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On the Number of Divisors of the Quadratic Form $m^2 + n^2$

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Abstract. For an integer n, let $d(n)$ denote the ordinary divisor function. This paper studies the asymptotic behavior of the sum

$$S(x) := \sum_{m\le x, n\le x} d(m^2+n^2).$$

It is proved in the paper that, as $x \to \infty$,

$$S(x) := A_1x^2\log x + A_2x^2 + O_\epsilon(x^{\frac{3}{2}+\epsilon}),$$

where A_1 and A_2 are certain constants and ϵ is any fixed positive real number.

The result corrects a false formula given in a paper of Gafurov concerning the same problem, and improves the error $O\big(x^{\frac{5}{3}}(\log x)^9\big)$ claimed by Gafurov.

1 Introduction

For an integer n, let $d(n)$ denote value of the ordinary divisor function, *i.e.* the number of positive divisors of n. It is our interest here to study the asymptotic behavior, as $x \to \infty$, of the sum

$$S(x) := \sum_{m\le x, n\le x} d(m^2+n^2).$$

Gafurov is the first person who studied this problem, and he claimed in [4] that

$$S(x) := A_1x^2\log x + A_2x^2 + O\big(x^{\frac{5}{3}}(\log X)^9\big),$$

where A_1 and A_2 are certain constants.

Unfortunately, Gafurov's paper contains a critical mistake in its initial decomposition, and thus his computation for A_2 is not correct. The purpose of this paper is to give the correct asymptotic main term for $S(x)$ and an error term sharper than what Gafurov claimed for his estimation.

Since there is a certain regularity in the distribution of the roots of the congruence

$$\nu^2 + 1 \equiv 0 \pmod{k}$$

for variable k, an exponential sum involving the ratio ν/k can be well estimated. We shall combine some results of Gafurov [4] with estimates of exponential sums to give the correct

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asymptotic formula with an error term $O(x^{\frac{3}{2}+\epsilon})$, for any positive ϵ. Exactly, we shall prove the following theorem.

Theorem 1 *For $x \geq 1$, we have*

$$S(x) = A_1 x^2 \log x + A_2 x^2 + O_\epsilon(x^{3/2+\epsilon}),$$

where ϵ is any positive number and A_1 and A_2 are the constants given by

$$A_1 = \frac{\pi}{2L(2,\chi)},$$

and

$$A_2 = 4\int_1^\infty \frac{E(t)}{t^3}\,dt + \frac{\pi^2 - 4\pi + 2\pi\log 2}{8L(2,\chi)},$$

where χ is the nontrivial character modulo 4 and function $E(t)$ is given in Lemma 3.

Remark 1 It should be noted that, because the sum S_2 in formula (4) of [4] is larger than the correct one, the value of A_2 Gafurov got is $\frac{\pi(\pi-2)}{8L(2,\chi)}$ smaller than ours.

We shall decompose $S(x)$ into three sums in Section 4, and give different treatments for the sums in the following three sections. We shall relate one of the three sums to the classic circle problem and make use of existing results about the circle problem to give an asymptotic formula for this sum with an error term $O(x^{1+\alpha+\epsilon})$, where $\alpha < 1/3$ is any exponent for the error term involved in the circle problem. For example, by appealing to the work of Iwaniec and Mozzochi [6], we have an error $O(x^{\frac{29}{22}+\epsilon})$ in this case.

One may naturally ask for the asymptotic formula of the sum

$$W(x) := \sum_{m^2+n^2\leq x} d(m^2+n^2).$$

Note that

$$W(x) = \sum_{n\leq x} d(n)\Big(\frac{1}{4}r_2(n)\Big) - \sum_{n\leq\sqrt{x}} d(n^2),$$

where $r_2(n)$ is the usual representation function of n as a sum of two integral squares, namely the number of lattice points (u,v) satisfying $u^2+v^2 = n$, we are concerned with two sums of multiplicative functions. Thus the sum $W(x)$ is much nicer than $S(x)$. Without proof, we state the asymptotic formula for $W(x)$ with an error $O(x^{\frac{1}{2}+\epsilon})$ as follows. This theorem, which can be easily proved via Perron's formula, is hardly the best that can be obtained for $W(x)$.

Theorem 2 *For sufficiently large x, we have*

$$W(x) = B_1 x\log x + B_2 x + O(x^{\frac{1}{2}+\epsilon}),$$

where

$$B_1 = \frac{\pi^2}{16L(2,\chi)},$$

and

$$B_2 = \frac{\pi}{16L(2,\chi)}\Big(2\gamma\pi - \pi + 8L'(1,\chi) - 2\pi\frac{L'}{L}(2,\chi)\Big),$$

where $\gamma = 0.577\cdots$ is the Euler constant.

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2 Notation and Conventions

Throughout the paper, $[t]$ denotes the integral part of the real number t and $\{t\}$ denotes its fractional part. As usual, $\psi(t) = \{t\} - 1/2$ and $\|t\| = \min\{\{t\}, 1 - \{t\}\}$. We use $e(t)$ as an abbreviation for $\exp(2\pi it)$. For integers m and n, we shall use (m, n) to denote the great common factor of m and n. Landau's symbol O and Vinogradov's symbol $\ll$ are often used.

For a complex valued function f and a real, positive valued function g, $f = O(g)$ or $f \ll g$ means there exists a positive constant c such that $|f| \leq cg$. With $n \sim N$ we mean there are positive constants c_1, c_2 such that $c_1N < n \leq c_2N$, and the notation is only used at where the values of c_1 and c_2 are not important.

By ϵ we shall always mean a small fixed constant, not necessarily the same on each occurrence. A simple notation $u \,(\mathrm{mod}\, d)$ means u runs over the residue classes modulo d, and $u \,(\mathrm{mod}\, d)^*$ means u runs over the reduced residue classes modulo d With $\rho_{k,l}(d)$ we define the arithmetic function

$$\rho_{k,l}(d) = \sum_{\substack{\nu \,(\mathrm{mod}\, d) \\ \nu^2+l^2 \equiv 0 \,(\mathrm{mod}\, d)}} e(\nu k/d).$$

Thus $\rho_l(d) := \rho_{0,l}(d)$ defines the number of solutions of the congruence $\nu^2+l^2 \equiv 0 \,(\mathrm{mod}\, d)$ for fixed l. We use another notation $\rho(d)$ to define the number of solutions of the congruence $u^2 + v^2 \equiv 0 \,(\mathrm{mod}\, d)$ subject to $0 < u, v \leq d$. We have $\phi(m)$ the Euler function; $\mu(m)$ the Möbius function; and $\chi(m)$ the non-trivial character mod 4. By $r(n)$ we denote the number of representations of n as the sum of two squares, with the order being important. For example, $r(3) = 0$, $r(2) = 1$ and $r(5) = 2$. One should be notified that $r(n)$ here differs from $r_2(n)$ defined in the introduction.

3 Lemmas

In this section, we list some lemmas which are needed in the proof of the theorem.

Lemma 1 *Suppose α is a real number, a and $q > 0$ are two coprime integers such that*

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2}.$$

Then

$$\sum_{t \leq T} \min\left(\frac{N}{t}, \frac{1}{\|\alpha t\|}\right) \ll (Nq^{-1} + T + q)\log(2qT). \tag{3.1}$$

Proof This estimate is most classic. One may refer to [1, pp. 143–144], for example.

Lemma 2 *Let*

$$S(D, H, N) := \sum_{d \sim D} \sum_{\substack{\nu \,(\mathrm{mod}\, d) \\ \nu^2+1 \equiv 0 \,(\mathrm{mod}\, d)}} \sum_{h \leq H} \frac{1}{h} \left|\sum_{n \leq N} e\left(\frac{hn\nu}{d}\right)\right|.$$

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Then for D sufficiently large, $H, N > 3$, we have

$$S(D,H,N) \ll_\epsilon (\mathrm{DNH})^\epsilon (D+N)\sqrt{D}. \tag{3.2}$$

for any $\epsilon > 0$.

Proof First by summing the geometric progression, we have

$$S(D,H,N) \ll \sum_{d\sim D} \sum_{\substack{\nu \,(\mathrm{mod}\, d)\\ \nu^2+1\equiv 0\,(\mathrm{mod}\, d)}} \sum_{h\le H} \frac{1}{h}\min\Big(N, \frac{1}{\|h\nu/d\|}\Big). \tag{3.3}$$

Note for each d which makes a positive contribution, there exist some primitive representation of d as the sum of two squares,

$$d = r^2 + s^2 \quad \text{with} \quad (r,s) = 1 \quad \text{and} \quad -s < r \le s.$$

There is a one-to-one correspondence between solutions $\nu \pmod d$ to $\nu^2+1 \equiv 0 \pmod d$ and representations of d as r^2+s^2, where $(r,s)=1$ and $-s<r\le s$. Given r, s, we can take for the corresponding $\nu \pmod d$ the residue class $r\bar{s} \pmod d$, where $0 \le \bar{s} < d$ is the only solution of the congruence $sx \equiv 1 \pmod d$. Thus we may write s_ν and r_ν instead of s and r respectively to indicate the corresponding ν. From the relation above, Iwaniec found that

$$\frac{\nu}{d} \equiv \frac{r_\nu}{s_\nu d} - \frac{\overline{r_\nu}}{s_\nu} \pmod 1,$$

where $r_\nu\overline{r_\nu} \equiv 1\,(\mathrm{mod}\, s_\nu)$ and $0 \le \overline{r_\nu} < s_\nu$. Note now we have

$$\left|\frac{\nu}{d} + \frac{\overline{r_\nu} + \alpha(\nu,d)s_\nu}{s_\nu}\right| = \frac{|r_\nu|}{s_\nu d} \le \frac{1}{2{s_\nu}^2},$$

where $\alpha(\nu,d)$ is -1, 0 or 1. Thus after (3.3), by appealing to Lemma 1, we have

$$\begin{aligned}
S(D,H,N) &\ll \log H \max_{J\ll H} J^{-1} \sum_{d\sim D} \sum_{\substack{\nu \,(\mathrm{mod}\, d)\\ \nu^2+1\equiv \,(\mathrm{mod}\, d)}} \sum_{h\sim J} \min\Big(\frac{NJ}{h}, \frac{1}{\|h\nu/d\|}\Big)\\
&\ll (\mathrm{DNH})^\epsilon \max_{J\ll H} J^{-1} \sum_{d\sim D} \sum_{\substack{\nu \,(\mathrm{mod}\, d)\\ \nu^2+1\equiv \,(\mathrm{mod}\, d)}} (NJ{s_\nu}^{-1} + J + s_\nu)\\
&\ll (\mathrm{DNH})^\epsilon \sum_{d\sim D} \sum_{\substack{\nu \,(\mathrm{mod}\, d)\\ \nu^2+1\equiv 0\,(\mathrm{mod}\, d)}} (Nd^{-\frac12} + 1 + d^{-\frac12})\\
&\ll (\mathrm{DNH})^\epsilon (D+N)\sqrt{D},
\end{aligned}$$

by noting $s_\nu \asymp \sqrt{d}$. This is what we wanted.

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Remark 2 Lemma 2 will be used in estimating errors of $R(x)$ and $T(x)$. From the context, it turns out to be the dominant factor for the exponent "3/2". Some similar sums have been discussed in [2], [3]. With the large sieve, essentially the same bounds as Lemma 2 are proved there. However, because of the problem arising from the coefficients, their results can not be directly applied in our case. The two formally different methods give the same result because both of them basically rely on the spacing property of ratios $\frac{\nu}{k}$.

Lemma 3 *Suppose $y \geq 3$. We have*

$$\sum_{d\leq y} \rho(d) = Ay^2 + O\big(y^{4/3}(\log y)^2\big);$$

$$\sum_{d\leq y} \frac{\rho(d)}{d} = 2Ay + O\big(y^{1/3}(\log y)^2\big);$$

$$\sum_{d\leq y} \frac{\rho(d)}{d^2} = 2A\log y + 2\int_1^\infty \frac{E(t)}{t^3}\,dt + A + O\big(y^{-2/3}(\log y)^2\big);$$

where

$$E(t) = \sum_{d\leq t} \rho(d) - At^2 \ll t^{4/3}(\log t)^2,$$

and

$$A = \frac{\pi}{8L(2,\chi)}$$

with

$$L(2,\chi) = \sum_1^\infty \frac{\chi(n)}{n^2}.$$

Proof This comes from Lemmas 4, 6 and 7 of Gafurov [4].

Lemma 4 **(Kusmin-Landau)** *If f is continuously differentiable, f' is monotonic, and $\|f'\| \geq \lambda > 0$ on I then*

$$\sum_{n\in I} e\big(f(n)\big) \ll \lambda^{-1},$$

where the implied constant is absolute.

Fourier expansion of function $\psi(t)$ will be needed. A truncated form given by Vaaler [7] has proved to be useful in applications. For each positive integer H, there is a trigonometric polynomial ${\psi^*}_H$ of degree H which satisfies

$$|\psi(t) - {\psi^*}_H(t)| \leq \frac{1}{2H+2} \sum_{|h|\leq H} \left(1 - \frac{|h|}{H+1}\right) e(ht),$$

where

$${\psi^*}_H(t) = \sum_{1\leq |h|\leq H} g(h)e(ht)$$

with the complex coefficients $g(h)$ satisfying $g(h) < |h|^{-1}$.

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4 Transformation of the Sum

Note we have

$$S(x) = \sum_{\substack{m\le x, n\le x\\ m^2+n^2=kl}} 1.$$

For such a sum, with the classic method as used to the Dirichlet divisor problem, we first sum over the one of k and l which is $\le \sqrt{2}x$ and then subtract the repeated part that raises from both k and l being $\le \sqrt{2}x$. Thus we can split $S(x)$ into three sums as follows.

$$S(x) = 2\sum_{k\le\sqrt{2}x} \sum_{\substack{m\le x, n\le x\\ m^2+n^2\equiv 0 \,(\mathrm{mod}\, k)}} 1 - \sum_{k\le x/\sqrt{2}} \sum_{\substack{m^2+n^2\le kx\sqrt{2}\\ m^2+n^2\equiv 0 \,(\mathrm{mod}\, k)}} 1 - \sum_{x/\sqrt{2}<k\le x\sqrt{2}} \sum_{\substack{m\le x, n\le x\\ m^2+n^2\le kx\sqrt{2}\\ m^2+n^2\equiv 0 \,(\mathrm{mod}\, k)}} 1$$

$$= 2R(x) - Q(x) - T(x), \quad \text{say,} \tag{4.1}$$

where we have written the subtracting part as the sum of $Q(x)$ and $T(x)$. And one should be noted that, for $Q(x)$, the conditions $m \le x$, $n \le x$ have naturally been satisfied.

In [4], Gafurov decomposed $S(x)$ into two sums in the similar way. He got

$$S(x) = 2S_1(x) - S_2(x)$$

where $S_1(x)$ is exactly the $R(x)$ in (4.1), and

$$S_2(x) = \sum_{k\le\sqrt{2}x} \sum_{\substack{m^2+n^2\equiv 0 \,(\mathrm{mod}\, k)\\ 1\le m,n\le\sqrt{\sqrt{2}kx-1}}} 1.$$

The ranges for m and n involved in $S_2(x)$ are obviously not correct. $S_2(x)$ is over counted and thus is greater than $Q(x) + T(x)$.

Among all of the three sums, $Q(x)$ is especially easy. We shall transform it into a sum involving the numbers of lattice points in circles and take advantage of the existing results of the circle problem to give the asymptotic formula with an error better than those we get for $R(x)$ and $T(x)$. For $R(x)$ and $T(x)$, we shall directly divide them into main terms and error terms, and estimate certain exponential sums to give upper bounds for the errors.

5 Treatment of $R(x)$

Suppose $m^2 + n^2 \equiv 0 \,(\mathrm{mod}\, k)$ and $(n^2, k) = ab^2$, where a is squarefree. Then we have $ab \mid (m, n)$. Thus we can rewrite $R(x)$ as

$$R(x) = \sum_{\substack{ab^2 d\le\sqrt{2}x\\ (d,a)=1}} \mu^2(a) \sum_{\substack{m\le x/ab\\ (m,d)=1}} \sum_{\substack{n\le x/ab\\ m^2+n^2\equiv 0 \,(\mathrm{mod}\, d)}} 1. \tag{5.1}$$

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Therefore,

$$(5.2)\qquad R(x) = \sum_{\substack{ab^2d\le\sqrt{2}x\\(d,a)=1}} \mu^2(a) \sum_{\substack{m\le x/ab\\(m,d)=1}} \sum_{\substack{\nu\ (\mathrm{mod}\ d)\\ m^2+\nu^2\equiv 0\ (\mathrm{mod}\ d)}} \sum_{\substack{n\le x/ab\\ n\equiv\nu\ (\mathrm{mod}\ d)}} 1$$

$$= \sum_{a,b,d} \mu^2(a) \sum_{\substack{m\le x/ab\\(m,d)=1}} \sum_{\substack{\nu\ (\mathrm{mod}\ d)\\ \nu^2+1\equiv 0\ (\mathrm{mod}\ d)}} \sum_{\substack{n\le x/ab\\ n\equiv m\\ \nu\ (\mathrm{mod}\ d)}} 1$$

$$= \sum_{a,b,d} \mu^2(a) \sum_{\substack{m\le x/ab\\(m,d)=1}} \sum_{\substack{\nu\ (\mathrm{mod}\ d)\\ \nu^2+1\equiv 0\ (\mathrm{mod}\ d)}} \left(\left[\frac{x/ab-m\nu}{d}\right]-\left[\frac{-m\nu}{d}\right]\right)$$

$$= x\sum_{a,b,d}\frac{\rho_1(d)\mu^2(a)}{abd}\sum_{\substack{m\le x/ab\\(m,d)=1}} 1$$

$$+ \sum_{a,b,d} \mu^2(a) \sum_{\substack{m\le x/ab\\(m,d)=1}} \sum_{\substack{\nu\ (\mathrm{mod}\ d)\\ \nu^2+1\equiv 0\ (\mathrm{mod}\ d)}} \left(\psi\Big(\frac{-m\nu}{d}\Big)-\psi\Big(\frac{x/ab-m\nu}{d}\Big)\right)$$

$$= R_0(x) + E_R(x), \quad \text{say.}$$

From the Fourier approximation of $\psi(t)$ given in Section 3, for any integer $H = H(a,b,x) \ge 1$, we have

$$\sum_{d}\sum_{\substack{m\le x/ab\\(m,d)=1}} \sum_{\substack{\nu\ (\mathrm{mod}\ d)\\ \nu^2+1\equiv 0\ (\mathrm{mod}\ d)}} \left(\psi\Big(\frac{-m\nu}{d}\Big)-\psi\Big(\frac{x/ab-m\nu}{d}\Big)\right)$$

$$(5.3)\qquad = \sum_{d}\sum_{\substack{m\le x/ab\\(m,d)=1}} \sum_{\substack{\nu\ (\mathrm{mod}\ d)\\ \nu^2+1\equiv 0\ (\mathrm{mod}\ d)}} \left({\psi^*}_H\Big(\frac{-m\nu}{d}\Big)-{\psi^*}_H\Big(\frac{x/ab-m\nu}{d}\Big)\right)$$

$$+ O\left(\Big|\sum_{d}\sum_{\substack{m\le x/ab\\(m,d)=1}} \sum_{\substack{\nu\ (\mathrm{mod}\ d)\\ \nu^2+1\equiv 0\ (\mathrm{mod}\ d)}} \sum_{1\le|h|\le H} g(h,d)e\Big(\frac{-hm\nu}{d}\Big)\Big| + \frac{x^{2+\epsilon}}{Ha^2b^3}\right),$$

where

$$g(h,d) = \frac{1}{H}\left(1-\frac{|h|}{H+1}\right)\left(1+e\Big(\frac{hx}{abd}\Big)\right).$$

Thus, from the series representation of ${\psi^*}_H(t)$ given in Section 3, we are concerned with the sum

$$\sum_{d}\sum_{\substack{m\le x/ab\\(m,d)=1}} \sum_{\substack{\nu\ (\mathrm{mod}\ d)\\ \nu^2+1\equiv 0\ (\mathrm{mod}\ d)}} \sum_{1\le|h|\le H} g'(h,d)e\Big(\frac{-hm\nu}{d}\Big),$$

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where $g'(h,d) = g(h,d)$ or $g(h)\big(1 - e(hx/abd)\big)$ with $g(h)$ given in Section 3. Since $|g'(h,d)| \le 2|h|^{-1}$, from Lemma 2, such a sum is

$$\ll \Bigg| \sum_{\substack{rl \le \sqrt{2}x/ab^2 \\ (rl,a)=1}} \frac{\mu(l)\rho_1(rl)}{\rho_1(r)} \sum_{n \le x/abl} \sum_{\substack{\nu \,(\mathrm{mod}\, r) \\ \nu^2+1 \equiv 0 \,(\mathrm{mod}\, r)}} \sum_{1 \le |h| \le H} g'(h,rl) e\Big(\frac{-hn\nu}{r}\Big) \Bigg|$$

$$\ll x^{\epsilon} \sum_{l \le x/ab^2} \sum_{r \le x/ab^2 l} \sum_{\nu} \sum_{h} \frac{1}{|h|} \Big| \sum_{n \le x/abl} e\Big(\frac{-hn\nu}{r}\Big) \Big| \tag{5.4}$$

$$\ll x^{\epsilon} \sum_{l} \Big(\frac{x}{abl} + \frac{x}{ab^2 l}\Big) \sqrt{\frac{x}{ab^2 l}} \ll \frac{x^{3/2+\epsilon}}{a^{\frac{3}{2}} b^2}.$$

Thus, by choosing $H = 1 + \Big[\frac{\sqrt{x}}{ab^2}\Big]$, we have

$$E_R(x) \ll \sum_{ab^2 \le \sqrt{2}x} \frac{x^{3/2+\epsilon}}{a^{\frac{3}{2}} b^2} + x^{\frac{3}{2}+\epsilon} \ll x^{\frac{3}{2}+\epsilon}. \tag{5.5}$$

Note from the simple formula

$$\sum_{\substack{m \le x/ab \\ (m,d)=1}} 1 = \frac{\phi(d)x}{abd} + O(d^{\epsilon}),$$

we have

$$R_0(x) = x^2 \sum_{\substack{ab^2 d \le \sqrt{2}x \\ (d,a)=1}} \frac{\rho_1(d)\phi(d)\mu^2(a)}{(abd)^2} + O(x^{1+\epsilon}). \tag{5.6}$$

By considering the great common factor of μ^2 and k in the equation $\mu^2 + \nu^2 \equiv 0 \,(\mathrm{mod}\, k)$, one can easily deduce that

$$\rho(k) = \sum_{\substack{ab^2 d = k \\ (a,d)=1}} \mu^2(a) b^2 \rho_1(d)\phi(d),$$

$$\sum_{k \le \sqrt{2}x} \frac{\rho(k)}{k^2} = \sum_{\substack{ab^2 d \le \sqrt{2}x \\ (d,a)=1}} \frac{\rho_1(d)\phi(d)\mu^2(a)}{(abd)^2}. \tag{5.7}$$

Thus, combining this with Lemma 3 and (5.6), and from (5.2) and (5.5), we get

$$R(x) = x^2 \Big(2A \log x + 2 \int_1^{\infty} \frac{E(t)}{t^3}\, dt + A(\log 2 + 1) \Big) + O(x^{\frac{3}{2}+\epsilon}), \tag{5.8}$$

where the constant A and function $E(t)$ are given in Lemma 3.

6 Treatment of $Q(x)$

It's well known that if n is not a perfect square, then

$$r(n) = \sum_{d|n} \chi(d).$$

Thus we have

$$\begin{aligned}
Q(x) &= \sum_{k\le x/\sqrt{2}} \sum_{n\le\sqrt{2}x} r(nk) \\
&= \sum_{k\le x/\sqrt{2}} \sum_{n\le\sqrt{2}x} \sum_{d|nk} \chi(d) + O(x^{1+\epsilon}) \qquad (6.1)\\
&= \sum_{d\le x^2} \chi(d) \sum_{k\le x/\sqrt{2}} \sum_{\substack{n\le\sqrt{2}x \\ nk\equiv 0 \pmod d}} 1 + O(x^{1+\epsilon}).
\end{aligned}$$

Let $(n,d) = d_1$, and write $d = d_1 d_2$, $n = d_1 l$, then we have

$$\begin{aligned}
Q(x) &= \sum_{d_1\le\sqrt{2}x} \chi(d_1) \sum_{d_2\le x/\sqrt{2}} \chi(d_2) \sum_{\substack{l\le\sqrt{2}x/d_1 \\ (l,d_2)=1}} \sum_{\substack{k\le x/\sqrt{2} \\ d_2|k}} 1 + O(x^{1+\epsilon}) \qquad (6.2)\\
&= \sum_{d_2\le x/\sqrt{2}} \chi(d_2) \Big[\frac{x}{\sqrt{2}d_2}\Big] \sum_{\substack{l\le\sqrt{2}x \\ (l,d_2)=1}} \sum_{d_1\le\sqrt{2}x/l} \chi(d_1) + O(x^{1+\epsilon}).
\end{aligned}$$

We eliminate the restriction $(l,d_2) = 1$ and, letting $d_2 = ms$, $l = mt$, we get

(6.3)

$$Q(x) = \sum_{m\le x/\sqrt{2}} \mu(m)\chi(m) \sum_{s\le x/\sqrt{2}m} \chi(s) \Big[\frac{x}{\sqrt{2}ms}\Big] \sum_{t\le\sqrt{2}x/m} \sum_{d_1\le\sqrt{2}x/mt} \chi(d_1) + O(x^{1+\epsilon}).$$

For the inner double sum we have

$$\begin{aligned}
\sum_{t\le\sqrt{2}x/m} \sum_{d_1\le\sqrt{2}x/mt} \chi(d_1) &= \sum_{n\le\sqrt{2}x/m} \sum_{d|n} \chi(d) = \frac{1}{4} \sum_{n\le\sqrt{2}x/m} r_2(n) \qquad (6.4)\\
&= \frac{1}{4} \sum_{\substack{i^2+j^2\le\sqrt{2}x/m \\ (i,j)\in\mathbb{Z}^2}} 1 = \frac{\sqrt{2}\pi x}{4m} + O\Big(\Big(\frac{x}{m}\Big)^{\alpha+\epsilon}\Big),
\end{aligned}$$

where $\alpha < 1/3$. The same argument shows that

$$\begin{aligned}
\sum_{s\le x/\sqrt{2}m} \chi(s) \Big[\frac{x}{\sqrt{2}ms}\Big] &= \sum_{s\le x/\sqrt{2}m} \chi(s) \sum_{r\le x/\sqrt{2}ms} 1 = \frac{1}{4} \sum_{n\le x/\sqrt{2}m} r_2(n) \qquad (6.5)\\
&= \frac{1}{4} \sum_{\substack{i^2+j^2\le x/\sqrt{2}m \\ (i,j)\in\mathbb{Z}^2}} 1 = \frac{\pi x}{4\sqrt{2}m} + O\Big(\Big(\frac{x}{m}\Big)^{\alpha+\epsilon}\Big).
\end{aligned}$$

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Therefore, from (6.3)–(6.5) we have

$$Q(x) = \frac{\pi^2 x^2}{16} \sum_{m \le x/\sqrt{2}} \frac{\mu(m)\chi(m)}{m^2} + O(x^{1+\alpha+\epsilon}) = \frac{\pi^2 x^2}{16 L(2,\chi)} + O(x^{1+\alpha+\epsilon}). \tag{6.6}$$

7 Treatment of $T(x)$

First we have

$$T(x) = \sum_{\frac{x}{\sqrt{2}} < k \le \sqrt{2}x} \sum_{m \le x} \sum_{\substack{n \le x \\ m^2+n^2 \equiv 0 \,(\mathrm{mod}\, k)}} 1 - \sum_{\frac{x}{\sqrt{2}} < k \le \sqrt{2}x} \sum_{\sqrt{\sqrt{2}kx - x^2} < m \le x} \sum_{\substack{\sqrt{\sqrt{2}kx - m^2} < n \le x \\ m^2+n^2 \equiv 0 \,(\mathrm{mod}\, k)}} 1 \tag{7.1}$$

$$= T_1(x) - T_2(x), \quad \text{say.}$$

Since $T_1(x)$ is in the same shape of $R(x)$, we can deal with it in the same way as that for $R(x)$; we will just give the result in the end of this section.

By writing $(m^2, k) = ab^2$ with a squarefree, we can rewrite $T_2(x)$ as

(7.2)

$$T_2(x) = \sum_{\substack{\frac{x}{\sqrt{2}} < ab^2 d \le \sqrt{2}x \\ (d,a)=1}} \mu^2(a) \sum_{\substack{\sqrt{\sqrt{2}ab^2\, dx - x^2}/ab < m \le \frac{x}{7}ab \\ (m,d)=1}} \sum_{\substack{\sqrt{\sqrt{2}\, dx/a - m^2} < n \le xab \\ (n,d)=1 \\ m^2+n^2 \equiv 0 \,(\mathrm{mod}\, d)}} 1$$

$$= \sum_{\substack{\frac{x}{\sqrt{2}} < ab^2 d \le \sqrt{2}x \\ (d,a)=1}} \mu^2(a) \sum_{m} \sum_{\substack{\nu \,(\mathrm{mod}\, d) \\ m^2+\nu^2 \equiv 0 \,(\mathrm{mod}\, d)}} \sum_{n \equiv \nu \,(\mathrm{mod}\, d)} 1$$

$$= \sum_{\substack{\frac{x}{\sqrt{2}} < ab^2 d \le \sqrt{2}x \\ (d,a)=1}} \mu^2(a) \sum_{m} \sum_{\substack{\nu \,(\mathrm{mod}\, d) \\ m^2+\nu^2 \equiv 0 \,(\mathrm{mod}\, d)}} \left(\left[\frac{x/ab - \nu}{d} \right] - \left[\frac{\sqrt{\sqrt{2}\, dx/a - m^2} - \nu}{d} \right] \right)$$

$$= \sum_{a,b,d} \frac{\mu^2(a)}{d} \sum_{m,\nu} \left(\frac{x}{ab} - \sqrt{\sqrt{2}\, dx/a - m^2} \right)$$

$$+ \sum_{a,b,d} \mu^2(a) \sum_{m} \sum_{\nu} \psi\left(\frac{\sqrt{\sqrt{2}\, dx/a - m^2} - \nu}{d} \right)$$

$$- \sum_{a,b,d} \mu^2(a) \sum_{m} \sum_{\nu} \psi\left(\frac{x/ab - \nu}{d} \right)$$

$$= T_{21}(x) + T_{22}(x) - T_{23}(x), \quad \text{say.}$$

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We shall show that $T_{21}(x)$ gives the main term of $T_2(x)$ and the others turn out to be remainders. We are going to give detailed estimation for $T_{22}(x)$. Estimation for $T_{23}(x)$ is easier, and is similar to that for $E_R(x)$, thus we shall just point out how it goes after we finish estimating $T_{22}(x)$. Note the main contribution for $T_{22}(x)$ comes from the terms when ab^2 is small: we suppose $ab^2 \leq x^{1/2-\epsilon}$ and the discarded terms contribute $O(x^{\frac{3}{2}+\epsilon})$. We write

$$T_{22}(x) = \sum_{a,b} T_{22}(a,b,x). \tag{7.3}$$

Similar to the detailed discussion concerning the Fourier approximation of $E_R(x)$ given in Section 5, for some parameter $H_1 = H_1(a,b)$ which satisfies $x^{\epsilon} \ll H_1 \ll x^{\frac{1}{2}-\epsilon}$ and will be fixed later, we have

$$\begin{aligned} T_{22}(a,b,x) &\ll \Big| \sum_d \sum_m \sum_\nu \sum_{|h_1|\leq H_1} \tilde{g}(h_1) e\Big(\frac{h_1\sqrt{\sqrt{2}\,dx/a - m^2} - h_1\nu}{d}\Big)\Big| \\ &\quad + \sum_d \sum_m \sum_\nu \frac{\log H_1}{H_1} \\ &= |T'_{22}(a,b,H_1,x)| + O\Big(\frac{x^{2+\epsilon}}{a^2b^3H_1}\Big), \quad \text{say,} \end{aligned} \tag{7.4}$$

where $\tilde{g}(h_1)$ is certain complex number satisfying $|\tilde{g}(h_1)| \ll 1/|h_1|$. We further have

(7.5)

$$\begin{aligned} &T'_{22}(a,b,H_1,x) \\ &= \sum_d \sum_{h_1} \tilde{g}(h_1) \sum_{\substack{\mu,\nu \,(\mathrm{mod}\, d)^* \\ d|(\mu^2+\nu^2)}} e\Big(\frac{-h_1\nu}{d}\Big) \sum_{m\equiv\mu\,(\mathrm{mod}\,d)} e\Big(\frac{h_1\sqrt{\sqrt{2}\,dx/a - m^2}}{d}\Big) \\ &= \sum_d \frac{1}{d} \sum_{-d/2<h_2\leq d/2} \sum_{h_1} \tilde{g}(h_1) \sum_{\substack{\mu,\nu \,(\mathrm{mod}\, d)^* \\ d|(\mu^2+\nu^2)}} e\Big(\frac{h_2\mu - h_1\nu}{d}\Big) \sum_m e\Big(\frac{h_1\sqrt{\sqrt{2}\,dx/a - m^2} - h_2 m}{d}\Big) \\ &= \sum_d \frac{1}{d} \sum_{h_2} \sum_{h_1} \tilde{g}(h_1) \sum_{\substack{\nu\,(\mathrm{mod}\,d) \\ \nu^2+1\equiv 0\,(\mathrm{mod}\,d)}} R(h_2 - h_1\nu; d) \sum_m e\Big(\frac{h_1\sqrt{\sqrt{2}\,dx/a - m^2} - h_2 m}{d}\Big), \end{aligned}$$

where $R(w;d)$ is the Ramanujan sum

$$R(w;d) := \sum_{\substack{0<u\leq d \\ (u,d)=1}} e\Big(\frac{wu}{d}\Big) = \sum_{s|(w,d)} s\mu(d/s). \tag{7.6}$$

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Note we may shorten the range for d to $(x+\sqrt{x})/\sqrt{2} < ab^2d \le \sqrt{2}x$, and this gives us an error $O(x^{\frac{3}{2}+\epsilon}/(ab)^2)$. We use (7.6) for the Ramanujan sum $R(h_2-h_1\nu;d)$, and thus the contribution from those h_2's satisfying $|h_2| \le H_1x^\epsilon$ is

$$\ll \frac{x}{ab}\sum_d \frac{1}{d}\sum_{|h_2|\le H_1x^\epsilon}\sum_{h_1}\frac{1}{|h_1|}\sum_\nu |R(h_2-h_1\nu;d)|$$

$$\ll \frac{x}{ab}\sum_d \frac{1}{d}\sum_{|h_2|\le H_1x^\epsilon}\sum_{h_1}\frac{1}{|h_1|}\sum_\nu \sum_{s|(d,h_2-h_1\nu)} s.$$

Since $\nu^2+1\equiv 0 \pmod{s}$, from $s|(h_2-h_1\nu)$ we have $s \mid ({h_1}^2+{h_2}^2)$, thus this is

$$\ll \frac{x}{ab}\sum_d \frac{1}{d}\sum_{|h_2|\le H_1x^\epsilon}\sum_{h_1}\frac{1}{|h_1|}\sum_{s|(d,{h_1}^2+{h_2}^2)} s$$

$$= \frac{x}{ab}\sum_{h_1}\frac{1}{|h_1|}\sum_{|h_2|\le H_1x^\epsilon}\sum_{\substack{s|({h_1}^2+{h_2}^2)\\ s\le\frac{\sqrt{2}x}{ab^2}}}\sum_{l\sim\frac{x}{ab^2s}}\frac{1}{l}$$

$$\ll \frac{x^{1+\epsilon}H_1}{ab}.$$

Hence we have

$$T'_{22}(a,b,H_1,x) \ll x^\epsilon \sum_d \frac{1}{d}\max_{\substack{H'_1\ll H_1\\ H_1x^\epsilon\ll H_2\le\frac{d}{4}}}\frac{1}{H'_1}\sum_{h_1\sim H'_1}\sum_{H_2<|h_2|\le 2H_2}\sum_\nu |R(h_2-h_1\nu;d)| \cdot\Bigl|\sum_m e\Bigl(\frac{h_1\sqrt{\sqrt{2}\,dx/a-m^2}-h_2m}{d}\Bigr)\Bigr| + \frac{x^{1+\epsilon}H_1}{ab}. \tag{7.7}$$

Note for fixed H'_1, H_2 and d, we can divide the range of m into two parts, say Ω_1 and Ω_2, such that

$$\left\|\frac{\frac{h_1m}{\sqrt{\sqrt{2}\,dx/a-m^2}}+h_2}{d}\right\| < \frac{H_1}{d}\quad \text{if} m\in\Omega_1,$$

and otherwise if $m\in\Omega_2$. It's clear that Ω_1 and Ω_2 are respectively consist of at most $O(1+H'_1x^{1/4}/d)$ continuous segments, by noting that when $d>(x+\sqrt{x})/\sqrt{2}ab^2$ we have $\sqrt{\sqrt{2}\,dx/a-m^2} > x^{3/4}/ab$. For the subsum over each segment contained in Ω_2, Lemma 4 provides the estimate $O(d/H_1)$, and thus the summation over Ω_2 is bounded by $O(x^{\frac{1}{4}}+d/H_1)$.

Since $H_1x^\epsilon \ll 2|h_2|\le d$ and h_1 is positive, we have

$$\Omega_1 = \bigcup_{0\le j\ll H'_1x^{\frac{1}{4}}}\Omega_{1j},$$

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where Ω_{1j} are pairwise disjoint segments such that $m \in \Omega_{1j}$ if and only if

$$\left|\frac{\frac{h_1 m}{\sqrt{\sqrt{2}\,dx/a-m^2}} + h_2}{d} - j\right| < \frac{H_1}{d}. \tag{7.8}$$

It can be seen that (7.8) is equivalent to

$$\frac{\sqrt{2}\,dx}{a}\left(1 + \frac{{h_1}^2}{(jd - h_2 - H_1)^2}\right)^{-1} < m^2 < \frac{\sqrt{2}\,dx}{a}\left(1 + \frac{{h_1}^2}{(jd - h_2 + H_1)^2}\right)^{-1} \tag{7.9}$$

Note $d \geq 2|h_2| \gg H_1 x^\epsilon$, thus from (7.9), the length of Ω_{1j} is

$$\begin{aligned}
&\ll {h_1}^2\sqrt{\frac{dx}{a}}\left(\frac{1}{(jd - h_2 - H_1)^2} - \frac{1}{(jd - h_2 + H_1)^2}\right)\\
&\ll \sqrt{\frac{dx}{a}} \cdot \frac{H_1 {h_1}^2}{|jd - h_2|^3}.
\end{aligned}$$

So the total length of Ω_1 is $O(\sqrt{\frac{dx}{a}}\frac{H_1 {H_1'}^2}{{H_2}^3})$, and it should be noted that Ω_{10} makes the main contribution. Combining this with the estimate over Ω_2, we get

$$\sum_m e\left(\frac{h_1\sqrt{\sqrt{2}\,dx/a - m^2} - h_2 m}{d}\right) \ll \sqrt{\frac{dx}{a}}\frac{H_1 {H_1'}^2}{{H_2}^3} + x^{\frac{1}{4}} + \frac{d}{H_1}. \tag{7.10}$$

Now from (7.6), (7.7) and (7.10) we have

(7.11)

$$\begin{aligned}
&T'_{22}(a, b, H_1, x)\\
&\ll x^\epsilon \sum_d \frac{1}{d} \max_{\substack{H_1' \ll H_1\\ H_1 x^\epsilon \ll H_2 \leq \frac{d}{4}}} \left(\sqrt{\frac{dx}{a}}\frac{H_1 H_1'}{{H_2}^3} + \frac{x^{\frac{1}{4}}}{H_1'} + \frac{d}{H_1 H_1'}\right)\sum_\nu \sum_{h_1}\sum_{h_2}\sum_{s|(d, h_2 - h_1\nu)} s + \frac{x^{1+\epsilon}H_1}{ab}\\
&\ll x^\epsilon \max_{\substack{H_1' \ll H_1\\ H_1 x^\epsilon \ll H_2 \ll \frac{x}{ab^2}}} \sum_d \frac{1}{d}\left(\frac{xH_1 H_1'}{abH_2^3} + \frac{x^{\frac{1}{4}}}{H_1'} + \frac{x}{ab^2 H_1 H_1'}\right)\sum_{h_1}\sum_{h_2}\sum_{s|(d, {h_1}^2 + {h_2}^2)} s + \frac{x^{1+\epsilon}H_1}{ab}\\
&\ll x^\epsilon \max_{\substack{H_1' \ll H_1\\ H_1 x^\epsilon \ll H_2 \ll \frac{x}{ab^2}}} \left(\frac{xH_1 H_1'}{abH_2^3} + \frac{x^{\frac{1}{4}}}{H_1'} + \frac{x}{ab^2 H_1 H_1'}\right)\sum_{h_1}\sum_{h_2}\sum_{s|({h_1}^2 + {h_2}^2)}\sum_{l \sim \frac{x}{ab^2 s}} \frac{1}{l} + \frac{x^{1+\epsilon}H_1}{ab}\\
&\ll x^\epsilon \max_{H_1', H_2}\left(\frac{xH_1 {H_1'}^2}{abH_2^2} + x^{\frac{1}{4}}H_2 + \frac{xH_2}{ab^2 H_1}\right) + \frac{x^{1+\epsilon}H_1}{ab}\\
&\ll \frac{x^{\frac{3}{2}+\epsilon}}{ab} + \frac{x^{2+\epsilon}}{ab^2 H_1} + \frac{x^{1+\epsilon}H_1}{ab}.
\end{aligned}$$

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Let $H_1 = x^{\frac{1}{2}}/b$, then we have, from (7.4) and (7.11), that

$$T_{22}(a,b,x) \ll \frac{x^{\frac{3}{2}+\epsilon}}{ab}. \tag{7.12}$$

So from (7.3) we have proved that

$$T_{22}(x) \ll x^{\frac{3}{2}+\epsilon}. \tag{7.13}$$

For $T_{23}(x)$, we are concerned with a sum of form

$$\sum_{a,b,d} \sum_{\nu^2+1\equiv 0 \pmod d} \sum_h a(h) \sum_m e\Big(\frac{hx/ab - hm\nu}{d}\Big).$$

This is of the same shape as the sum involved in $E_R(x)$. By using Lemma 3, as what we have done for $E_R(x)$, one can prove that

$$T_{23}(x) \ll x^{\frac{3}{2}+\epsilon}. \tag{7.14}$$

Now we turn to the main term of $T_2(x)$, namely $T_{21}(x)$. For brevity, we write

$$f(a,b,d,m) = \frac{x}{ab} - \sqrt{\sqrt{2}\,dx/a - m^2}.$$

Then we have

(7.15)

$$\begin{aligned}
T_{21}(x) &= \sum_{\substack{\frac{x}{\sqrt{2}} < ab^2 d \le \sqrt{2}x \\ (d,a)=1}} \frac{\mu^2(a)}{d} \sum_{\substack{\mu,\nu \in (\mathbb{Z}/d\mathbb{Z})^* \\ \mu^2+\nu^2 \equiv 0 \pmod d}} \sum_{\substack{\sqrt{\sqrt{2}ab^2\,dx - x^2}/ab < m \le x/ab \\ m \equiv \mu \pmod d}} f(a,b,d,m) \\
&= \sum_{a,b,d} \frac{\mu^2(a)}{d^2} \sum_{\mu,\nu} \sum_{-d/2 < h \le d/2} e\Big(\frac{-h\mu}{d}\Big) \sum_{\sqrt{\sqrt{2}ab^2\,dx - x^2}/ab < m \le x/ab} f(a,b,d,m) e\Big(\frac{hm}{d}\Big) \\
&= \sum_{a,b,d} \frac{\mu^2(a)\rho_1(d)\phi(d)}{d^2} \sum_m f(a,b,d,m) \\
&\qquad + \sum_{a,b,d} \frac{\mu^2(a)\rho_1(d)}{d^2} \sum_{h \ne 0} R(-h;d) \sum_m f(a,b,d,m) e\Big(\frac{hm}{d}\Big) \\
&= {T_{21}}^{(1)}(x) + {T_{21}}^{(2)}(x), \quad \text{say.}
\end{aligned}$$

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Since for $0 < |h| \le d/2$,

$$\sum_{m\le t} e\Big(\frac{hm}{d}\Big) \ll \frac{d}{|h|},$$

using the Abel summation formula, we get

$$\sum_{m} f(a,b,d,m) e\Big(\frac{hm}{d}\Big) \ll \frac{dx}{ab|h|}.$$

Hence with (7.6) we have

$$\begin{aligned} T_{21}{}^{(2)}(x) &\ll x^{1+\epsilon} \sum_{a,b,d} \frac{1}{abd} \sum_{h\le d/2} \frac{1}{h} \sum_{s|(d,h)} s \\ &\ll x^{1+\epsilon} \sum_{a,b} \frac{1}{ab} \sum_{s\le \frac{x}{ab^2}} \frac{1}{s} \sum_{l\sim \frac{x}{ab^2 s}} \frac{1}{l} \sum_{h'\le l/2} \frac{1}{h'} \\ &\ll x^{1+\epsilon}. \end{aligned} \tag{7.16}$$

Therefore, from (7.2), (7.13)–(7.16), we have

(7.17)

$$\begin{aligned} T_2(x) &= \sum_{\substack{\frac{x}{\sqrt{2}} < ab^2 d \le \sqrt{2}x \\ (d,a)=1}} \frac{\mu^2(a)\rho_1(d)\phi(d)}{d^2} \sum_{\sqrt{\sqrt{2}ab^2\,dx - x^2}/ab < m \le x/ab} f(a,b,d,m) + O(x^{\frac{3}{2}+\epsilon}) \\ &= \sum_{a,b,d} \frac{\mu^2(a)\rho_1(d)\phi(d)}{d^2} \#\Big\{ m \le \frac{x}{ab}, n \le \frac{x}{ab} : m^2 + n^2 > \frac{\sqrt{2}\,dx}{a} \Big\} + O(x^{\frac{3}{2}+\epsilon}). \end{aligned}$$

With the same method as we used to deal with $R(x)$, we have

$$T_1(x) = x^2 \sum_{\substack{\frac{x}{\sqrt{2}} < ab^2 d \le \sqrt{2}x \\ (d,a)=1}} \frac{\mu^2(a)\rho_1(d)\phi(d)}{a^2 b^2 d^2} + O(x^{\frac{3}{2}+\epsilon}). \tag{7.18}$$

Thus, (7.1), (7.17) and (7.18) imply that

(7.19)

$$T(x) = \sum_{\substack{\frac{x}{\sqrt{2}} < ab^2 d \le \sqrt{2}x \\ (d,a)=1}} \frac{\mu^2(a)\rho_1(d)\phi(d)}{d^2} \#\Big\{ m \le \frac{x}{ab}, n \le \frac{x}{ab} : m^2 + n^2 \le \frac{\sqrt{2}\,dx}{a} \Big\} + O(x^{\frac{3}{2}+\epsilon})$$

$$= \sum_{\substack{\frac{x}{\sqrt{2}}<ab^2d\le\sqrt{2}x\\(d,a)=1}} \frac{\mu^2(a)\rho_1(d)\phi(d)}{(abd)^2}\#\{m\le x, n\le x : m^2+n^2\le\sqrt{2}ab^2\,dx\}$$
$$+ O\Big(x\sum_{a,b,d}\frac{\mu^2(a)\rho_1(d)\phi(d)}{abd^2}\Big) + O(x^{\frac{3}{2}+\epsilon})$$
$$= \sum_{x\sqrt{2}<k\le\sqrt{2}x} \frac{\rho(k)}{k^2}\#\{m\le x, n\le x : m^2+n^2\le\sqrt{2}kx\} + O(x^{\frac{3}{2}+\epsilon}),$$

where the last equality holds because of (5.7).

By counting the number of lattice points, we get

$$\#\{m\le x, n\le x : m^2+n^2\le\sqrt{2}kx\} \tag{7.20}$$
$$= \frac{kx}{\sqrt{2}}\arccos\sqrt{\frac{2x}{k}\Big(\sqrt{2}-\frac{x}{k}\Big)} + x\sqrt{\sqrt{2}kx-x^2} + O(x).$$

And from Lemma 3 we have

$$\sum_{x\sqrt{2}<k\le\sqrt{2}x}\frac{\rho(k)}{k^2}\cdot\frac{kx}{\sqrt{2}}\arccos\sqrt{\frac{2x}{k}\Big(\sqrt{2}-\frac{x}{k}\Big)} \tag{7.21}$$
$$= \frac{x}{\sqrt{2}}\int_{x/\sqrt{2}}^{\sqrt{2}x}\arccos\sqrt{\frac{2x}{t}\Big(\sqrt{2}-\frac{x}{t}\Big)}\, d\Big(\sum_{k\le t}\frac{\rho(k)}{k}\Big)$$
$$= \frac{x}{\sqrt{2}}\Big(\sum_{k\le t}\frac{\rho(k)}{k}\Big)\arccos\sqrt{\frac{2x}{t}\Big(\sqrt{2}-\frac{x}{t}\Big)}\,\Big|_{x/\sqrt{2}}^{\sqrt{2}x}$$
$$- \frac{x}{\sqrt{2}}\int_{x/\sqrt{2}}^{\sqrt{2}x}\Big(\sum_{k\le t}\frac{\rho(k)}{k}\Big)\frac{-\sqrt{2}xt^{-2}+2x^2t^{-3}}{\sqrt{1-2\sqrt{2}xt^{-1}+2x^2t^{-2}}\sqrt{2\sqrt{2}xt^{-1}-2x^2t^{-2}}}\,dt$$
$$= -\frac{\pi x}{2\sqrt{2}}\sum_{k\le x/\sqrt{2}}\frac{\rho(k)}{k} + \sqrt{2}Ax\int_{x/\sqrt{2}}^{\sqrt{2}x}\frac{\sqrt{2}xt^{-1}}{\sqrt{2\sqrt{2}xt^{-1}-2x^2t^{-2}}}\,dt + O(x^{\frac{4}{3}+\epsilon})$$
$$= -\frac{A\pi x^2}{2} + \sqrt{2}Ax\cdot\frac{x}{\sqrt{2}}\int_1^2\frac{2u^{-1}}{\sqrt{4u^{-1}-4u^{-2}}}\,du + O(x^{\frac{4}{3}+\epsilon})$$
$$= -\frac{A\pi x^2}{2} + Ax^2\int_1^2\frac{1}{\sqrt{u-1}}\,du + O(x^{\frac{4}{3}+\epsilon})$$
$$= A\Big(2-\frac{\pi}{2}\Big)x^2 + O(x^{\frac{4}{3}+\epsilon}).$$

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Also from Lemma 3 we have

(7.22)

$$\begin{aligned}
\sum_{x\sqrt{2}<k\leq\sqrt{2}x} \frac{\rho(k)}{k^2}\cdot\sqrt{\sqrt{2}kx - x^2}
&= \sqrt{\sqrt{2}tx - x^2}\sum_{k\leq t}\frac{\rho(k)}{k^2}\Big|_{x/\sqrt{2}}^{\sqrt{2}x} - \frac{x}{\sqrt{2}}\int_{x/\sqrt{2}}^{\sqrt{2}x}(\sqrt{2}tx - x^2)^{-\frac{1}{2}}\sum_{k\leq t}\frac{\rho(k)}{k^2}\,dt\\
&= (2A\log\sqrt{2}x + B)x - \frac{x}{\sqrt{2}}\int_{x/\sqrt{2}}^{\sqrt{2}x}(\sqrt{2}tx - x^2)^{-\frac{1}{2}}(2A\log t + B)\,dt + O(x^{\frac{1}{3}+\epsilon})\\
&= (2A\log\sqrt{2}x + B)x - x\int_1^2 \frac{A\log\frac{xu}{\sqrt{2}} + B/2}{\sqrt{u-1}}\,du + O(x^{\frac{1}{3}+\epsilon})\\
&= (2A\log 2)x - Ax\int_0^1\frac{\log(1+u)}{\sqrt{u}}\,du + O(x^{\frac{1}{3}+\epsilon})\\
&= (4-\pi)Ax + O(x^{\frac{1}{3}+\epsilon})
\end{aligned}$$

where

$$B = A + 2\int_1^\infty \frac{E(t)}{t^3}\,dt$$

as in Lemma 3.

Now from (7.19)–(7.22) we conclude that

$$T(x) = \Big(6 - \frac{3\pi}{2}\Big)Ax^2 + O(x^{\frac{3}{2}+\epsilon}). \tag{7.23}$$

8 Completion of Proof of Theorem 1

From (4.1), (5.8), (6.6) and (7.23), we have proved that

$$\begin{aligned}
S(x) &= 4Ax^2\log x + \Big(4\int_1^\infty\frac{E(t)}{t^3}\,dt + 2A + 2A\log 2 - 6A + \frac{3A\pi}{2} - \frac{\pi^2}{16L(2,\chi)}\Big)x^2\\
&\qquad + O(x^{\frac{3}{2}+\epsilon})\\
&= \frac{\pi}{2L(2,\chi)}x^2\log x + \Big(4\int_1^\infty\frac{E(t)}{t^3}\,dt + \frac{\pi^2 - 4\pi + 2\pi\log 2}{8L(2,\chi)}\Big)x^2 + O(x^{\frac{3}{2}+\epsilon}),
\end{aligned}$$

which proves the theorem.

9 A Further Remark

For any given primitive binary quadratic form $f(u,v) = au^2 + buv + cv^2$ of discriminant D, an interesting problem is to study the asymptotic behavior of the sum

$$S_f(\Omega_x) := \sum_{(m,n)\in\Omega_x} d\big(f(m,n)\big),$$

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where Ω_x is a bounded area of dimension 2 with a linear parameter x. (A typical case is a square with side length x). When D is not a perfect square, it is reasonable to guess that an asymptotic formula of form

$$S_f(\Omega_x) = A(D)|\Omega_x| \log x + B(D)|\Omega_x| + o(|\Omega_x|) \tag{9.1}$$

holds (uniformly if Ω is regular). A possible proof for such a result could essentially depend on the study of the uniform distribution of the ratios ν/k in the variable k, where ν runs over the primitive roots of the quadratic polynomial $f(x, 1)$ modulo k. In [5], Hooley deals with such a problem and he has gotten the "uniformity" with certain error term. Thus, except for some minor technical problems, including the computation of the main terms, one could expect a complete proof for (9.1) with a good error estimate.

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