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DENSE SETS AND FAR FIELD PATTERNS FOR ACOUSTIC WAVES IN AN INHOMOGENEOUS MEDIUM*

by DAVID COLTON

1. Introduction

In this paper, we shall obtain two results on the class of far field patterns corresponding to the scattering of time harmonic acoustic plane waves by an inhomogeneous medium of compact support. Although the problem of characterizing the class of far field patterns is of basic importance in inverse scattering theory, very little is known about this class other than the fact that the far field patterns are entire functions of their independent (complex) variables for each positive fixed value of the wave number. In particular, the class of far field patterns is not all of $L^2(\partial\Omega)$ where $\partial\Omega$ is the unit sphere and this implies that the inverse scattering problem is improperly posed since the far field patterns are, in practice, determined from inexact measurements. The purpose of this paper is to show that while the class of far field patterns corresponding to the scattering of time harmonic plane waves by an inhomogeneous medium is not all of $L^2(\partial\Omega)$, it is dense in $L^2(\partial\Omega)$ for sufficiently small values of the wave number. In addition, a related result will be obtained for a special translation of the class of far field patterns. Analogous results for the scattering of time harmonic acoustic waves by a homogeneous scattering obstacle have recently been obtained by Colton [1], Colton and Kirsch [2], Colton and Monk [3, 4] and Kirsch [8].

We now want to be more precise on what we are going to prove. Consider the scattering due to an inhomogeneous medium of compact support of the incident plane wave

$$u^i(\mathbf{x}, t) = \exp[ik\mathbf{x}\cdot\boldsymbol{\alpha} - i\omega t] \tag{1.1}$$

where $k>0$ is the wave number, ω is the frequency and $\boldsymbol{\alpha}, |\boldsymbol{\alpha}|=1$, is the direction of propagation. Then for $\mathbf{x}\in R^3$ the scattered field has the asymptotic behavior

$$u^s(\mathbf{x}, t) = \frac{e^{i(kr-\omega t)}}{r} F(\hat{\mathbf{x}}; k, \boldsymbol{\alpha}) + O\left(\frac{1}{r^2}\right) \tag{1.2}$$

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where $\hat{\mathbf{x}}=\mathbf{x}/|\mathbf{x}|$ and $r=|\mathbf{x}|$. The function $F(\hat{\mathbf{x}};k,\boldsymbol{\alpha})$ is known as the *far field pattern* corresponding to the incident wave (1.1). Now let $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\ldots$ be a countable dense set of vectors on the unit sphere and define the sets

$$\begin{aligned}\mathbf{F}&=\operatorname{span}\{F(\hat{\mathbf{x}};k,\boldsymbol{\alpha}_n): n=1,2,\ldots\}\\ \mathbf{S}&=\operatorname{span}\{F(\hat{\mathbf{x}};k,\boldsymbol{\alpha}_n)-F(\hat{\mathbf{x}};k,\boldsymbol{\alpha}_1): n=1,2,\ldots\}.\end{aligned} \tag{1.3}$$

The aim of this paper is to show that for k sufficiently small, we have that $\overline{\mathbf{F}}=L^2(\partial\Omega)$ and to furthermore explicitly characterize the orthogonal complement $\mathbf{S}^{\perp}$ of $\mathbf{S}$.

2. Wave propagation in an inhomogeneous medium

Consider the propagation of the acoustic time harmonic plane wave (1.1) through an inhomogeneous medium of compact support. Let $c(\mathbf{x})$, $\mathbf{x}\in R^3$, denote the local speed of sound and assume that $c(\mathbf{x})=c_0>0$ for $r=|\mathbf{x}|>a$ where c_0 is a constant. Then, if $k=\omega/c_0>0$ is the wave number, $n(\mathbf{x})=(c_0/c(\mathbf{x}))^2$, and we factor out the term $e^{-i\omega t}$, under appropriate assumptions (cf. [6]) the mathematical problem we are faced with is to determine the velocity potential $u(\mathbf{x})$ of the total field such that

$$\Delta_3 u+k^2 n(\mathbf{x})u=0 \quad \text{in } R^3 \tag{2.1}$$

$$u(\mathbf{x})=\exp[ik\mathbf{x}\cdot\boldsymbol{\alpha}]+u^s(\mathbf{x}) \tag{2.2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r}-iku^s\right)=0 \tag{2.3}$$

where $u^s(\mathbf{x})$ denotes the scattered field and the Sommerfeld radiation condition (2.3) is assumed to hold uniformly for $\hat{\mathbf{x}}$ on the unit sphere $\partial\Omega$. We shall make the assumption that $n(\mathbf{x})$ is positive, continuously differentiable and that

$$B=\{\mathbf{x}\in R^3: n(\mathbf{x})\neq 1\} \tag{2.4}$$

is simply connected and contains the origin. In particular, this implies that for $\mathbf{x}\in B$ either $c(\mathbf{x})>c_0$ or $0<c(\mathbf{x})<c_0$.

The scattering problem (2.1)–(2.3) is easily seen to be equivalent to the integral equation

$$u(\mathbf{x})=\exp[ik\mathbf{x}\cdot\boldsymbol{\alpha}]-k^2\iint_B \Phi(\mathbf{x},\mathbf{y})m(\mathbf{y})u(\mathbf{y})\,d\mathbf{y} \tag{2.5}$$

where

$$m(\mathbf{x})=1-n(\mathbf{x}) \tag{2.6}$$

and

$$\Phi(\mathbf{x},\mathbf{y})=\frac{\exp[ik|\mathbf{x}-\mathbf{y}|]}{4\pi|\mathbf{x}-\mathbf{y}|}. \tag{2.7}$$

Noting that for $\mathbf{x}\in B$, $m(\mathbf{x})$ is either always positive or always negative, we assume without loss of generality that $m(\mathbf{x})$ is positive for $\mathbf{x}\in B$ and define the Hilbert space $L_m^2(B)$ by

$$L_m^2(B)=\left\{u(\mathbf{x}): u(\mathbf{x}) \text{ measurable}, \iint_B m(\mathbf{x})|u(\mathbf{x})|^2\,d\mathbf{x}<\infty\right\} \tag{2.8}$$

with inner product and norm given by

$$\begin{aligned}(f,g)&=\iint_B m(\mathbf{x})f(\mathbf{x})\overline{g(\mathbf{x})}\,d\mathbf{x}\\ \|u\|&=\left[\iint_B m(\mathbf{x})|u(\mathbf{x})|^2\,d\mathbf{x}\right]^{1/2}.\end{aligned} \tag{2.9}$$

(If $m(\mathbf{x})$ is negative for $\mathbf{x}\in B$ then $m(\mathbf{x})$ must be replaced by $-m(\mathbf{x})$ in the above definitions.) We can now write the integral equation (2.5) in operator notation as

$$f_\alpha=u+k^2\mathbf{T}u \tag{2.10}$$

where $\mathbf{T}:L_m^2(B)\to L_m^2(B)$ and $f_\alpha(\mathbf{x})=\exp[ik\mathbf{x}\cdot\boldsymbol{\alpha}]$. In [5], it was shown that

$$\|\mathbf{T}\|\leqq\frac{2Ma^2}{\sqrt{6}} \tag{2.11}$$

where

$$M=\max_{\mathbf{x}\in B} m(\mathbf{x}) \tag{2.12}$$

and hence by the contraction mapping principle (2.10) is uniquely solvable for $k^2<\sqrt{6}/2Ma^2$. Letting $\mathbf{x}$ tend to infinity in (2.5), we see that for $k^2<\sqrt{6}/2Ma^2$ the far field pattern, as defined by (1.2), is given by

$$F(\hat{\mathbf{x}};k,\boldsymbol{\alpha})=-\frac{k^2}{4\pi}\iint_B \exp[-ik\hat{\mathbf{x}}\cdot\mathbf{y}]m(\mathbf{y})u(\mathbf{y})\,d\mathbf{y}. \tag{2.13}$$

Finally, we note that the adjoint of $\mathbf{T}$ in $L_m^2(B)$ is given by

$$\mathbf{T}^*v=\iint_B \overline{\Phi}(\mathbf{x},\mathbf{y})m(\mathbf{y})v(\mathbf{y})\,d\mathbf{y} \tag{2.14}$$

where $\bar{\Phi}(\mathbf{x},\mathbf{y})$ denotes the complex conjugate of $\Phi(\mathbf{x},\mathbf{y})$.

3. Far field patterns

We shall now show that for $k^2<\sqrt{6}/2Ma^2$ we have that $\bar{\mathbf{F}}=L^2(\partial\Omega)$ and, in addition, shall characterize $\mathbf{S}^{\perp}$.

Definition. Let $j_l(k|\mathbf{x}|)$ denote a spherical Bessel function and $Y_l^m(\hat{\mathbf{x}})$ a spherical harmonic. Then H is the vector space

$$H=\operatorname{span}\{j_l(k|\mathbf{x}|)Y_l^m(\hat{\mathbf{x}}): l=0,1,2,\ldots,\ -l\leqq m\leqq l\}$$

and $\bar{H}$ is the closure of H in $L_m^2(B)$.

Theorem. *Let* $k^2<\sqrt{6}/2Ma^2$. *Then* $\bar{\mathbf{F}}=L^2(\partial\Omega)$.

Proof. Let $g\in L^2(\partial\Omega)$. We must show that if

$$\int_{\partial\Omega} F(\hat{\mathbf{x}};k,\boldsymbol{\alpha}_n)\overline{g(\mathbf{x})}\,ds(\hat{\mathbf{x}})=0 \tag{3.1}$$

for $n=1,2,\ldots$ then $g(\hat{\mathbf{x}})$ is identically zero. Suppose (3.1) is true. Then from (2.10) and (2.13) we have

$$\begin{aligned}0&=\iint_B m(\mathbf{y})u(\mathbf{y})\overline{v(\mathbf{y})}\,dy\\&=\iint_B m(\mathbf{y})[(\mathbf{I}+k^2\mathbf{T})^{-1}f_{\alpha_n}](\mathbf{y})\overline{v(\mathbf{y})}\,d\mathbf{y}\\&=\iint_B m(\mathbf{y})\,e^{iky\cdot\boldsymbol{\alpha}_n}\overline{[(\mathbf{I}+k^2\mathbf{T}^*)^{-1}v](\mathbf{y})}\,d\mathbf{y}\end{aligned} \tag{3.2}$$

where $v(\mathbf{y})$ is the *Herglotz wave function*

$$v(\mathbf{y})=\int_{\partial\Omega} g(\hat{\mathbf{x}})\exp[ik\hat{\mathbf{x}}\cdot\mathbf{y}]\,ds(\hat{\mathbf{x}}) \tag{3.3}$$

with *Herglotz kernel* $g(\mathbf{x})$ (cf. [7]). From the expansion

$$\exp[ik\mathbf{y}\cdot\boldsymbol{\alpha}]=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l j_l(k|\mathbf{y}|)Y_l^m(\hat{\mathbf{y}})\overline{Y_l^m(\boldsymbol{\alpha})} \tag{3.4}$$

and the fact that the set $\{\boldsymbol{\alpha}_n\}$ has an accumulation point, we see from (3.2) that $w=(\mathbf{I}+k^2\mathbf{T}^*)^{-1}v\in H^{\perp}$. Since v can be approximated in $L^2(B)$ by a function in H (approximate $g(\hat{\mathbf{x}})$ by a finite series of spherical harmonics) and convergence in $L^2(B)$ implies convergence in $L_m^2(B)$, we can conclude that $v\in\bar{H}$.

Now let $\mathbf{P}: L_m^2(B) \to H^\perp$ be the projection operator from $L_m^2(B)$ onto $H^\perp$. Then from the above discussion, we have that

$$0 = \mathbf{P}v = \mathbf{P}w + k^2\mathbf{P}\mathbf{T}^*w$$

$$= w + k^2\mathbf{P}\mathbf{T}^*w \tag{3.5}$$

and since $\|\mathbf{P}\| = 1$ and $\|\mathbf{T}\| = \|\mathbf{T}^*\|$ we can conclude from (2.11) and the hypothesis of the theorem that $k^2\mathbf{P}\mathbf{T}^*$ is a contraction mapping. Hence, from (3.5) we have that $w = 0$. Since $v = w + k^2\mathbf{T}^*w$, we also have that $v = 0$ and hence $g = 0$ ([7]). This establishes the theorem.

If $k^2 \geqq \sqrt{6}/2Ma^2$ it is not in general true that $\overline{\mathbf{F}} = L^2(\partial\Omega)$. For examples in the case of a spherically stratified medium, see [5].

We now proceed to the characterization of $\mathbf{S}^\perp$. In order to do this, we need to consider the *interior transmission problem* defined as follows: Find $v \in C^2(\Omega_b) \cap C^1(\bar{\Omega}_b)$ and $w \in C^2(\Omega_b) \cap C^1(\bar{\Omega}_b)$ where $\Omega_b = \{\mathbf{x} : |\mathbf{x}| < b\}$, $b > a$, such that

$$\Delta_3 v + k^2 v = 0 \quad \text{in } \Omega_b \tag{3.6}$$

$$\Delta_3 w + k^2 n(\mathbf{x}) w = 0 \quad \text{in } \Omega_b \tag{3.7}$$

$$w(\mathbf{x}) - v(\mathbf{x}) = \frac{e^{-ikr}}{r} \quad \text{on } \partial\Omega_b \tag{3.8}$$

$$\frac{\partial w}{\partial r}(\mathbf{x}) - \frac{\partial v}{\partial r}(\mathbf{x}) = \frac{\partial}{\partial r} \frac{e^{-ikr}}{r} \quad \text{on } \partial\Omega_b. \tag{3.9}$$

If $k^2 < \sqrt{6}/2Ma^2$, the existence of a unique (weak) solution of (3.6)–(3.9) was established in [5].

Theorem. *Let $k^2 < \sqrt{6}/2Ma^2$ and $\{v, w\}$ be the unique (weak) solution of the interior transmission problem. Then*

(a) *If $v(\mathbf{x})$ is a Herglotz wave function with Herglotz kernel $g(\hat{\mathbf{x}})$, then $\mathbf{S}^\perp = \operatorname{span}\{g\}$.*

(b) *If $v(\mathbf{x})$ is not a Herglotz wave function, then $\mathbf{S}^\perp = \{0\}$.*

Proof. Let $g \in L^2(\partial\Omega)$ be such that

$$\int_{\partial\Omega} [F(\hat{\mathbf{x}}; k, \boldsymbol{\alpha}_n) - F(\hat{\mathbf{x}}; k, \boldsymbol{\alpha}_1)]\overline{g(\hat{\mathbf{x}})}\, ds(\hat{\mathbf{x}}) = 0 \tag{3.10}$$

for $n = 1, 2, \ldots$. Then, as in the previous theorem, we can conclude that $w_0 = (\mathbf{I} + k^2\mathbf{T}^*)^{-1} v_0 \in M^\perp$ where v_0 is the Herglotz wave function defined by the right-hand side of (3.3),

$$M = \operatorname{span}\{e^{ik\mathbf{x}\cdot\boldsymbol{\alpha}_n} - e^{ik\mathbf{x}\cdot\boldsymbol{\alpha}_1} : n = 1, 2, \ldots\} \tag{3.11}$$

and $M^{\perp}$ is the orthogonal complement of M in $L^2_m(B)$. From (3.4) we can conclude that M is a subspace of $\bar{H}$, but is not all of $\bar{H}$ since $j_0(k|\mathbf{x}|)$ is not in M. If $\mathbf{M}^{\perp}$ denotes the orthogonal complement of M in $\bar{H}$, then $\mathbf{M}^{\perp}$ has dimension one since from the proof of the previous theorem we have that

$$\bar{H} = \overline{\operatorname{span}\{e^{ik\mathbf{x}\cdot\alpha_n}: n=1,2,\ldots\}}. \tag{3.12}$$

In particular, if $\mathbf{P}:\bar{H}\to\mathbf{M}^{\perp}$ is the projection operator from $\bar{H}$ onto $\mathbf{M}^{\perp}$ then

$$\mathbf{M}^{\perp} = \operatorname{span}\{\mathbf{P}j_0\}. \tag{3.13}$$

Finally,

$$\begin{aligned}\iint_B m(\mathbf{y})\, e^{ik\mathbf{y}\cdot\alpha_n}\overline{(\mathbf{P}j_0)(\mathbf{y})}\,d\mathbf{y} &= \iint_B m(\mathbf{y})\, e^{ik\mathbf{y}\cdot\alpha_n}\overline{(\mathbf{P}j_0)(\mathbf{y})}\,d\mathbf{y}\\ &= \text{constant}\end{aligned} \tag{3.14}$$

for $n=1,2,\ldots$ and hence from (3.4) we have that

$$\iint_B m(\mathbf{y}) j_l(k|\mathbf{y}|) Y_l^m(\hat{\mathbf{y}})(\mathbf{P}j_0)(\mathbf{y})\,d\mathbf{y} = 0 \tag{3.15}$$

for $l=1,2,\ldots,\ -l\leq m\leq l$.

Now let $\mathbf{x}\in R^3\backslash\bar{B}$. Then from the addition formula for Bessel functions and (3.15) we have that

$$\begin{aligned}(\mathbf{T}^*\mathbf{P}j_0)(\mathbf{x}) &= \iint_B \bar{\Phi}(\mathbf{x},\mathbf{y})m(\mathbf{y})(\mathbf{P}j_0)(\mathbf{y})\,d\mathbf{y}\\ &= \frac{e^{-ikr}}{r}\iint_B m(\mathbf{y}) j_0(k|\mathbf{y}|)(\mathbf{P}j_0)(\mathbf{y})\,d\mathbf{y}\\ &= \overline{(j_0, \mathbf{P}j_0)}\,\frac{e^{-ikr}}{r}.\end{aligned} \tag{3.16}$$

But since $\mathbf{P}$ is a projection operator, $\mathbf{P}=\mathbf{P}^2$, $\mathbf{P}^*=\mathbf{P}$, and hence

$$(j_0, \mathbf{P}j_0) = (j_0, \mathbf{P}^2 j_0) = (\mathbf{P}j_0, \mathbf{P}j_0) = \|\mathbf{P}j_0\|^2. \tag{3.17}$$

Since $\mathbf{P}j_0$ is not zero, we can conclude that

$$c = \overline{(j_0, \mathbf{P}j_0)} \neq 0. \tag{3.18}$$

Returning now to the beginning of the proof, we have that

$$\begin{aligned} w_0 &= (\mathbf{I}+k^2\mathbf{T}^*)^{-1}v_0 \\ &= h+\gamma \mathbf{P}j_0 \end{aligned} \tag{3.19}$$

where $h \in H^\perp$ and γ is a constant. Hence, $v_0=(\mathbf{I}+k^2\mathbf{T}^*)w_0$ and for $\mathbf{x}\in\partial\Omega_b$ we have

$$\begin{aligned} w_0(\mathbf{x})-v_0(\mathbf{x}) &= -k^2(\mathbf{T}^*w_0)(\mathbf{x}) \\ &= -k^2\gamma(\mathbf{T}^*\mathbf{P}j_0)(\mathbf{x}) \\ &= -k^2\gamma c\frac{e^{-ikr}}{r} \end{aligned} \tag{3.20}$$

and similarly

$$\frac{\partial w_0}{\partial r}(\mathbf{x})-\frac{\partial v_0}{\partial r}(\mathbf{x})=-k^2\gamma c\frac{\partial}{\partial r}\frac{e^{-ikr}}{r}. \tag{3.21}$$

Since v_0 is a solution of (3.6), $w_0=(\mathbf{I}+k^2\mathbf{T}^*)^{-1}v_0$ is a solution of (3.7). If $\gamma=0$, then by the uniqueness of the solution to the interior transmission problem we have that $v_0=0$ and hence $g=0$ whereas if γ does not equal zero then $v=-(1/k^2\gamma c)v_0$, $w=-(1/k^2\gamma c)w_0$ is the unique solution of the interior transmission problem. This establishes the theorem.

We note that in the case of a spherically stratified medium, $v(\mathbf{x})$ is in fact a Herglotz wave function and hence $\mathrm{S}^\perp=\{g\}$ where $g(\mathbf{x})$ is the Herglotz kernel of $v(\mathbf{x})$ ([5]).

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Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19716
U.S.A.