the three-fold fiber product of Y with itself over X. Morphisms in this category are defined to be maps $s_1 \to s_2$ compatible with isomorphisms between their pull-backs on $Y \underset{X}{\times} Y$. We have the obvious functor from $G(X)$ to this category of descent data. We require that this functor is an equivalence of categories.

EXAMPLES

1. Consider the presheaf of categories $G(X) := C_X$.

LEMMA. *This presheaf of categories is a stack if and only if*

(1) *For every $X \in C$, $Y \in C_X$, the presheaf on C_X given by $Z \in C_X \to \mathrm{Hom}_{C_X}(Z, Y)$ is a sheaf.*

(2) *For every sheaf of sets S on C_X the fact that for some $Y \in C_X$ covering X, $S|C_Y$ is representable implies that S is representable as well.*

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The proof follows directly from the definitions.

All the sites in this paper will satisfy the conditions of the above Lemma.

2. If S is a sheaf of groups, we can define $G(X) = \mathrm{Tors}_{C_X}(S|C_X)$ (torsors over $S|C_X$ in the category C_X). This presheaf of categories is always a stack, which we denote by Tors_S.

1.5.4. *Gerbes*. Let once again C be a site with a final object X_0 and let S be a sheaf of abelian groups. Let G be a stack on C endowed with the following additional structure:

(1) Each $G(X)$ is acted on by the monoidal category $\mathrm{Tors}_{C_X}(S|C_X)$.

(2) For each $\alpha: Y \to X$ we are given a natural transformation between two functors $\mathrm{Tors}_{C_X}(S|C_X) \times G(X) \to G(Y)$

$$G_\alpha \circ \mathrm{Action}_X \to \mathrm{Action}_Y \circ (\mathrm{Tors}_{S_\alpha} \times G_\alpha),$$

which is compatible with the natural transformations of 1.5.1(2) and with composition of restrictions.

Suppose that for each $X \in C, G(X)$ is a gerbe over $\mathrm{Tors}_{C_X}(S|C_X)$ and that there exists a covering X of X_0 such that $G(X)$ is nonempty. We say then that G is a gerbe bound by S.

Functors between gerbes bound by a sheaf of abelian groups S and natural transformations between such functors are defined in a natural fashion.

Remark. Let S be a sheaf of abelian groups and let G be a stack such that if $s_1, s_2 \in G(X)$, there exists a covering Y of X such that the pull-backs of s_1 and s_2 on $G(Y)$ become isomorphic. Then G is a gerbe bound by S if for every $X, s \in G(X), \mathrm{Aut}(s)$ is isomorphic to $\Gamma(X, S)$ functorially in X and in s.

EXAMPLES

1. Let S be in $\mathrm{Sh}(C)$. A basic example of a gerbe bound by S is the stack Tors_S of 1.5.3 (Example 2) above.

It is an easy observation that a gerbe G is equivalent to Tors_S if and only if $G(X_0)$ is nonempty.

2. Let $S_1 \to S_2$ be a map of sheaves of abelian groups. If G is a gerbe bound by S_1, we can construct an induced gerbe G' bound by S_2.

3. (cf. [D-Ill], [BB]) Let $0 \to S \to K_0 \to K_1 \to \mathbb{Z}_{X_0} \to 0$ be an exact sequence of sheaves on C. Let $K^\bullet$ denote the 2-complex $K_0 \to K_1$. To this 2-complex we can associate a gerbe $G(K^\bullet)$ bound by S in a canonical way by

setting: $G(K^\bullet)(X) =$ the category of extensions $0 \to K_1|C_X \to E \to \mathbb{Z}_X \to 0$ of sheaves over C_X endowed with a map of complexes

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & K_0|C_X & \longrightarrow & E & \longrightarrow & \mathbb{Z}_X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
0 & \longrightarrow & S & \longrightarrow & K_0|C_X & \longrightarrow & K_1|C_X & \longrightarrow & \mathbb{Z}_X & \longrightarrow & 0
\end{array}$$

It is easy to verify that $G(K^\bullet)$ defined in this way is indeed a gerbe. The following assertion follows from the definitions:

LEMMA

$$\pi_0(G(K^\bullet)(X)) \simeq \mathrm{Hom}_{D(\mathrm{Sh}(C_X))}(\mathbb{Z}_X[-1], K^\bullet|C_X).$$

If now $\alpha: K^\bullet \to K'^\bullet$ is a quasi-isomorphism of 2-complexes, we get a canonical functor between the corresponding gerbes $G(K^\bullet)$ and $G(K'^\bullet)$. This means that the operation of assigning a gerbe to a 2-complex is well defined on the derived category $D(\mathrm{Sh}(C))$.

1.5.5. The following proposition is not difficult to prove:

PROPOSITION. *The assignment $K^\bullet \to G(K^\bullet)$ establishes a one-to-one correspondence between the set isomorphism classes of objects $K^\bullet$ in $D(\mathrm{Sh}(C))$ with nontrivial cohomologies only in degrees 0 and 1, such that $H^0(K^\bullet) \simeq S$, $H^1(K^\bullet) \simeq \mathbb{Z}_{X_0}$ and the set of equivalence classes of gerbes G bound by S.*

In particular, since the set of isomorphism classes of 2-complexes of the above type in the derived category is $\mathrm{Ext}^2(\mathbb{Z}_{X_0}, S) \simeq H^2(S)$, to any gerbe G bound by S we can associate a well defined class in $H^2(S)$ that vanishes if and only if $G(X_0)$ of this gerbe is nonempty.

2. The site $C_X(A)$

2.0. As it has been explained in the introduction, our bridge between deformations and cohomology is based on considering sheaves on the site $C_X(A)$ which we are about to define. Throughout this paper, by a scheme we will mean a separated scheme. It is not difficult, however, to generalize all our results to the case of an arbitrary scheme.

If $f: X \to Y$ is a morphism of schemes, f_{us}^* and f_{us*} will denote the usual inverse and direct image functors on the categories of quasi-coherent sheaves of $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ modules.

Remark. For an affine scheme we will usually make no distinction between a quasi-coherent sheaf and its global sections.

2.1. Let X be a scheme and let Zar_X denote the Zariski site of X, whose objects are disjoint finite unions of open subsets of X and whose morphisms are maps of schemes over X. A morphism in Zar_X is a covering if it is surjective. Let A be a quasi-coherent sheaf of associative algebras on X.

2.1.1. DEFINITION of $C_X(A)$.

Objects: triples (U, B_U, ϕ), with $U \in \mathrm{Zar}_X$, B_U is a quasi-coherent sheaf of associative algebras on U and $\phi\colon B_U \to A|U$ is a map of sheaves of associative algebras. Here $A|U$ is the restriction of A on U. When no confusion can be made, we will omit ϕ.

Morphisms: $\mathrm{Hom}((V, D_V), (U, B_U))$ is a set of pairs (j, α), where $j \in \mathrm{Hom}_{\mathrm{Zar}_X}(V, U)$ and $\alpha\colon D_V \to B_U|V$ (restriction by means of j).

The category $C_X(A)$ is easily seen to have fiber products.

Topology: $(j, \alpha)\colon (V, D_V) \to (B, B_U)$ is said to be a covering map if j is a covering in Zar_X, and if α is an epimorphism.

Sometimes when no confusion can be made we will write A instead of $\Gamma(X, A)$ for X being an affine scheme.

2.1.2. *Variant*. When X is an affine scheme $X = \mathrm{Spec}(R)$, the site $C_X(A)$ will be often denoted by $C_X^{\mathrm{new}}(A)$ to emphasize the difference between $C_X^{\mathrm{new}}(A)$ and $C_X^{\mathrm{old}}(A)$:

DEFINITION of $C_X^{\mathrm{old}}(A)$

Objects: R-algebras B with a map to A.
Morphisms: Algebra homomorphisms commuting with structure maps to A.
Topology: Covering maps are defined to be just epimorphisms of algebras.

2.1.3. $\mathrm{Sh}_X(A)$ will denote the category of sheaves of abelian groups over $C_X(A)$. For X affine, $X = \mathrm{Spec}(R)$, this category will also be denoted by $\mathrm{Sh}_X^{\mathrm{new}}(A)$, whereas $\mathrm{Sh}_X^{\mathrm{old}}(A)$ will denote the category of sheaves of abelian groups over $C_X^{\mathrm{old}}(A)$.

EXAMPLE 1. Let $(U, B_U) \in C_X^{\mathrm{new}}(A)$. According to 1.3.1(2) we can consider the sheaf $\mathbb{Z}_{(U, B_U)} \in \mathrm{Sh}_X^{\mathrm{new}}(A)$.

EXAMPLE 2. A similar construction can be carried out in the case of an affine scheme $X = \mathrm{Spec}(R)$ for $C_X^{\mathrm{old}}(A)$. For a projective R-module V, let $\mathrm{Free}(V)$

denote the free associative algebra built on V. If now $\mathrm{Free}(V) \in C_X^{\mathrm{old}}(A)$, the sheaf $\mathbb{Z}_{\mathrm{Free}(V)}$ is a projective object of $\mathrm{Sh}_A^{\mathrm{old}}(X)$. This is because every covering of $\mathrm{Free}(V)$ admits a section.

2.1.4. LEMMA. *The presheaf of categories over* $C_X(A)$ *given by* $G((U,B_U)) := C_U(B_U)$ *is a stack.*

Proof. Observe first of all, that $C_U(B_U) \simeq C_X(A)_{(U,B_U)}$ and by 1.5.3(1) we must check that the following two conditions are verified:

(1) For every $(U,B_U) \in C_X(A)$, the presheaf of sets on $C_X(A)$ given by

$$(V,D_V) \in C_X(A) \to \mathrm{Hom}_{C_X(A)}((V,D_V),(U,B_U)),$$

is in fact a sheaf.

(2) If S is a sheaf of sets on $C_X(A)$ becomes representable after restriction to some $(U,B_U) \in C_X(A)$ covering (X,A), then it is representable.

The first point is obvious. In order to treat the second one, let us decompose the map $(U,B_U) \to (A,X)$ as a composition

$$(U,B_U) \to (U,A|U) \to (X,A),$$

and it becomes sufficient to treat separately the case when X is affine with $U = X$ and the case when $B_U \simeq A|U$. In both cases, the assertion is straightforward. □

2.2. Let $f\colon Y \to X$ be a morphism of schemes. Let us be given quasi-coherent sheaves of algebras A on X and A' on Y. Assume also be given a map of sheaves of algebras $\phi\colon A' \to f_{us}^*(A)$. We say then that (f,ϕ) is a map from the pair (Y,A') to the pair (X,A).

2.2.1. We have a functor denoted (f,ϕ) or just f

$$C_X(A) \to C_Y(A')\colon (U,B_U) \text{ goes to } (U \underset{X}{\times} Y, f_{us}^*(B) \underset{f_{us}^*(A|U)}{\times} (A'|U \underset{X}{\times} Y)).$$

This functor is strict if f is flat.

In the case when both X and Y are affine schemes, we have also the functor $C_X^{\mathrm{old}}(A) \to C_Y^{\mathrm{old}}(A')$. Having said this, we possess the following collection of functors between categories of sheaves on X and on Y.

(1) $f_\bullet\colon \mathrm{Sh}_Y(A') \to \mathrm{Sh}_X(A)$.

(2) The left adjoint of $f_\bullet$: $f^\bullet\colon \mathrm{Sh}_X(A) \to \mathrm{Sh}_Y(A')$. By 1.3, this functor is exact if f is flat and is moreover faithful if $Y \to X$ is onto and $A' \to f_{us}^*(A)$ is epimorphic.

(3) (for X and Y affine) $f_*\colon \mathrm{Sh}_Y^{\mathrm{old}}(A') \to \mathrm{Sh}_X^{\mathrm{old}}(A)$. This functor is exact if $A' \to f_{us}^*(A)$ is an isomorphism, by 1.3(1).

(4) (also for X and Y affine) The left adjoint of the previous functor, denoted by f^*. This functor is also exact if f is flat and is moreover faithful if $Y \to X$ is onto and $A' \to f^*_{us}(A)$ is epimorphic, by 1.3.

2.3. Our next goal will be to define a certain subcategory $\mathrm{Sh}^{qc}_X(A)$ in $\mathrm{Sh}_X(A)$ which we will call the category of quasi-coherent sheaves. The category $\mathrm{Sh}^{qc}_X(A)$ will have properties analogous to those of the category of quasi-coherent sheaves of $\mathcal{O}(X)$-modules inside the category of all sheaves of $\mathcal{O}(X)$-modules over a scheme X. It will turn out that for an affine scheme, the category of quasi-coherent sheaves is equivalent to $\mathrm{Sh}^{\mathrm{old}}_X(A)$.

2.3.1. Let X be an affine scheme. We have the natural inclusion functor $C^{\mathrm{old}}_X(A) \to C^{\mathrm{new}}_X(A)\colon B \to (X, B)$. In this case there is the direct image functor denoted $(\mathrm{new} \to \mathrm{old})\colon \mathrm{Sh}^{\mathrm{new}}_X(A) \to \mathrm{Sh}^{\mathrm{old}}_X(A)$ given by

$$\Gamma(B, (\mathrm{new} \to \mathrm{old})(S)) = \Gamma((X, B), S),$$

and the left adjoint of $(\mathrm{new} \to \mathrm{old})$, denoted by $(\mathrm{old} \to \mathrm{new})$.

The functor $(\mathrm{old} \to \mathrm{new})$ is exact and the functor $(\mathrm{new} \to \mathrm{old})$ is left exact. $R^\bullet(\mathrm{new} \to \mathrm{old})$ will denote the right derived functor of $(\mathrm{new} \to \mathrm{old})$.

2.3.2. Let now f be a map $(Y, A') \to (X, A)$ with X and Y affine.

LEMMA

(1) *The functors* $f_* \circ (\mathrm{new} \to \mathrm{old})$ and $(\mathrm{new} \to \mathrm{old}) \circ f_\bullet\colon \mathrm{Sh}^{\mathrm{new}}_Y(A') \to \mathrm{Sh}^{\mathrm{old}}_X(A)$ *are canonically isomorphic.*
(2) *The same for the functors* $(\mathrm{old} \to \mathrm{new}) \circ f^*$ *and* $f^\bullet \circ (\mathrm{old} \to \mathrm{new})$ *from* $\mathrm{Sh}^{\mathrm{old}}_X(A)$ *to* $\mathrm{Sh}^{\mathrm{new}}_Y(A')$.

Proof. To prove the first statement it suffices to observe that each of the two functors identifies with the direct image functor corresponding to compositions:

$C^{\mathrm{old}}_X(A) \to C^{\mathrm{new}}_X(A) \to C^{\mathrm{new}}_Y(A')$ and $C^{\mathrm{old}}_X(A) \to C^{\mathrm{old}}_Y(A') \to C^{\mathrm{new}}_Y(A')$ that are naturally isomorphic.

The second statement follows from the first one by adjunction. □

2.3.3. We will now describe the functor $(\mathrm{old} \to \mathrm{new})$ more explicitly. The next result can be considered as an analog of Serre's Lemma.

THEOREM. *Let X be an affine scheme. Then the adjunction morphism of functors* $\mathrm{Id}_{\mathrm{Sh}^{\mathrm{old}}_X(A)} \to (\mathrm{new} \to \mathrm{old}) \circ (\mathrm{old} \to \mathrm{new})$ *is an isomorphism.*

Proof of the Theorem.

Step 1. Let X be an affine scheme and consider a full subcategory $C_X^{\text{new}}(A)_{\text{aff}}$ of $C_X^{\text{new}}(A)$ formed by pairs (U, B_U) with U affine. This subcategory carries a natural (induced) Grothendieck topology. Let emb be the embedding functor emb: $C_X^{\text{new}}(A)_{\text{aff}} \to C_X^{\text{new}}(A)$. By definition, this is a functor between sites in the sense of 1.1.2.

The following lemma is easy to prove.

LEMMA. *The functor* $\text{emb}_\bullet\colon \text{Sh}(C_X^{\text{new}}(A)) \to \text{Sh}(C_X^{\text{new}}(A)_{\text{aff}})$ *is an equivalence of categories. In particular*

$$\text{emb}_\bullet \circ \text{emb}^\bullet \simeq \text{Id and } \text{emb}^\bullet \circ \text{emb}_\bullet \simeq \text{Id}.$$

Step 2. Let us start now with a sheaf $S \in \text{Sh}_X^{\text{old}}(A)$ and consider the following presheaf S' on $C_X^{\text{new}}(A)_{\text{aff}}$.

For $j\colon (U, B_U) \to (X, A)$ we set

$$\Gamma((U, B_U), S') \simeq \Gamma((U, B_U), j^*(S)).$$

We claim that this presheaf is in fact a sheaf.

Let $j\colon (U, B_U) \to (X, A)$ be a covering, and let $j_i\colon (U_i, B_{U\,i}) \to (X, A)$ denote the map of the $i+1$-fold fiber product of (U, B_U) with itself over (X, A) to (X, A).

Without restricting generality it suffices to check that the complex

$$0 \to \Gamma((X, A), S')) \to \Gamma((U, B_U), S') \to \Gamma((U_1, B_{U\,1}), S'),$$

is exact at first two places.

Since the functor of taking sections is left exact, it is enough to prove the following lemma:

LEMMA. *The canonical complex (called the Čech complex of S with respect to (U, B_U))*

$$0 \to S \to j_{0*}{j_0}^*(S) \to j_{1*}j_1^*(S) \to j_{2*}j_2^*(S) \to \cdots,$$

is exact.

Proof of the Lemma.

Step 1. Assume first, that the map $(X, A) \to (U, B_U)$ admits a section. In this case our complex is exact since we can write down an explicit homotopy operator.

Step 2. In the general case, by 2.2.1(4), it is enough to prove that our complex is exact after applying the functor j^*. However, when we do that, the complex obtain is the Čech complex of $j^*(S)$ over (U, B_U) with respect to $(U_1, B_{U\,1})$. Therefore, we find ourselves in the situation of Step 1, since the projection $(U_1, B_{U\,1}) \to (U, B_U)$

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admits a section. □

Step 3. We claim now, that the sheaf S' constructed above is in fact canonically isomorphic to $\mathrm{emb}_\bullet(\mathrm{new} \to \mathrm{old})(S)$. For this we must establish an isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}(C_X^{\mathrm{new}}(A)_{\mathrm{aff}})}(S', \mathrm{emb}_\bullet(M)) \to \mathrm{Hom}_{\mathrm{Sh}_X^{\mathrm{old}}(A)}(S, (\mathrm{new} \to \mathrm{old})(M)),$$

for any $M \in \mathrm{Sh}_X^{\mathrm{new}}(A)$. But this is clear from from the construction of S'.

In order to finish the proof of the theorem it remains to observe that $(\mathrm{new} \to \mathrm{old}) \circ \mathrm{emb}^\bullet(S') \simeq S$, by definition. □

2.3.4. Let us now present several corollaries of the above theorem.

COROLLARY 1. *Assume X to be affine. The functor* $(\mathrm{old} \to \mathrm{new})$ *realizes* $\mathrm{Sh}_X^{\mathrm{old}}(A)$ *as a full Abelian subcategory of* $\mathrm{Sh}_X^{\mathrm{new}}(A)$ *stable under extensions.*

This is a formal consequence of the Theorem.

COROLLARY 2. *Let X and Y be affine and consider the canonical morphism of functors*

$$f^* \circ (\mathrm{new} \to \mathrm{old}) \to (\mathrm{new} \to \mathrm{old}) \circ f^\bullet,$$

that is given by the adjunction map of the functors f^* and f_*.

Then it is an isomorphism when applied to objects of $\mathrm{Sh}_X^{\mathrm{new}}(A)$ *of the form* $(\mathrm{old} \to \mathrm{new})(S)$ *for* $S \in \mathrm{Sh}_X^{\mathrm{old}}(A)$.

Proof of Corollary 2.

Indeed, on the left hand side we get $f^* \circ (\mathrm{new} \to \mathrm{old})(\mathrm{old} \to \mathrm{new})(S) \simeq f^*(S)$ whereas on the right-hand side we get

$$(\mathrm{new} \to \mathrm{old}) \circ f^\bullet \circ (\mathrm{old} \to \mathrm{new})(S) \simeq$$

$$(\mathrm{old} \to \mathrm{new}) \circ (\mathrm{old} \to \mathrm{new}) \circ f^*(S) \simeq f^*(S),$$

and it is easy to verify, that under these identifications the above natural transformation yields the identity morphism on $f^*(S)$. □

COROLLARY 3. *Let $X = \mathrm{Spec}(R)$ be an affine scheme.*

(1) $R^i(\mathrm{new} \to \mathrm{old})(\mathrm{old} \to \mathrm{new})(S) = 0$ for any $S \in \mathrm{Sh}_X^{\mathrm{old}}(A)$ and for any $i \geqslant 1$.
(2) *The functor* $(\mathrm{old} \to \mathrm{new}) : D(\mathrm{Sh}_X^{\mathrm{old}}(A)) \to D(\mathrm{Sh}_X^{\mathrm{new}}(A))$ *if fully faithful.*

Proof of Corollary 3.

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To prove the first point, we must show that if $0 \to S \to K_1 \to K_2 \to 0$ is an exact sequence of sheaves in $\mathrm{Sh}_X^{\mathrm{new}}(A)$ with $S \in \mathrm{Sh}_X^{\mathrm{old}}(A)$, the sequence

$$0 \to S \to ((\mathrm{new} \to \mathrm{old})(K_1)) \to ((\mathrm{new} \to \mathrm{old})(K_2)) \to 0,$$

is exact in $\mathrm{Sh}_X^{\mathrm{old}}(A)$.

For this it suffices to check that if $B \in C_X^{\mathrm{old}}(A)$ is a free algebra built on a projective R-module, the complex of sections $0 \to \Gamma(B, S) \to \Gamma(B, K_1) \to \Gamma(B, K_2) \to 0$ is exact.

Now, since $\mathrm{Sh}_X^{\mathrm{old}}(A) \subset \mathrm{Sh}_X^{\mathrm{new}}(A)$ is stable under extensions, $\mathrm{Ext}^1_{\mathrm{Sh}_X^{\mathrm{old}}(A)}(\mathbb{Z}_{(X,B)}, S) = 0$ implies

$$H^1_{\mathrm{Sh}_X^{\mathrm{new}}(A)}((X,B), S) \simeq \mathrm{Ext}^1_{\mathrm{Sh}_X^{\mathrm{new}}(A)}(\mathbb{Z}_{(X,B)}, S) = 0,$$

and the assertion follows.

The second point readily follows from the first one. □

2.3.5. Let us summarize some of the above results into a proposition:

PROPOSITION. *Let X be affine, $S \in \mathrm{Sh}_X^{\mathrm{new}}(A)$. The following conditions are equivalent:*

(1) $S \in \mathrm{Sh}_X^{\mathrm{old}}(A)$.
(2) *For some pair $f\colon (Y, A') \to (X, A)$ with Y affine and faithfully flat over X and with $A' \to f^*_{us}(A)$ surjective, $f^\bullet(S)$ belongs to $\mathrm{Sh}_Y^{\mathrm{old}}(A')$.*
(3) *For every $(j\colon (U, U_B) \to (X, A)) \in C_X^{\mathrm{new}}(A)$ with U affine, the canonical map $j^* \circ (\mathrm{new} \to \mathrm{old})(S) \to (\mathrm{new} \to \mathrm{old}) \circ j^\bullet(S)$ is an isomorphism.*

The proof follows immediately from Corollaries 1, 2 of the Theorem.

2.4. We arrive now to the definition of a quasi-coherent sheaf in $\mathrm{Sh}_X(A)$.

DEFINITION. Let X be first an affine scheme. A sheaf $S \in \mathrm{Sh}_X^{\mathrm{new}}(A)$ is said to be quasi-coherent if it belongs to $\mathrm{Sh}_X^{\mathrm{old}}(A)$.

Let X now be an arbitrary scheme. A sheaf $S \in \mathrm{Sh}_X(A)$ is said to be quasi-coherent if for every pair $(U, B_U) \in C_X^{\mathrm{new}}(A)$ with U affine, the restriction $S|(U, B_U)$ (cf. 1.3.1(4)) of S to (U, B_U) is quasi-coherent in the sense of the previous definition.

LEMMA

(1) *Quasi-coherent sheaves form a full abelian subcategory in $\mathrm{Sh}_X(A)$, stable under extensions. We will denote it by $\mathrm{Sh}_X^{qc}(A)$.*

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(2) *Let* $f\colon (Y, A') \to (X, A)$ *be a map. Then the direct image functor* $f_\bullet$ *maps* $\mathrm{Sh}^{qc}_Y(A')$ *to* $\mathrm{Sh}^{qc}_X(A)$.
(3) *Let once again* $f\colon (Y, A') \to (X, A)$ *be a map. Then the functor* $f^\bullet$ *also maps* $\mathrm{Sh}^{qc}_X(A)$ *to* $\mathrm{Sh}^{qc}_Y(A')$.

Proof. The first point follows readily from the definition and from the fact that the analogous statement holds for X affine, by 2.3.4(1).

To prove the third point we may assume that both X and Y are affine and then Lemma 2.3.2 implies the statement.

In order to prove the second point, as in the usual theory of schemes, it is sufficient to check the statement in the case when both X and Y are affine.

Let $(\text{old} \to \text{new})(S) \in \mathrm{Sh}^{\text{new}}_Y(A')$. By 2.3.5(3), we must show that for every $(j\colon (U, U_B) \to (X, A)) \in C^{\text{new}}_X(A)$ with U affine, the canonical map

$$j^* \circ (\text{new} \to \text{old}) \circ f_\bullet \circ (\text{old} \to \text{new})(S) \to$$

$$(\text{new} \to \text{old}) \circ j^\bullet \circ f_\bullet \circ (\text{old} \to \text{new})(S),$$

is an isomorphism.

Indeed, let j' and f' denote the maps from $U \underset{X}{\times} Y$ to Y and to U and we have

$$j^* \circ (\text{new} \to \text{old}) \circ f_\bullet \circ (\text{old} \to \text{new})(S) \simeq$$

$$j^* \circ f_* \circ (\text{new} \to \text{old}) \circ (\text{old} \to \text{new})(S) \simeq$$

$$j^* \circ f_*(S) \simeq f'_* \circ j'^*(S) \simeq$$

$$f'_* \circ (\text{new} \to \text{old}) \circ (\text{old} \to \text{new}) \circ j'^*(S) \simeq$$

$$(\text{new} \to \text{old}) \circ f'_\bullet \circ j'^\bullet \circ (\text{old} \to \text{new}) \simeq$$

$$(\text{new} \to \text{old}) \circ j^\bullet \circ f_\bullet \circ (\text{old} \to \text{new}),$$

where the above isomorphisms follow by applying Lemma 2.3.2 and Theorem 2.3.3. □

By analogy with the affine situation, we will denote the inclusion functor $\mathrm{Sh}^{qc}_X(A) \to \mathrm{Sh}_X(A)$ also by $(\text{old} \to \text{new})$. Note, however, that when X is not affine there is no functor analogous to $(\text{new} \to \text{old})\colon \mathrm{Sh}_X(A) \to \mathrm{Sh}^{qc}_X(A)$ possessing good properties.

As before, restrictions of the functors $f_\bullet$, $f^\bullet$ to the category of quasi-coherent sheaves will be denoted by f_*, f^*. By definition, we have

$$f_\bullet \circ (\text{old} \to \text{new}) \simeq (\text{old} \to \text{new}) \circ f_*,$$

and

$$f^\bullet \circ (\text{old} \to \text{new}) \simeq (\text{old} \to \text{new}) \circ f^*.$$

2.4.1. EXAMPLE. Let $(X,B)\in C_X(A)$, then $\mathbb{Z}_{(X,B)}\in \mathrm{Sh}_X^{qc}(A)$.

2.4.2. The category $\mathrm{Sh}_X^{qc}(A)$ has enough injective objects because every $\mathrm{Sh}_U^{qc}(B_U)$ with U affine does (cf. [Ar], [Gr]).

2.5. We have the natural functor $(\text{old}\to\text{new})\colon D(\mathrm{Sh}_X^{qc}(A))\to D(\mathrm{Sh}_X(A))$ that sends $D(\mathrm{Sh}_X^{qc}(A))$ to the subcategory $D_{qc}\mathrm{Sh}_X(A)$ that consists of objects of $D(\mathrm{Sh}_X(A))$ with quasi-coherent cohomologies.

THEOREM. *The above functor induces an equivalence of categories:* $(\text{old}\to\text{new})\colon D^b(\mathrm{Sh}_X^{qc}(A))\to D^b_{qc}\mathrm{Sh}_X(A)$.

Corollary 3 of Theorem 2.3.3 implies the assertion for X affine, as well as the following fact:

PROPOSITION. *Let $f\colon(Y,A')\to(X,A)$ be a map. Then the natural map $R^\bullet f_\bullet\circ(\text{old}\to\text{new})\to(\text{old}\to\text{new})\circ R^\bullet f_*$ is an isomorphism of functors.*

Proof of the Proposition.

In order to prove this statement, we may assume that both X and Y are affine. A standard devissage shows that it is sufficient to show that if $I\in\mathrm{Sh}_Y^{qc}(A')$ is an injective object, $R^if_\bullet\circ(\text{old}\to\text{new})(I)=0$ for $i>0$.

We know, that $R^if_\bullet\circ(\text{old}\to\text{new})(I)$ is the sheaf associated to the presheaf

$$(U,B_U)\to H^i_{\mathrm{Sh}_Y(A')}((f^{-1}(U),f^*_{us}(B_U)\underset{f^*_{us}(A)|f^{-1}(U)}{\times}A'|f^{-1}(U)),I).$$

However, we know by 2.3.4(3) that

$$H^i_{\mathrm{Sh}_Y(A')}((f^{-1}(U),f^*_{us}(B_U)\underset{f^*_{us}(A)|f^{-1}(U)}{\times}A'|f^{-1}(U)),I)\simeq,$$

$$H^i_{\mathrm{Sh}_Y^{qc}(A')}((f^{-1}(U),f^*_{us}(B_U)\underset{f^*_{us}(A)|f^{-1}(U)}{\times}A'|f^{-1}(U)),I)=0,$$

since I is injective. □

Proof of the Theorem.

Since any complex is glued from its cohomologies, it suffices to prove that for $S_1,S_2\in\mathrm{Sh}_X^{qc}(A)$ the map

$$(\text{old}\to\text{new})\colon \mathrm{Ext}^i_{\mathrm{Sh}_X^{qc}(A)}(S_1,S_2)\to$$
$$\mathrm{Ext}^i_{\mathrm{Sh}_X(A)}((\text{old}\to\text{new})(S_1),(\text{old}\to\text{new})(S_2)),$$

is an isomorphism.

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Choose $j: U \to X$ to be a covering in Zar_X with U affine. Choose also an embedding $j^\bullet(S_2) \to I$ where I is an injective object of $\text{Sh}_Y^{qc}(A|U))$.

$S \to j_\bullet(I)$ is an injection and let K denote the cokernel. The above proposition implies

$$\text{Ext}^i_{\text{Sh}_X^{qc}(A)}(S_1, f_*(I)) \simeq \text{Ext}^i_{\text{Sh}_X(A)}(S_1, f_\bullet(I)) = 0,$$

and we have a commutative diagram for every $i > 1$

$$\begin{array}{ccc} \text{Ext}^{i-1}_{\text{Sh}_X^{qc}(A)}(S_1, K) & \xrightarrow{(\text{old}\to\text{new})} & \text{Ext}^{i-1}_{\text{Sh}_X(A)}((\text{old} \to \text{new})(S_1), (\text{old} \to \text{new})(K)) \\ \downarrow \sim & & \downarrow \sim \\ \text{Ext}^{i}_{\text{Sh}_X^{qc}(A)}(S_1, S_2) & \xrightarrow{(\text{old}\to\text{new})} & \text{Ext}^{i}_{\text{Sh}_X(A)}((\text{old} \to \text{new})(S_1), (\text{old} \to \text{new})(S_2)) \end{array}$$

The needed assertion follows by induction on i, since for $i = 0, 1$ it is already known (Lemma 2.4(1)). □

3. *A*-Bimodules and sheaves on $C_X(A)$

3.0. In this section we will study the connection between the category of quasi-coherent sheaves of A-bimodules and that of quasi-coherent sheaves on $C_X(A)$. The material here is parallel to the one of Section 3 in [Ga]. The category of quasi-coherent sheaves of A-bimodules will be denoted by A_{qc}-mod.

3.1. Let us recall several definitions from [Ga]. If B is a quasi-coherent sheaf of algebras on a scheme X, we denote by I_B the sheaf of B-bimodules given by $I_B = \ker(B \otimes B \to B)$ (the map here is the multiplication).

If M is a quasi-coherent sheaf of B-bimodules, we will denote by $\Omega_{\mathcal{O}(X)}(B, M)$ the group $\text{Hom}_B(I_B, M)$.

3.2. Let now X be a scheme and let A be a quasi-coherent sheaf of associative algebras on X. We will construct a localization functor $\Im$: A_{qc}-mod $\to \text{Sh}_X^{qc}(A)$:

Let $M \in A_{qc}$-mod. Consider the presheaf $\Im(M)$ on $C_X(A)$ given by

$$\Gamma((U, B_U), \Im(M)) = \Omega_{\mathcal{O}(U)}(B_U, M|U).$$

The following Lemma follows directly from the definition.

LEMMA. *This presheaf is in fact a sheaf.*

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3.2.1. If X is an affine scheme, similar constructions can be carried out in the category $C_X^{\text{old}}(A)$. In this case we denote the localization functor by $\Im^{\text{old}}$.

LEMMA. *The functors*

$$(\text{new} \to \text{old}) \circ \Im \quad \text{and} \quad \Im^{\text{old}}: A_{qc}\text{-mod} \to \text{Sh}_X^{qc}(A),$$

are canonically isomorphic.

The proof follows immediately from the definition.

3.3. Let us describe $\Im(M)$ in a slightly different way. Consider the sheaf of algebras $A \oplus M$ over X, $(X, A \oplus M) \in C_X(A)$. This is a group-like object in this category

$$\text{Hom}((U, B_U), (X, A \oplus M)) = \Omega(B_U, M|U),$$

and $\Im(M)$ is a sheaf given by $\Gamma((U, B_U), \Im(M))) = \text{Hom}((U, B_U), (X, A \oplus M))$. In other words, $\Im(M)$ is a group like object in the category of sheaves of sets with $\Im(M) = \text{Const}_{A \oplus M}$, as a sheaf of sets. (cf. 1.3.1(2)).

3.3.1. PROPOSITION

(1) *Let $f: (Y, A') \to (X, A)$ be a map with Y and X affine. Then the functors*

$$\Im^{\text{old}} \circ f_{us}^* \quad \textit{and} \quad f^* \circ \Im^{\text{old}}: A_{qc}\text{-mod}(X) \to \text{Sh}_{A'}^{\text{old}}(Y),$$

are canonically isomorphic.

(2) *For any X and Y $f: (Y, A') \to (X, A)$ the functors*

$$\Im \circ f_{us}^* \quad \textit{and} \quad f^\bullet \circ \Im: A_{qc}\text{-mod}(X) \to \text{Sh}_{A'}(Y),$$

are canonically isomorphic.

(3) *If X is affine the functor $\Im: A_{qc}\text{-mod} \to \text{Sh}_X^{\text{new}}(A)$ takes values in $\text{Sh}_X^{\text{old}}(A)$.*

(4) *Let once again X be affine. Then the functors*

$$(\text{old} \to \text{new}) \circ \Im^{old} \quad \textit{and} \quad \Im: A_{qc}\text{-mod} \to$$
$$\text{Sh}_A^{\text{new}}(X): A_{qc}\text{-mod}(X) \to \text{Sh}_A(X),$$

are canonically isomorphic.

(5) *For X arbitrary the functor $\Im: A_{qc}\text{-mod} \to \text{Sh}_A(X)$ takes values in $\text{Sh}_X^{qc}(A)$.*

(6) *The functor $\Im: A_{qc}\text{-mod} \to \text{Sh}_X^{qc}(A)$ is exact and faithful.*

Proof of the Proposition. The first two points are immediately deduced from the following general Lemma:

LEMMA 1. *Let $F: C_1 \to C_2$ be a functor between two sites. Let $A \in C_1$ be an (abelian) group-like object so that the sheaf of sets* Const_A *possesses a structure*

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of a sheaf of abelian groups. Suppose that $F(A)$ is an (abelian) group-like object in C_2 as well and that $F\colon \mathrm{End}(A) \to \mathrm{End}(F(A))$ is a homomorphism of groups. Then $F^\bullet_{ab}(\mathrm{Const}_A) \simeq \mathrm{Const}_{F(A)}$.

Proof of the Lemma 1.

Remark. Here the subscript ab is to emphasize that the inverse image functor is taken in the category of sheaves of abelian groups.

Observe first that for any $S \in \mathrm{Sh}(C_1)$,

$$\mathrm{Hom}_{\mathrm{Sh}_{ab}(C_1)}(\mathrm{Const}_A, S)) = \{\gamma \in \Gamma(A, S) | n \cdot \gamma = n^*(\gamma)\} \text{ for any } n \in \mathbb{Z},$$

where n on the right-hand side denotes the endomorphism $n \cdot \mathrm{Id}_A \in \mathrm{End}(A)$.

Therefore, for each $S \in \mathrm{Sh}_{ab}(C_2)$ we have

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Sh}_{ab}(C_1)}(\mathrm{Const}_A, F_\bullet(S)) \simeq \{\gamma \in \Gamma(A, F_\bullet(S)) | n \cdot \gamma \\ &\quad = n^*(\gamma)\} \text{ for any } n \in \mathbb{Z} \simeq \{\gamma \in \Gamma(F(A), S) | n \cdot \gamma \\ &\quad = n^*(\gamma)\} \text{ for any } n \in \mathbb{Z} \simeq \mathrm{Hom}_{\mathrm{Sh}_{ab}(C_2)}(\mathrm{Const}_{F(A)}, S), \end{aligned}$$

and that is what we wanted to prove. □

The third point follows from (1), (2), 3.2.1 and 2.3.5(3).

Now, (2), (3) imply (5), whereas (3), 3.2.1 and 2.3.3 imply (4).

In order to prove (6) we may assume X to be affine, and the assertion follows from the following lemma, whose proof is a straightforward verification.

LEMMA 2. *Let $X = \mathrm{Spec}(R)$ and let V be an R-module. Let also $\mathrm{Free}_R(V) \in C_X^{\mathrm{old}}(A)$ be the free associative algebra built on V. Then the functors*

$$M \to \mathrm{Hom}_R(V, M) \quad \textit{and} \quad M \to \Gamma(\mathrm{Free}_R(V), \Im(M))\colon A_{qc}\text{-mod} \to Ab,$$

are canonically isomorphic. □

3.3.2. Let f be a map from a pair $(Y, A' \simeq f^*_{us}(A))$ to the pair (X, A). We have then the direct image functor $f_{us*}\colon A'_{qc}$-mod $\to A_{qc}$-mod and the natural transformation

$$\Im_X \circ f_{us*} \to f_* \circ \Im_Y.$$

PROPOSITION

(1) *The above above map is an isomorphism of functors.*
(2) *Moreover, it induces an isomorphism of the derived functors*

$$\Im_X \circ R^\bullet(f_{us*}) \to R^\bullet(f_*) \circ \Im_Y \to R^\bullet(\Im_X \circ f_{us*}),$$

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from $D_{qc}(A) \overset{\text{def}}{\simeq} D(A_{qc}\text{-mod})$ *to* $D(\mathrm{Sh}_X^{qc}(A))$.

Proof. Let us first make the following observation:

LEMMA. *Let* $R \to R'$ *be a homomorphism of commutative rings. Let also* B *be an* R*-algebra, and* M *be a* $B \underset{R}{\otimes} R'$*-bimodule. Then*

$$\Omega_R(B, M) \simeq \Omega_{R'}(B \underset{R}{\otimes} R', M).$$

The first assertion follows now, since

$$\Gamma((U, B_U), \Im_X \circ f_{us*}(M)) \simeq \Omega_{\mathcal{O}(U)}(B_U, f_{us}(M)|U) \overset{\text{Lemma}}{\simeq}$$
$$\Omega_{\mathcal{O}(f^{-1}(U))}(f_{us}^*(B_U, M|f^{-1}(U)) \simeq \Gamma((f^{-1}(U), f_{us}^*(B_U)), f_* \circ \Im_Y(M)).$$

In order to prove the second statement, it suffices to assume that both X and Y are affine. In this case the functors f_{us*} and f_* are exact (cf. 2.2.1(3)), and since the functor $\Im$ is always exact, the assertion follows. □

3.3.3. *Remark*. When the condition $A' \simeq f_{us}^*(A)$ is not satisfied, the functor of direct image on quasi-coherent sheaves of A'-modules can still be defined but it will be neither left exact nor right exact. Therefore, the simply minded isomorphism 3.3.2(1) will be false in that situation. However, if one modifies the definition of direct image in order to get a correctly defined functor in the derived category, the above isomorphism in derived categories will still hold.

3.4. PROPOSITION–DEFINITION

(1) *The functor* $\Im\colon A_{qc}\text{-mod} \to \mathrm{Sh}_X^{qc}(A)$ *admits a left adjoint denoted by* $\mathcal{L}$: $\mathrm{Sh}_X^{qc}(A) \to A_{qc}\text{-mod}$.
(2) *Let* f *be a map from a pair* $(Y, A' \simeq f_{us}^*(A))$ *to the pair* (X, A). *The functors*

$$\mathcal{L}_Y \circ f^* \quad \textit{and} \quad f_{us}^* \circ \mathcal{L}_X\colon \mathrm{Sh}_A^{qc}(X) \to A'_{qc}\text{-mod}(Y),$$

are canonically isomorphic.

Proof. For any $S \in \mathrm{Sh}_X^{qc}(A)$ we must construct a quasi-coherent A-module $\mathcal{L}(S)$, satisfying

$$\mathrm{Hom}_{A_{qc}\text{-mod}}(\mathcal{L}(S), M) \simeq \mathrm{Hom}_{\mathrm{Sh}_X^{qc}(A)}(S, \Im(M)),$$

functorially in M.

Assume first, that for a given S, such $\mathcal{L}(S)$ exists. Proposition 3.3.2(1) implies then, that for every $f\colon (Y, f_{us}^*(A)) \to (X, A)$, $\mathcal{L}_Y(f^*(S))$ over $(Y, f_{us}^*(A))$ exists as well and satisfies

$$\mathcal{L}_Y(f^*(S)) \simeq f_{us}^* \circ \mathcal{L}(S).$$

Observe that this, on one hand, implies (2) modulo (1) and on the other hand reduces the construction of the functor $\mathcal{L}$ to the case when X is affine.

In these circumstances, every object in $\mathrm{Sh}_X^{qc}(A)$ is a quotient of a one of the type $\mathbb{Z}_{\mathrm{Free}(V)}$.

However, for $S = \mathbb{Z}_{\mathrm{Free}(V)}$, Lemma 2 of 3.3.1 implies that $\mathcal{L}(S) := F(V)$ satisfies our requirements, where $F(V)$ denotes the free A-module built on V. The proof follows in view of the following assertion:

SUB-LEMMA. *Let C_1 and C_2 be two Abelian categories and let $F: C_1 \to C_2$ be an additive left exact functor between them. Suppose that F admits a partially defined left adjoint functor which is however defined on a large collection of objects in C_2 (i.e. any object in C_2 is a quotient of a one from this collection). Then this left adjoint is defined on the whole of C_2.* □

3.4.1. Since the functor $\Im$ is exact, it can be prolonged to a functor between the corresponding derived categories: $D_{qc}(A) \overset{\text{def}}{\simeq} D(A_{qc}\text{-mod}) \to D(\mathrm{Sh}_X^{qc}(A))$ which will be also denoted by $\Im$. Our next aim is to show that the functor $\mathcal{L}$ (which is obviously right exact) can be also derived into a functor $L^\bullet\mathcal{L}: D^-(\mathrm{Sh}_X^{qc}(A)) \to D_{qc}^-(A)$, which will be the left adjoint functor to $\Im: D_{qc}^-(A) \to D^-(\mathrm{Sh}_X^{qc}(A))$. When X is affine, the argument of 3.4 proves also the existence of such $L^\bullet\mathcal{L}$, since the sheaves $\mathbb{Z}_{\mathrm{Free}(V)}$ with V being a projective $\mathcal{O}(X)$-module form a set of projective generators of $\mathrm{Sh}_X^{qc}(A)$. However, in order to treat the general case an additional argument is needed, since objects of the derived category cannot be reconstructed just from the local information.

3.4.3. THEOREM. *Let X be an arbitrary scheme and A be a quasi-coherent sheaf of algebras on A.Then*

(1) *The functor $\mathcal{L}$ admits a left derived functor $L^\bullet\mathcal{L}: D^-(\mathrm{Sh}_X^{qc}(A)) \to D_{qc}^-(A)$.*
(2) *$L^\bullet\mathcal{L}$ satisfies*

$$\mathrm{Hom}(L^\bullet\mathcal{L}(S^\bullet), M^\bullet) \simeq \mathrm{Hom}(S^\bullet, \Im(M^\bullet)),$$

functorially in $S^\bullet \in D^-(\mathrm{Sh}_X^{qc}(A))$ and in $M^\bullet \in D_{qc}^+(A)$.
(3) *Let $(Y, A') \to (X, A)$ be a map such that $f: Y \to X$ is flat and such that $A' \to f_{us}^*(A)$ is an isomorphism. Then*

$$L^\bullet\mathcal{L}_Y \circ f^* \simeq f_{us}^* \circ L^\bullet\mathcal{L}_X,$$

as functors $D^{qc-}(\mathrm{Sh}_X(A)) \to D_{qc_Y}^-(A')$.

Remark 1. In (3), one can drop the assumptions that Y is flat over X and that $A' \to f_{us}^*(A)$ is an isomorphism. In this case the functors $f_{us}^*: A_{qc}\text{-mod} \to A'_{qc}\text{-mod}$ and $f^*: \mathrm{Sh}_X^{qc}(A) \to \mathrm{Sh}_{A'}^{qc}(Y)$ will have to be replaced by appropriate derived functors.

Remark 2. The category $\mathrm{Sh}_X^{qc}(A)$ is lacking objects that would be acyclic for the functor $\mathcal{L}$. The situation here is similar to that in [Bo], [Be], when one wants to define the direct image functor for D-modules. As in [Bo], there are at least two ways to overcome this difficulty: a more straightforward one is to go beyond the category $\mathrm{Sh}_X^{qc}(A)$ and work with arbitrary sheaves. In this case there are enough acyclic objects for the functor $\mathcal{L}$, but the drawback of this approach is that we will have to rely on the equivalence of the categories $D_{qc}(A)$ and $D(A\text{-mod})$ with quasi-coherent cohomologies as well as on Theorem 2.5. Another way is the one described below:

Proof of the Theorem.

Step 1. First we will present a construction of a functor: $L'^{\bullet}\mathcal{L}\colon D^{-}(\mathrm{Sh}_X^{qc}(A)) \to D_{qc}^{-}(A)$.

Consider first a pair (Y, A') with Y is affine and let S be a quasi-coherent sheaf on $C_Y(A')$. We will construct a canonical sheaf $\mathrm{Can}'(S)$ mapping surjectively onto S with $\mathrm{Can}'(S)$ being acyclic for the functor $\mathcal{L}$. Namely, $\mathrm{Can}(S) = \oplus\Gamma(B,S)\otimes\mathbb{Z}_B$, the sum being taken over isomorphism classes of objects in $C_Y^{\mathrm{old}}(A')$ with B a free algebra on a projective $\mathcal{O}(Y)$-module. This construction has the following two properties:

(1) For any map of sheaves $S_1 \to S_2$ there is a canonical map $\mathrm{Can}'(S_1) \to \mathrm{Can}'(S_2)$.
(2) If $f\colon(Z, f_{us}^*(A')) \to (Y, A')$ is a morphism of pairs with Y and Z affine, there exists a canonical map $f^*(\mathrm{Can}'(S)) \to \mathrm{Can}'(f^*(S))$.

Thus any complex $S^{\bullet}$ of sheaves bounded from above in $\mathrm{Sh}_Y^{qc}(A')$ admits a admits a *canonical* quasi-isomorphism $\mathrm{Can}(S^{\bullet}) \to S^{\bullet}$ by a complex consisting of sheaves acyclic with respect to the functor $\Im$.

Let now $S^{\bullet}$ be a complex bounded from above on X giving rise to an object of $D^{-}(\mathrm{Sh}_X^{qc}(A))$, and choose $j\colon U \to X$ to be a covering in Zar_X with U affine. Put $A' \simeq j_{us}^*(A)$ and let also $j_i\colon U_i \to X$ be the $i+1$-st cartesian product of U with itself over X. All these schemes are affine since X is assumed to be separated.

For each U_i, fix the canonical resolution $\mathrm{Can}(S^{\bullet}|U_i)$ of $S^{\bullet}|U_i$ as above.

Then for each i we can form a complex $\mathcal{L}(\mathrm{Can}(j_i^*(S^{\bullet})))$ of quasi-coherent sheaves of $A|U_i$-bimodules on U_i.

There are $i+1$ maps from U_i to U_{i-1}, call them p_k^i, and for each $1 \leqslant k \leqslant i+1$ we have a map of complexes

$$p_{kus}^{i}{}^{*}\mathcal{L}(\mathrm{Can}(j_{i-1}^*(S^{\bullet}))) \to \mathcal{L}(\mathrm{Can}(j_i^*(S^{\bullet}))),$$

which is a quasi-isomorphism by 3.4(2), since the functor

$$p_k^{*i}\colon \mathrm{Sh}_{A|U_{i-1}}^{qc}(A|U_{i-1}) \to \mathrm{Sh}_{A|U_i}^{qc}(A|U_i),$$

is exact.

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The alternating sum of the maps p^i_{kus*} defines a complex of complexes

$$0 \to j_{us*}\mathcal{L}(\mathrm{Can}(j^*(S^\bullet))) \to j_{1us*}\mathcal{L}(\mathrm{Can}(j_1^*(S^\bullet)))$$
$$\to j_{2us*}(\mathrm{Can}(j_2^*(S^\bullet))) \to \cdots,$$

or, in other words, a double complex $\mathcal{L}'^{\bullet\bullet}(S^\bullet)$ in A_{qc}-mod, whose associated complex we denote by $\mathrm{Ass}(\mathcal{L}'^{\bullet\bullet}(S^\bullet))$.

Using Lemma 2.3.3 and Proposition 3.4(2) we see that the canonical map of complexes

$$\mathcal{L}(\mathrm{Can}(j^*(S^\bullet))) \to \mathrm{Ass}(\mathcal{L}'^{\bullet\bullet}(S^\bullet))|U,$$

is a quasi-isomorphism. This implies that the functor

$$S^\bullet \to \mathrm{Ass}(\mathcal{L}'^{\bullet\bullet}(S^\bullet)),$$

is a well-defined functor $D^-(\mathrm{Sh}_X^{qc}(A)) \to D^-_{qc}(A)$, which we denote by $L'^\bullet\mathcal{L}(S^\bullet)$.

Step 2. Let us prove now, that the functor $L'^\bullet\mathcal{L}(S^\bullet)$ we have constructed satisfies the adjunction property

$$\mathrm{Hom}(L'^\bullet\mathcal{L}(S^\bullet), M^\bullet) \simeq \mathrm{Hom}(S^\bullet, \Im(M^\bullet)). \qquad (*)$$

For this we must construct the adjunction morphisms

$$S^\bullet \to \Im \circ \mathrm{Ass}(\mathcal{L}'^{\bullet\bullet}(S^\bullet)),$$

and

$$\mathrm{Ass}(\mathcal{L}'^{\bullet\bullet}(\Im(M^\bullet))) \to M^\bullet.$$

This is done in the following way

$$\begin{CD}
\mathrm{Ass}[j_{i*}\mathrm{Can}(j_i^* S^\bullet)] @>{\sim}>> \mathrm{Ass}[j_{i*}j_i^*(S^\bullet)] @<{\text{by Lemma 2.3.3}}<{\sim}< S^\bullet \\
@VVV \\
\mathrm{Ass}[j_{i*}\Im\mathcal{L}(\mathrm{Can}(j_i^* S^\bullet))] @>{\text{by Lemma 3.3.2}}>{\sim}> \Im\,\mathrm{Ass}[j_{ius*}\mathcal{L}(\mathrm{Can}(j_i^* S^\bullet))]
\end{CD}$$

for the first adjunction map, and

$$\begin{CD}
\mathrm{Ass}[j_{ius*}\mathcal{L}(j_i^*(\Im(M^\bullet)))] @<<< \mathrm{Ass}[j_{ius*}\mathcal{L}\,\mathrm{Can}(j_i^*(\Im(M^\bullet)))] \\
@V{\text{by 3.4(2)}}VV \\
\mathrm{Ass}[j_{ius*}j_{i\,us}^*(\mathcal{L}\Im(M^\bullet))] @>>> \mathrm{Ass}[j_{ius*}j_{i\,us}^*(M^\bullet)] @<{\text{by Lemma 2.3.3}}<{\sim}< M^\bullet
\end{CD}$$

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for the second one. It is now easy to verify, that the adjunction maps constructed above give rise to $(*)$.

Step 3. The adjunction property $(*)$ established above implies that the functor $L'^{\bullet}\mathcal{L}(S^{\bullet})$ we have constructed is a derived functor of $\mathcal{L}$ as well as (3) of the Theorem in view of Proposition 3.3.2. □

3.5. DEFINITION. Let (X, A) be as before: a scheme with a quasi-coherent sheaf of algebras on it. We define $T^{\bullet}(A)$ to be the object of $D_{qc}(A)$ given by $L^{\bullet}\mathcal{L}(\mathbb{Z}(X, A))$.

From the fact that $\mathcal{L}$ is right exact we infer that $H^i(T^{\bullet}(A))$ vanishes for $i > 0$ and that $H^0(T^{\bullet}(A)) = I_A$.

$T^{\bullet}(A)$ will be called the cotangent complex of A. If M is a quasi-coherent sheaf of A-bimodules, we denote by $H^i_A(M)$ the groups $\mathrm{Ext}^i(T^{\bullet}(A), M)$.

3.5.1. EXAMPLE. Suppose that A is flat over $\mathcal{O}(X)$. It follows from the results of Quillen [Qu], that $T^{\bullet}(A) \simeq I_A$. Indeed, this is true for X affine, and then we apply 3.4.3(3).

4. Deformation theory

4.0. This section is almost a word by word repetition of [Ga], after we adopt certain modifications connected with the fact that we are working over a scheme.

4.1.0. For a scheme X, $\mathcal{O}_i(X)$ will denote the sheaf $\mathcal{O}[t]/t^{i+1} \cdot \mathcal{O}(X)$.

4.1.1. Let A be a quasi-coherent sheaf of associative algebras on X and let M be a quasi-coherent sheaf of A-bimodules endowed with a map of A-bimodules $\phi: A \to M$.

Consider the category $\mathrm{Ext}_{\mathrm{alg}}(A, M)$ defined as follows:

Objects: $\mathcal{O}_1$-algebras $\mathrm{ext}(A, M)$ such that $\ker(t: \mathrm{ext}(A, M) \to \mathrm{ext}(A, M)) = \mathrm{im}(t: \mathrm{ext}(A, M) \to \mathrm{ext}(A, M))$ with fixed isomorphisms

$$\mathrm{ext}(A, M)/t \cdot \mathrm{ext}(A, M) \simeq A \quad \text{and} \quad \mathrm{im}(t: \mathrm{ext}(A, M) \to \mathrm{ext}(A, M)) \simeq M,$$

and such that under the above identifications the action of t

$$A \simeq \mathrm{ext}(A, M)/t \cdot \mathrm{ext}(A, M) \xrightarrow{t} \mathrm{im}(t: \mathrm{ext}(A, M) \to \mathrm{ext}(A, M)) \simeq M,$$

coincides with ϕ.

Morphisms: Maps in this category are defined to be $\mathcal{O}_1$-algebra homomorphisms that induce identity maps on $M \simeq \mathrm{im}(t: \mathrm{ext}(A, M) \to \mathrm{ext}(A, M))$ and on $A \simeq \mathrm{ext}(A, M)/t \cdot \mathrm{ext}(A, M)$.

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LEMMA. *The category* $\mathrm{Ext}_{\mathrm{alg}}(A, M)$ *above is canonically equivalent to the category* $\mathrm{Tors}_{C_X(A)}(\Im(M))$ *of* $\Im(M)$*-torsors on* $C_X(A)$.

Proof. Indeed, to any $\mathrm{ext}(A, M)$ as above, we can assign a sheaf of sets whose sections over $(U, B_U) \in C_X(A)$ are algebra-homomorphisms $B_U \to \mathrm{ext}(A, M)|U$ that respect the projection to $A|U$. This set of sections is clearly a torsor over $\Omega_{\mathcal{O}(U)}(B_U, M|U) \simeq \Gamma((U, B_U), \Im(M))$.

The above assignment is an equivalence of categories by 2.1.4. □

4.1.2. Let us define the category $\mathrm{Deform}^i(A)$ to have as objects quasi-coherent sheaves of associative $\mathcal{O}_i(X)$-algebras A_i, endowed with an isomorphism $A_i/A_i \cdot t \simeq A$ such that $\mathrm{Tor}_1^{\mathcal{O}_i(X)}(A_i, \mathcal{O}(X)) = 0$. In other words, we need that

$$A \tilde{\to} \mathrm{im}(t^i\colon A_i \to A_i) \simeq \ker(t\colon A_i \to A_i).$$

Morphisms in this category are just $\mathcal{O}_i(X)$-algebras homomorphisms respecting the identifications with A modulo t. This category is obviously a groupoid. It is called the category of i-th level deformations of A.

We have natural functors $\mathrm{Deform}^{i+1}(A) \to \mathrm{Deform}^i(A)$ given by reduction modulo t^{i+1}. If A_i is an object in $\mathrm{Deform}^i(A)$, we denote by $\mathrm{Deform}^{i+1}_{A_i}(A)$ the category-fiber of the above functor. This category, which is obviously a groupoid too, is called the category of prolongations of A_i onto the $i+1$-st level.

Observe that for any object $A_{i+1} \in \mathrm{Deform}^{i+1}_{A_i}(A)$

$$\mathrm{Aut}_{\mathrm{Deform}^{i+1}_{A_i}(A)}(A_{i+1}) \simeq \Omega_{\mathcal{O}(X)}(A, A).$$

4.2. We are now ready to state the main result of the present paper:

THEOREM

(1) *The category* $\mathrm{Deform}^1(A)$ *is equivalent to the category* $\mathrm{Tors}_{C_X(A)}(\Im(A))$ *of* $\Im(A)$*-torsors on* $C_X(A)$.
(2) *To any* $A_i \in \mathrm{Deform}^i(A)$ *one can associate a gerbe* G_{A_i} *bound by* $\Im(A)$ *on* $C_X(A)$ *in such a way that* $G_{A_i}((X, A))$ *is canonically equivalent to* $\mathrm{Deform}^{i+1}_{A_i}(A)$.

Remark. Observe that the first point in the statement of the Theorem is a special cases of the second one for $i = 0$.

Proof. The first point of the Theorem is a special case of Lemma 4.1.1 above with $M \simeq A$, $\phi = \mathrm{id}$.

We define the gerbe G_{A_i} as follows:

$G_{A_i}((U, B_U))$ is the groupoid of $\mathcal{O}_i(U)$-algebras $B_{U\,i+1}$ with an isomorphism

$$B_{U\,i+1}/t^{i+1} \cdot B_{U\,i+1} \simeq B_U \underset{A|U}{\times} (A_i)|U,$$

such that $\ker(t^{i+1}: B_{U\,i+1} \to B_{U\,i+1}) = \operatorname{im}(t: B_{U\,i+1} \to B_{U\,i+1})$ and such that the map

$$B_U \simeq B_{U\,i+1}/t^{i+1} \cdot B_{U\,i+1} \to \operatorname{im}(t: B_{U\,i+1} \to B_{U\,i+1}),$$

factors through $A|U$ and gives rise to an isomorphism $A|U \to \operatorname{im}(t: B_{U\,i+1} \to B_{U\,i+1})$.

By definition, $G_{A_i}((X, A)) \simeq \operatorname{Deform}^{i+1}_{A_i}(A)$.

Functors $G_{A_i}((U, B_U)) \to G_{A_i}((V, D_V))$ for maps $(V, D_V) \to (U, B_U)$ are given by restricting sheaves to V and taking fiber products with D_V over $B_U|V$.

We must exhibit now the action of the category $\operatorname{Tors}_{C_U(B_U)}(\Im(A|U))$ on $G_{A_i}((U, B_U))$.

Recall, that by 4.1.1, $\operatorname{Tors}_{C_U(B_U)}(\Im(M|U)) \simeq \operatorname{Ext}_{\text{alg}}(B_U, A|U)$.

Let $\operatorname{ext}(B_U, A|U)$ be an object of $\operatorname{Ext}_{\text{alg}}(B_U, A|U)$ and we put

$$\begin{aligned}&\operatorname{Action}(B_{i+1} \times \operatorname{ext}(B_U, A|U)\\&\quad \simeq \ker(B_{i+1} \oplus \operatorname{ext}(B_U, A|U) \to B_U)/\operatorname{im}(t: A \to B_{i+1} \oplus \operatorname{ext}(B_U, A|U)).\end{aligned}$$

If now $B^0_{U\,i+1}$ is a fixed object of $G_{A_i}((U, B_U))$, the inverse functor

$$G_{A_i}((U, B_U)) \to \operatorname{Tors}_{C_U(B_U)}(\Im(A|U)),$$

is provided by setting for any other object $B^1_{U\,i+1} \in G_{A_i}((U, B_U))$ the corresponding torsor to have as the set of sections over (V, D_V) the set of $\mathcal{O}_{i+1}(V)$-algebra homomorphisms $(B^0_{U\,i+1})|(V, D_V) \to (B^1_{U\,i+1})|(V, D_V)$ that commute with the canonical projections to D_V.

In order to finish the proof, we must show that G_{A_i} is a stack, but this follows from 2.1.4. □

4.3. We will now translate the assertions of the above theorem into cohomological terms.

1-st Level Deformations. The set of isomorphism classes of the groupoid $\operatorname{Deform}^1(A)$ is canonically isomorphic to

$$\operatorname{Ext}^1(\mathbb{Z}_{(X,A)}, \Im(A)) \simeq R^1 \operatorname{Hom}_{A_{qc}\text{-mod}}(T^\bullet(A), A) \simeq H^1_A(A),$$

(cf. 1.5.2, 3.5, 4.2(1)).

Prolongation of Deformations 1. If A_i is an i-th level deformation, there exists a canonical class in $H^2_A(A)$ which is zero if and only if there exists a prolongation A_{i+1} of A_i. (cf. 1.5.5, 3.5, 4.2(2)).

Prolongation of Deformations 2. Suppose that for a given i-th level deformation A_i the category $\operatorname{Deform}^{i+1}_{A_i}(A)$ has an object. Then π_0 of this category is a torsor

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over the abelian group $H^1_A(A)$. (cf. 1.5.1, 3.5, 4.2(2)).

4.4. EXAMPLE. Suppose now that the sheaf A is flat as a sheaf of $\mathcal{O}(X)$-modules. From Example 3.5.1, it follows that deformations of A are controlled by $\mathrm{Ext}^i(I_A, A)$ (Ext s being taken in the category of quasi-coherent sheaves of A-bimodules) for $i = 1, 2$.

Acknowledgments

The author is deeply indebted to J. Bernstein under whose supervision the work was written. Also I would like to thank L. Breen, A. Joseph, and S. Shnider for interesting and stimulating discussions.

References

[Ar] Artin, M.: *Grothendieck Topologies*, Notes on a seminar by M. Artin, Harvard Math. Dept. Lecture Notes, 1962.
[BB] Beilinson, A. and Bernstein, J.: A proof of Jantzen Conjectures, in the collection: I. M. Gelfand Seminar (part 1), series: *Adv. in Soviet Math.* 16 (1993).
[D-Ill] Deligne, P. and Illusie, L.: Relèvements modulo p^2 et décomposition du complexe de De Rham, *Inventiones Mathematicae* 89 (1987).
[DM] Deligne, P. and Milne, J.: Tannakien categories in the collection: Hodge Cycles, Motives, Shimura varieties, series: *Lecture Notes in Mathematics* 900 (1982).
[Fo] Fox, T.: Introduction to algebraic deformation theory, *Journal of Pure and Applied Algebra* 84 (1993).
[Ga] Gaitsgory, D.: Operads, Grothendieck topologies and Deformation theory, preprint (1995), in: *alg-geom eprints*, 9502010.
[Ge-Sch1] Gerstenhaber, M.: The cohomology of presheaves of algebras, *Trans. Amer. Math. Soc.* 310(1) (1988).
[Ge-Sch2] Gerstenhaber, M. and Schack, S. D.: Algebraic cohomology and deformation theory, in the collection: Deformation Theory of Algebras and Structures and Applications (II-Ciocco, 1986) (1988), Kluwer academic publishers,
[Gr] Grothendieck, A.: SGA 4, parts I, II, III, *Lecture Notes in Mathematics* 269, 270, 305 (1972–1973).
[Ill] Illusie, L.: Complexe Cotangent et Déformations, parts I, II, *Lecture Notes in Mathematics* 239, 283 (1972–73).
[Ma] Markl, M.: A cohomology theory for $A(m)$-algebras and applications, *Journal of Pure and Applied Algebra* 83(2) (1992).
[Ma-St] Markl, M. and Stasheff, J.: Deformation theory via deviations, *Journal of Algebra* 170(1) (1994).
[Schl] Schlessinger, M.: PhD Thesis, Harvard (1965).
[St] Stasheff, J.: The intrinsic bracket on the deformation complex of an associative algebra, *Journal of Pure and Applied Algebra* 89(1, 2) (1993).
[Qu] Quillen, D.: On the (co-)homology of commutative rings, in the collection: Applications of Categorical Algebra, *Proceedings of Symposia in Pure Mathematics* 17 (1970).