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ON THE POINTS OF INFLECTION OF BESSEL FUNCTIONS OF POSITIVE ORDER, I

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1. Introduction. The primary concern addressed here is the variation with respect to the order $\nu > 0$ of the zeros $j''_{\nu k}$ of fixed rank of the second derivative of the Bessel function $J_\nu(x)$ of the first kind. It is shown that $j''_{\nu 1}$ increases $0 < \nu < \infty$ (Theorem 4.1) and that $j''_{\nu k}$ increases in $0 < \nu \leq 3838$ for fixed $k = 2, 3, \ldots$ (Theorem 10.1).

It is true, as one would expect, that $j''_{\nu k}$ increases throughout the entire interval $0 < \nu < \infty$, also for $k = 2, 3, \ldots$. This result has been achieved by R. Wong and T. Lang [**12**] who applied to (3.1) delicate asymptotic estimates with numerical estimates of the remainder terms to establish monotonicity for $\nu \geq 10$ when $k \geq 2$. For $-1 < \nu \leq 0$ (and hence in combination with the present paper and [**12**] for $-1 < \nu < \infty$) monotonicity for $j''_{\nu k}, k = 3, 4 \ldots$, have also been demonstrated, but by different methods [**6**]. These cover analogous properties for $k = 1, 2$, as well.

The study of variation with respect to $\nu > 0$ is based on a formula for $dj''_{\nu k}/d\nu$, given in Theorem 3.1.

This is analogous to one for $dj_{\nu k}/d\nu$ enunciated by Schläfli [**11**, § 15.6(2), p. 508] and one for $dj'_{\nu k}/d\nu$ due to Schafheitlin [**8**, p. 274] from which monotonicities of $j_{\nu k}$ and $j'_{\nu k}$ follow. As usual, $j_{\nu k}$ and $j'_{\nu k}$ denote the respective k-th positive zeros of $J_\nu(x)$ and $J'_\nu(x)$. Their formulas have been generalized by Watson [**11**, § 15.6(3), p. 508, § 15.6(4), p. 510]. A corresponding extension of the formula for $dc''_{\nu k}/d\nu$ would be desirable, with $c''_{\nu k}$ the k-th positive zero of the second derivative of the general cylinder function $C_\nu(x)$.

2. Some preliminary remarks. Before deriving the formula for $dj''_{\nu k}/d\nu$, some elementary comments are in order. The general cylinder function $C_\nu(x)$ satisfies the differential equations

$$(2.1)\qquad x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

and [**3**, § 7.3(67), p. 13]

$$(2.2)\qquad x^2(x^2 - \nu^2)y''' + x(x^2 - 3\nu^2)y'' + (x^4 - (2\nu^2 + 1)x^2 + \nu^4 - \nu^2)y' = 0.$$

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We need an observation which establishes explicitly what is depicted in graphs of Bessel functions:

LEMMA 2.1. *Each point of inflection of $C_\nu(x)$ for which ${c''_{\nu k}}^2 > \nu^2$, lies in the "second half" of the arch, i.e., between the extremum and the following zero. There is only one point of inflection in each arch when ${c''_{\nu k}}^2 > \nu^2$.*

Proof. Putting $x = c''_{\nu k}$ in (2.1) shows that $C'_\nu(c''_{\nu k})$ and $C_\nu(c''_{\nu\kappa})$ are of opposite sign when ${c''_{\nu k}}^2 > \nu^2$. This is equivalent to our first assertion.

From equation (2.1) it is clear that the second half of each arch must contain an odd number of points of inflection whose abscissae exceed $|\nu|$. From equation (2.2) it follows that the square of the first such abscissa in the arch exceeds the (larger) positive zero of the coefficient of y', i.e.,

$$(2.3) \qquad {c''_{\nu k}}^2 > \nu^2 + \frac{1}{2} + \frac{1}{2}(8\nu^2+1)^{\frac{1}{2}}.$$

However, were there a second point of inflection in the same half arch, equation (2.2) would imply that the square of its abscissa would be less than that zero which is impossible of course. This proves the remaining conclusion of the lemma.

Thus, for $0 < \nu \le 1, j''_{\nu k} > j'_{\nu k} > \nu$, $k = 1, 2, 3, \ldots$. For $\nu > 1$, a new point of inflection comes into being in the first half of the first arch [**11**, § 15.3, pp. 486–487], easily seen to be unique. This causes an awkward renumbering of the $j''_{\nu k}$ so that [**11**, § 15.3(6), p. 487] for $k = 1, 2, \ldots$,

$$(2.4) \qquad j''_{\nu 1} < \sqrt{\nu^2 - 1}, \quad j'_{\nu k} < j''_{\nu k+1} < j_{\nu k}, \quad \nu > 1.$$

Also, as will be used later,

$$(2.5) \qquad j''_{\nu 1} > j'_{\nu+1,k-1}, \ \nu > 1, \ k = 2, 3, \ldots.$$

This follows from the un-numbered formula immediately preceding [**11**, p. 487(6)] which becomes, on putting $x = j''_{\nu k}$,

$$J'_{\nu+1}(j''_{\nu k}) = -\nu\left\{1 - \frac{\nu^2 - 1}{(j''_{\nu k})^2}\right\} J_\nu(j''_{\nu k}).$$

This shows that $J'_{\nu+1}(j''_{\nu k})$ and $J_\nu(j''_{\nu k})$ are of opposite sign for $\nu > 1$, $k = 2, 3, \ldots$, and that

$$(2.6) \qquad j''_{\nu k} > j'_{\nu+1,k}, \ 0 < \nu \le 1, \ k = 1, 2, \ldots.$$

3. The basic formula: a representation for $dj''_{\nu k}/d\nu$. The differentiablility of $j''_{\nu k}$ with respect to ν when $\nu > 0$ and k is fixed can be established by the same argument F. W. J. Olver used [**7**, Lemma 6.1, Theorem 6.3, p. 246] *mutatis mutandis* to verify that $j_{\nu k}$ and $j'_{\nu k}$ are differentiable functions of $\nu > 0$ for fixed k.

Our objective is, then, to obtain an expression for that derivative and later to prove it positive.

THEOREM 3.1. *If $\nu > 0$ and the rank k of $j'' = j''_{\nu k}$ is fixed, then*

$$(3.1)\qquad \frac{dj''}{d\nu} = \frac{2\nu}{(j'')^2 J_\nu(j'') J_\nu{'''}(j'')} \left\{ \int_0^{j''} \frac{J_\nu^2(t)}{t}\, dt - J_\nu^2(j'') \right\}.$$

Proof. By definition, $J_\nu{''}(j'') = 0$ whose derivative with respect to ν (of which j'' is also a function) yields

$$(3.2)\qquad J_\nu'''(j'') \frac{dj''}{d\nu} + \left[\frac{\partial J_\nu''(x)}{\partial \nu} \right]_{x=j''} = 0.$$

We use the well-known formula [**7**, § 6.4, p. 247]:

$$(3.3)\qquad \int \frac{J_\nu^2(x)}{x}\, dx = \frac{x}{2\nu} \left\{ J_\nu(x) \frac{\partial J_\nu'(x)}{\partial \nu} - J_\nu'(x) \frac{\partial J_\nu(x)}{\partial \nu} \right\}.$$

Differentiating both sides of (3.3) with respect to x:

$$(3.4)\qquad \frac{1}{x} J_\nu^2(x) = \frac{1}{2\nu} \left\{ J_\nu(x) \frac{\partial J_\nu'(x)}{\partial \nu} - J_\nu'(x) \frac{\partial J_\nu(x)}{\partial \nu} \right\} + \frac{x}{2\nu} \left\{ J_\nu(x) \frac{\partial J_\nu''(x)}{\partial \nu} - J_\nu''(x) \frac{\partial J_\nu(x)}{\partial \nu} \right\}.$$

We use (3.3) to replace the first right-hand term of (3.4) to yield:

$$(3.5)\qquad \frac{1}{x} J_\nu^2(x) = \frac{1}{x} \int \frac{J_\nu^2(x)}{x}\, dx + \frac{x}{2\nu} \left\{ J_\nu(x) \frac{\partial J_\nu''(x)}{\partial \nu} - J_\nu''(x) \frac{\partial J_\nu(x)}{\partial \nu} \right\}.$$

We now multiply by x to obtain:

$$(3.6)\qquad J_\nu^2(x) - \int \frac{J_\nu^2(x)}{x} dx = \frac{x^2}{2\nu} \left\{ J_\nu(x) \frac{\partial J_\nu''(x)}{\partial \nu} - J_\nu''(x) \frac{\partial J_\nu(x)}{\partial \nu} \right\}.$$

To investigate the behaviour of the right side of (3.6) as $x \longrightarrow 0+$, we use [**7**, § 9.3, p. 57], [**7**, § 5.2(5.06), p. 243]:

$$(3.7)\qquad \begin{cases} J_\nu(x) = (\tfrac{1}{2}x)^\nu \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{4^s s! \Gamma(\nu+s+1)} \\ \frac{\partial J_\nu(x)}{\partial \nu} = (\tfrac{1}{2}x)^\nu \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{4^s s! \Gamma(\nu+s+1)} \left\{ \ln \tfrac{1}{2}x - \psi(\nu+s+1) \right\}, \end{cases}$$

where ψ denotes the logarithmic derivative of the Gamma function.

In what follows, we must take ν positive and distinguish two cases, $\nu \neq 1$ and $\nu = 1$.

CASE 1. $\nu \neq 1, \nu > 0$.

$$\text{(3.8)} \qquad \begin{cases} J_\nu(x) = O(x^\nu),\ J_\nu''(x) = O(x^{\nu-2}), \\ \frac{\partial J_\nu(x)}{\partial \nu} = O(x^\nu \ln x),\ \frac{\partial J_\nu''(x)}{\partial \nu} = O(x^{\nu-2} \ln x), \quad \text{as } x \longrightarrow 0+. \end{cases}$$

CASE 2. $\nu = 1$.

$$\text{(3.9)} \qquad \begin{cases} J_1(x) = O(x),\ J_1''(x) = O(x), \\ \frac{\partial J_1(x)}{\partial \nu} = O(x \ln x),\ \frac{\partial J_1''(x)}{\partial \nu} = O(1/x), \text{ as } x \longrightarrow 0+. \end{cases}$$

Using (3.8) and (3.9) in the right side of (3.6), we have for the right side:

CASE 1. $\nu \neq 1, \nu > 0$.

$$O(x^2)\left\{O(x^\nu)O(x^{\nu-2}\ln x) + O(x^{\nu-2})O(x^\nu \ln x)\right\} = O(x^{2\nu}\ln x) \longrightarrow 0$$

as $x \longrightarrow 0+$.

CASE 2. $\nu = 1$.

$$O(x^2)\left\{O(x)\cdot O(\frac{1}{x}) + O(x)\cdot O(x\ln x)\right\} = O(x^2) + O(x^4 \ln x) \longrightarrow 0$$

as $x \longrightarrow 0+$.

Thus, under the restriction $\nu > 0$, we may write (3.6) as:

$$\text{(3.10)} \qquad J_\nu^2(j'') - \int_0^{j''} \frac{J_\nu^2(t)}{t}dt = \frac{(j'')^2}{2\nu}\cdot J_\nu(j'')\cdot\left[\frac{\partial J_\nu''(x)}{\partial \nu}\right]_{x=j''}.$$

Hence,

$$\text{(3.11)} \qquad \left[\frac{\partial J_\nu''(x)}{\partial \nu}\right]_{x=j''} = \frac{2\nu}{j''^2 J_\nu(j'')}\left\{J_\nu{}^2(j'') - \int_0^{j''}\frac{J_\nu^2(t)}{t}dt\right\}.$$

Substituting (3.11) into (3.2) yields (3.1) and thereby proves the theorem.

THEOREM 3.2. *If* $\nu > 0$, $j'' = j''_{\nu k}$, $k = 2, 3, \ldots$, *and, when* $0 < \nu \leq 1$, *also for* $k = 1$, *then*

$$\text{(3.12)} \qquad \frac{2\nu}{(j'')^2 J_\nu(j'')J_\nu'''(j'')} > 0.$$

Proof. Under the hypotheses, $j''_{\nu k} > \nu$. From Lemma 2.1, $J_\nu''(x)$ changes from negative to positive at $x = j''$ when $J_\nu(j'') > 0$, from positive to negative when $J_\nu(j'') < 0$. Hence $J_\nu'''(j'')$ has the same sign as $J_\nu(j'')$ and our assertion is established.

The remaining case is settled by a similar argument.

THEOREM 3.3. *If* $\nu > 1$, *then*

$$(3.13)\qquad \frac{2\nu}{(j''_{\nu 1})^2 J_\nu(j''_{\nu 1}) J'''_\nu(j''_{\nu 1})} < 0.$$

Proof. Here $0 < j'' < j'_{\nu 1}$ while $J''_\nu(x) > 0$, for sufficiently small $x > 0$, [**11**, § 15.3, p. 486]. Thus, $J'''_\nu(j''_{\nu 1}) < 0$ while $J_\nu(j''_{\nu 1}) > 0$, and this assertion too is proved.

Together, these theorems show that the sign of $dj''/d\nu$ is determined by the sign of

$$(3.14)\qquad G(x) = \int_0^x \frac{J_\nu{}^2(t)}{t}\,dt - J_\nu{}^2(x)$$

when $x = j''_{\nu k}$.

4. The first point of inflection: the case $\nu > 1$**.** What is here the first point of inflection comes into existence in the *first* half of the first arch only when $\nu > 1$. When $0 < \nu \leq 1$, there is no such point in $0 < x < j'_{\nu 1}$. This special j'' can be discussed more easily than the remaining $j''_{\nu k}$ all of which occur in the *second* half of each arch, including the first. We prove

THEOREM 4.1. *The function* $j''_{\nu 1}$ *increases for all* ν, $1 < \nu < \infty$.

Proof. In view of (3.1) and (3.13), this conclusion will follow from the inequality $G(j''_{\nu 1}) < 0$, where $G(x)$ is defined by (3.14) and $G(0) = 0$, $\nu > 0$.

Now,

$$G'(x) = -x\frac{d}{dx}\left\{\frac{J_\nu{}^2(x)}{x}\right\} = -2xz(x)z'(x),$$

where $z(x) = x^{-1/2}J_\nu(x)$.

The function $z(x)$ satisfies the differential equation [**11**, § 4.31 (19), p. 98]

$$(4.1)\qquad x^2z'' + 2xz' + (x^2 - \nu^2 + \frac{1}{4})z = 0.$$

Moreover, $z(0) = 0$, since $\nu > 1$; $z(x) > 0$, $0 < x < j_{\nu 1}$.

The first zero, $a'_{\nu 1}$, of its derivative $z'(x)$ therefore yields a positive maximum of $z(x)$. Putting $x = a'_{\nu 1}$ in (4.1), it follows that

$$(4.2)\qquad a'_{\nu 1} > \sqrt{\nu^2 - \frac{1}{4}},\quad \nu > 1.$$

Obviously, $G'(x) < 0, 0 < x < a'_{\nu 1}$.

From (2.4) and (4.2) we have

$$j''_{\nu k} < \sqrt{\nu^2 - 1} < \sqrt{\nu^2 - \frac{1}{4}} < a'_{\nu 1}, \quad \nu > 1,$$

whence $G'(x) < 0, 0 < x \le j''_{\nu 1}$.

Therefore, $G(j''_{\nu 1}) < 0$, as required to complete the proof since $G(0) = 0$.

5. The other points of inflection: $0 < \nu \le \frac{1}{2}$.

THEOREM 5.1. *The function* $j''_{\nu k}$ *increases in* $0 < \nu \le \frac{1}{2}$, *for each fixed* $k = 1, 2, \ldots$.

We offer two proofs. The first is in the fashion of that of Theorem 4.1. The second serves as an introduction to the method employed in §§6–10.

First Proof. For $0 < \nu \le 1$, an even larger ν-interval, $j''_{\nu k} > j'_{\nu k}$, $k = 1, 2, \ldots$, and Theorem 3.2 implies that this conclusion of this theorem as well follows from $G(j''_{\nu k}) > 0$. We establish this first for $k = 1$; this will yield readily also the cases $k = 2, 3, \ldots$.

For $\nu > 0$, $G(0) = 0$ and so we consider again

$$G'(x) = -x\frac{d}{dx}\left\{\frac{J_\nu{}^2(x)}{x}\right\}.$$

Now, $x^{-\frac{1}{2}}J_\nu(x) = x^{\nu - 1/2}[x^{-\nu}J_\nu(x)]$ and [**11**, §3.2, p. 45]

$$\frac{d}{dx}\frac{J_\nu(x)}{x^\nu} = -\frac{J_{\nu+1}(x)}{x^\nu} < 0, \ 0 < x < j_{\nu+1,1}.$$

Thus, $x^{-\nu}J_\nu(x)$ decreases for $0 < x < j_{\nu 1} < j_{\nu+1,1}$ and so obviously must $x^{-1/2}J_\nu(x)$, hence also $J_\nu^2(x)/x$, for $0 < \nu \le \frac{1}{2}$.

Therefore, $G'(x) > 0, 0 < x < j_{\nu 1}$ so that $G(x)$ increases (from 0) in $0 < x < j_{\nu 1}$. In particular, $G(j''_{\nu 1}) > 0$, establishing the theorem's assertion when $k = 1$; also $G(j'_{\nu 1}) > 0$. Clearly, $G(j''_{\nu k}) > G(j'_{\nu k}), 0 < \nu < \frac{1}{2}, k = 1, 2, \ldots$. If it be shown that $G(j'_{\nu k}) > G(j'_{\nu 1})$, $k = 2, 3, \ldots$, this would complete the proof.

The first term in the definition of $G(j'_{\nu k})$ obviously increases with k. The subtrahend $J_\nu{}^2(j'_{\nu k})$ decreases, as a consequence of the Sonin-Butlewski-Pólya Theorem [**1**; **9**, §7.31, p. 166 fn] (recorded in full following the statement of Theorem 11.1 below), since the Bessel differential equation can be written $(xy')' + x^{-1}(x^2 - \nu^2)y = 0$. Here $j''_{\nu 1} > \nu$, making $x^2 - \nu^2 = x[x^{-1}(x^2 - \nu^2)]$ the increasing product of positive functions in $j''_{\nu 1} \le x < \infty$.

Second Proof. This is based on a formula [**11**, §5.51(5), p. 152] which will be used also for greater values of ν:

$$(5.1) \qquad G(x) = \left[\frac{1}{2\nu} - 1\right] J_\nu{}^2(x) + \frac{1}{\nu} \sum_{m=1}^{\infty} J^2_{\nu+m}(x),$$

whence $G(x) > 0$, $0 < \nu \leq \frac{1}{2}$, for all $x > 0$, proving the theorem.

6. The other points of inflection: $\frac{1}{2} \leq \nu \leq 22.7$. The following lemma is used in the proof of Theorem 6.1.

LEMMA 6.1. *For $\nu \geq \frac{1}{2}$, the function $f(x) = x - \frac{\nu(\nu-1)}{x}$ increases for $\nu < x < \infty$.*

Proof. For $\nu \geq 1$ the conclusion is obvious, even for all $x > 0$. So we consider ν in $\frac{1}{2} \leq \nu < 1$. Now,

$$f'(x) = 1 - \frac{\nu(1-\nu)}{x^2}.$$

This will be positive if $x^2 > \nu - \nu^2$. This is the case here, since $\nu - \nu^2 \leq \frac{1}{4}$, $\frac{1}{2} \leq \nu < 1$, while $x > \frac{1}{2}$.

THEOREM 6.1. *The function $j''_{\nu k}$ increases with ν, $\frac{1}{2} \leq \nu \leq 22.79$, for each fixed $k = 1, 2, \ldots$.*

Proof. Let j''_ν denote any $j''_{\nu k}$ larger than ν. For $\frac{1}{2} \leq \nu \leq 1$, this includes $j''_{\nu 1}$; for $\nu > 1$, j''_ν must be at least $j''_{\nu 2}$. This does not restrict the generality, remembering Theorem 4.1 in the latter cases.

Proof will be accomplished by showing that $G(j''_{\nu k}) > 0$ under the current hypotheses. From (5.1)

$$G(x) > (\frac{1}{2\nu} - 1)J_\nu{}^2(x) + \frac{1}{\nu} J_{\nu+1}{}^2(x) = G_1(x).$$

These terms can be combined when $x = j''_{\nu 1}$ since [**11**, §15.3, p. 487]

$$(6.1) \qquad J_{\nu+1}(j''_\nu) = \left[j''_\nu - \frac{\nu(\nu-1)}{j''_\nu}\right] J_\nu(j''_\nu).$$

Hence,

$$\begin{aligned} G_1(j''_\nu) &= \left[\frac{1}{2\nu} - 1 + \frac{1}{\nu}\left[j''_\nu - \frac{\nu(\nu-1)}{j''_\nu}\right]^2\right] J_\nu{}^2(j''_\nu) \\ &= G_2(j''_\nu) J_\nu{}^2(j''_\nu), \end{aligned}$$

so that $G_1(j_\nu'') > 0$ if $G_2(j_\nu'') > 0$.

We have assumed that $j_\nu'' > \nu$ and so Lemma 6.1 applies with $x = j_\nu''$, i.e., $G_2(j_\nu'') > G_2(t)$ if $t < j_\nu''$. It is recorded in (2.4) and (2.6) for $\nu > 1$ and $0 < \nu \leq 1$, respectively, that $j_\nu'' > j_{\nu+1,1}'$. Now, [**11**, § 15.3(8), p. 487],

$$(6.2) \qquad j_{\nu+1,1}' > \sqrt{(\nu+1)(\nu+3)}, \quad \nu > 0.$$

Hence,

$$G_2(j_\nu'') > G_2(\sqrt{(\nu+1)(\nu+3)}) = \frac{-2\nu^3+43\nu^2+58\nu+21}{2\nu(\nu+1)(\nu+3)} > 0,$$

$0 \leq \nu \leq 22.79$.

This establishes the theorem which, together with previous theorems, demonstrates the increasing character of $j_{\nu k}''$ for $0 < \nu \leq 22.79$, $k = 1, 2, \ldots$.

7. The other points of inflection: $3 < \nu \leq 32.8$. When $\nu > 3$, the inequality (6.2) can be sharpened [**11**, § 15.3(9), p. 487] to

$$(7.1) \qquad j_{\nu+1,1}' > \sqrt{(\nu+1)(\nu+4)}, \quad \nu > 3.$$

Now that monotonicity of $j_{\nu k}''$ has been verified for ν even larger, the proof of Theorem 6.1 can be modified so as to extend the result yet further.

Using (7.1) where (6.2) had been employed, we have

$$G_2(j_\nu'') > G_2(\sqrt{(\nu+1)(\nu+4)}) = \frac{-2\nu^3+63\nu^2+93\nu+36}{2\nu(\nu+1)(\nu+3)} > 0,$$

$3 \leq \nu \leq 32.8$.

In sum, we have

THEOREM 7.1. *The function $j_{\nu k}''$ increases with ν, $0 < \nu \leq 32.8$, $k = 1, 2, \ldots$. When $\nu > 1$, $j_{\nu 1}''$ increases for $1 < \nu < \infty$.*

8. The other points of inflection: $3 < \nu \leq 151.03$. It is possible to proceed further by this method, although not to infinity with ν, since $G_2(j_\nu'') \longrightarrow -1$ as $\nu \longrightarrow \infty$, for $j_\nu'' > \nu$. This follows from (2.4) together with Tricomi's asymptotic formulas for $j_{\nu k}$ and $j_{\nu k}'$, k fixed, $\nu \longrightarrow \infty$ (cf. [**3**], §7.9, p. 60]). Indeed, these asymptotics show that adding a *fixed* finite number of additional terms from (5.1) cannot establish the monotonicity of $j_{\nu k}''$ for all ν (k fixed) since any such section will become negative for sufficiently large ν.

Monotonicity of $j'' = j_{\nu k}''$, for fixed $k = 2, 3, \ldots$, can be established for larger ν-intervals by using additional terms of the infinite series of positive terms in (5.1) defining $G(x)$ and also by employing more precise lower bounds for j''. In this section we introduce one additional term of (5.1).

This term can be combined with the previous ones thanks to a standard recursion formula [**11**, § 3.2(4), p. 45], with $x = j''$:

$$j'' J_{\nu+2}(j'') = (\nu+1) J_{\nu+1}(j'') - j'' J'_{\nu+1}(j'').$$

The first term on the right has been represented already in (6.1) in terms of $J_\nu(j''_\nu)$. Moreover [**11**, § 15.3, p. 487]

$$J'_{\nu+1}(j'') = -\nu\left(1 - \frac{\nu^2-1}{j''^2}\right) J_\nu(j'').$$

Therefore,

$$J_{\nu+2}(j'') = \left\{2\nu + 1 - \frac{2\nu(\nu^2-1)}{j''^2}\right\} J_\nu(j''). \tag{8.1}$$

The factor in braces is positive, since $j'' > \nu$, and, like $G_2(j'')$, will clearly be diminished if j'' is replaced by a lower estimate.

Thus,

$$G(j'') > \left[\frac{1}{2\nu} - 1 + \frac{1}{\nu}\left[j'' - \frac{\nu(\nu-1)}{j''}\right]^2 + \frac{1}{\nu}\left\{2\nu + 1 - \frac{2\nu(\nu^2-1)}{(j'')^2}\right\}^2\right] J_\nu^2(j'') = G_3(j'') J_\nu^2(j'').$$

$G_3(j'')$ is made smaller when j'' is replaced by a smaller value. Keeping this in mind, it is convenient to rewrite $G_3(j'')$ as

$$G_3(j'') = \frac{3}{2\nu} + \frac{j''^2}{\nu} + 2\nu + 5 - \frac{(7\nu^2 + 13\nu + 4)(\nu - 1)}{(j'')^2} + \frac{4\nu(\nu^2-1)^2}{(j'')^4}. \tag{8.2}$$

We extend the range of ν for which $G(j'') > 0$ by determining the ν-interval for which $G_3(j'') > 0$.

The addition made of one more term of the infinite series in (5.1) permits a substantial extension of the ν-interval covered in Theorem 7.1 even when the same lower bound (7.1) is used for $j''_\nu > j'_{\nu+1,1}$. Hence, for $\nu > 3$,

$$\begin{aligned} G_3(j'') &> G_3\left(\sqrt{(\nu+1)(\nu+4)}\right) \\ &= \frac{-\nu^5 + 147.5\nu^4 + 530\nu^3 + 689.5\nu^2 + 396\nu + 88}{\nu(\nu^2+5\nu+4)^2} \\ &> 0, \end{aligned}$$

$3 < \nu \leq 151.03$.

This yields a substantial extension of Theorem 7.1:

THEOREM 8.1. *The function $j''_{\nu k}$ increases with ν for fixed $k = 1, 2, \ldots, 0 < \nu \leq 151.03$. When $\nu > 1$, $j''_{\nu 1}$ increases for $1 < \nu < \infty$.*

9. The other points of inflection; the effect of using a better bound for j''_ν

Preliminaries. To establish the monotonicity of $j''_{\nu k}$, $k = 2, 3, \ldots$, for considerably larger ν, we rely upon some deeper inequalities for j''_ν which will be used in conjunction with (8.2).

Together with (2.5), we shall use now instead of the elementary bound (7.1), a sharper one due to L. Gatteschi and A. Laforgia [**4**, p. 422(27)]:

$$(9.1) \qquad j'_{\nu 1} > \nu \exp(2^{-1/3} a'_1 \nu^{-2/3} - 1.06 \nu^{-4/3}), \qquad \nu \geq 20,$$

where $a'_1 = 1.01879297$.

Needed also in this connection is a result obtain by A. Elbert and A. Laforgia [**2**, Corollary 4.1, p. 778]:

$$(9.2) \qquad j'_{\nu+1,k} > \nu + j'_{\mu+1,k} - \mu,$$

where $k = 1, 2, \ldots$, and $\nu > \mu$.

The inequality (9.1) will be used with ν replaced by $\nu + 1$ throughout, in view of (2.5), and with μ constant in each separate application, giving (9.2) the structure $j''_{\nu k} > \nu + \alpha_\mu$. In the actual calculations we use values slightly smaller than the ones specified in (9.1), thereby obtaining slightly smaller lower bounds for $j'_{\nu 1}$. We replace $2^{-1/3} a'_1$ by 0.80861647, the exponents $\frac{2}{3}$ and $\frac{4}{3}$ by 0.666666666667 and 1.333333333333, and then use the first five decimal places of the result.

10. Monotonicity for $0 < \nu \leq 3838$; $k = 2, 3, \ldots$.

For this range monotonicity will be established in stages. From (9.2) and (2.5) it follows, for $k = 2, 3, \ldots$, that

$$(10.1) \quad j'' = j''_{\nu k} > \nu + \alpha_\mu, \quad \nu > \mu,$$

where the constant $\alpha_\mu = j'_{\mu+1,1} - \mu$, μ being kept fixed. As noted, $G_3(j'')$, whose positivity for appropriate ν we seek to establish, decreases if j'' is replaced by a smaller quantity. Hence, $G_3(j'') > G_3(\nu + \alpha_\mu)$.

Thus, $G_3(j'')$ will be positive at least for those ν for which $G_3(\nu + \alpha_\mu) > 0$. To ascertain these values, we replace j'' by $\nu + \alpha_\mu$ in (8.2) and rewrite the function. This gives

$$(10.2) \quad \begin{cases} \nu(\nu + \alpha_\mu)^4 G_3(j'') > \nu(\nu + \alpha_\mu)^4 G_3(\nu + \alpha_\mu) \\ \quad = -\nu^5 + (20\alpha_\mu{}^2 + 8\alpha_\mu + \frac{5}{2})\nu^4 \\ \qquad + (28\alpha_\mu{}^3 + 24\alpha_\mu{}^2 + 24\alpha_\mu + 4)\nu^3 \\ \qquad + (17\alpha_\mu{}^4 + 20\alpha_\mu{}^3 + 18\alpha_\mu{}^2 + 8\alpha_\mu + 4)\nu^2 \\ \qquad + (6\alpha_\mu{}^5 + 5\alpha_\mu{}^4 + 6\alpha_\mu{}^3 + 4\alpha_\mu{}^2)\nu + \alpha_\mu{}^6 + \frac{3}{2}\alpha_\mu{}^4. \end{cases}$$

This is a polynomial in ν, positive at $\nu = 0$, becoming eventually negative. It has but one variation in sign and so has exactly one positive zero in view of Descartes' Rule of Signs. It is therefore positive for all positive values smaller than the root.

We know already (Theorem 8.1) that $G_3(j''_{\nu k}) > 0$, $0 < \nu \leq 151.03$, $k = 2, 3, \ldots$, and so we start our calculations with $\mu = 151$. Here, for $\nu > 151$,

$$\begin{aligned} j''_{\nu k} &> j'_{\nu+1,1} \\ &> \nu + j'_{152,1} - 151 \\ &> \nu + 5.17306 \\ &= \nu + \alpha_{151} \end{aligned}$$

since $j'_{152,1} > 156.173063272$ from (9.1).

Putting in (10.2) $\alpha_\mu = 5.17306$, it turns out that the resulting polynomial is positive for $0 \leq \nu \leq 587.05$.

Now we begin again, this time with $\mu = 587$ so that, for $\nu > 587$,

$$\begin{aligned} j''_{\nu k} &> j'_{\nu+1,1} \\ &> \nu + j'_{588,1} - 587 \\ &> \nu + 7.68555 \\ &= \nu + \alpha_{587}, \end{aligned}$$

again from (9.1). With $\alpha_\mu = 7.68555$, the polynomial (10.2) is positive for $0 \leq \nu \leq 1256.7$.

The next value of μ is 1256. The same procedure leads to $\alpha_{1256} = 9.65825$ and thence to a polynomial (10.2) which is positive for $0 \leq \nu \leq 1959.5$.

The method is now clear and is summarized in the Table below. Column 1 records the value of μ, Column 2 of α_μ. Column 3 lists a value of ν (slightly smaller than the largest, due to numerical caution) for which the polynomial (10.2) generated by the corresponding μ remains positive when five decimal places are used from the entry in Column 2.

Together with earlier theorems, the entries in this Table imply the following result:

THEOREM 10.1. *The function $j''_{\nu k}$ is an increasing function of ν, for fixed $k = 1, 2, 3, \ldots$, in the interval $0 < \nu \leq 3838$. For $1 < \nu < \infty$, $j''_{\nu 1}$ is an increasing function of ν.*

It will be noted that, although α_μ increases with μ (as established in [**2**]), the ν-intervals lengthen more slowly as μ increases. This is in conformity with the

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basic result of [**2**] which establishes that $j'_{\nu k}$ is a concave function of ν for each fixed k.

Column 1	Column 2	Column 3
151	5.173063272	578.05
587	7.685556889	1256.7
1256	9.65825044	1959.5
1959	11.06058306	2553.8
2553	11.99913859	2995.5
2995	12.60597476	3299.8
3299	12.98925528	3499.6
3499	13.22867726	3627.4
3627	13.37713598	3707.8
3707	13.46815011	3757.5
3757	13.52436834	3788.4
3788	13.55897268	3807.5
3807	13.58008824	3819.1
3819	13.59338808	3826.5
3826	13.60113339	3830.8
3830.5	13.60610754	3833.55
3833.5	13.60942145	3835.39
3835.3	13.61140896	3836.48
3836.4	13.61262326	3837.16
3837.15	13.61345106	3837.62
3837.6	13.61394765	3837.89
3837.88	13.61425664	3838.06
3838.06	13.61445529	3838.17

11. Ordinates of the points of inflection. Monotonicity problems arise also for the ordinates $J_\nu(j''_{\nu k})$ of the points of inflection. Just as $j''_{\nu k}$ appears to have properties analogous to those of $j'_{\nu k}$, so too conjectures for $J^2_\nu(j''_{\nu k})$ can be motivated by such results as the complete monotonicity of $\{J^2_\nu(j'_{\nu k})\}$ $k = 1, 2, 3, \ldots$, $\nu \geq 0$ fixed [**5**, Theorem 7.2].

Here ν is kept constant and we consider the resulting sequence arising from $k = 1, 2, \ldots$, apparently reversing the approach of the previous sections in which ν varied. However, the formula for $dj''/d\nu$ established in Theorem 3.1 reveals a potentially useful interplay. From that formula it is clear that if the sequence $\{J^2_\nu(j''_{\nu k}), k = 2, 3, \ldots$, decreases for fixed $\nu > 0$, then $\{dj''_{\nu k}/d\nu\}$ would increase, $k = 2, 3, \ldots$ for fixed $\nu > 0$.

If this monotonicity of $\{dj''_{\nu k}/d\nu\}$, $k = 2, 3, \ldots$, could be established, then proving that $dj''_{\nu 2}/d\nu > 0$, $0 < \nu < \infty$ (indeed, $3838 < \nu < \infty$ would suffice), as R. Wong and T. Lang have shown otherwise for $10 < \nu < \infty$ [**12**], would demonstrate that $dj''_{\nu 2}/d\nu > 0$, for all $k \geq 2$. Unfortunately, we have not been able to prove the indicated conjecture, but only to establish a partial result.

THEOREM 11.1. *If* $j''_{\nu\lambda} \geq 2^{1/2}\nu > 0$, *then*
(i) $\{|J'_\nu(j''_{\nu k})|\}$ *and*
(ii) $\{|J_\nu(j''_{\nu k})|\}$ *are decreasing sequences for* $k = \lambda, \lambda+1, \lambda+2, \ldots$, *with* ν *constant. When* $0 < \nu \leq \nu_0 = 7.61690139\ldots$, *the sequences (i) and (ii) both decrease for* $k = 2, 3, \ldots$, *and, when* $0 < \nu \leqq 1$, *for* $k = 1, 2, \ldots$.

The proof depends principally on the Sonin-Butlewski-Pólya Theorem [**1**; **9**, §7.31, p. 166 fn]. This states:

Given the differential equation $(g(x)y')' + f(x)y = 0$, $g(x) > 0$, $f(x) > 0$, $g'(x)$, $f'(x)$ *both continuous, and suppose that the product* $g(x)f(x)$ *increases [decreases] in* $a < x < b \leq \infty$. *Then the relative maxima of* $|y(x)|$ *form a decreasing [increasing] sequence in* $a < x < b$.

Proof of Theorem 11.1. The differential equation (2.2) can be written, with $y' = z$ and $p_\nu(x) = x^4 - (2\nu^2+1)x^2 + \nu^4 - \nu^2$, as

$$x^2(x^2-\nu^2)z'' + x(x^2-3\nu^2)z' + p_\nu(x)z = 0, \tag{11.1}$$

and is satisfied by $z = J'_\nu(x)$. It assumes the form

$$[g(x)z'(x)]' + f(x)z(x) = 0$$

when $g(x) = x^3(x^2-\nu^2)^{-1}$ and $f(x) = x(x^2-\nu^2)^{-1}p_\nu(x)$.

Clearly, $g(x) > 0$ for $x > \nu > 0$. To use the Sonin-Butlewski-Pólya Theorem, we need to know that $f(x) > 0$ for $x \geq j''_{\nu k} > \nu$ when j''_ν is chosen greater than ν and that $f(x)g(x)$ increases for $x \geq j''_\nu$.

For $0 < \nu \leq 1$, and $j''_\nu = j''_{\nu 1}$, it follows from (11.1) that $p_\nu(j''_{\nu 1}) > 0$, since here $j''_{\nu 1} > j'_{\nu 1} > \nu$.

But for these ν, the graph of $p_\nu(x)$ crosses the positive x-axis only once and so $p_\nu(x) > 0$, $x \geq j''_{\nu 1}$. Therefore, $f(x) > 0$, $x \geq j''_{\nu 1}$, $0 < \nu \leq 1$.

For $\nu > 1$, we may take $j''_\nu = j''_{\nu 2}(> \nu)$. Here $J'''_\nu(j''_{\nu 2}) > 0$, $J'_\nu(j''_{\nu 2}) < 0$. Hence $p_\nu(j''_{\nu 2}) > 0$, from (11.1).

For these ν, the graph of $p_\nu(x)$ crosses the positive x-axis twice, first when

$$x^2 = \nu^2 + \frac{1}{2} - \frac{1}{2}(8\nu^2+1)^{1/2} < \nu^2$$

and again when

$$x^2 = \nu^2 + \frac{1}{2} - \frac{1}{2}(8\nu^2+1)^{1/2} > \nu^2.$$

Since $j''_{\nu 2} > \nu$, it follows that $j''_{\nu 2}$ is found after the second crossing so that $p_\nu(x) > 0$, $x \geq j''_{\nu 2}$, $\nu > 1$.

Thus, in both cases, $p_\nu(x) > 0$, $x \geq j''_\nu$, $0 < \nu < \infty$, as required.

To check the monotonicity of $f(x)g(x)$, we seek where

$$\begin{aligned} h_\nu(x) &= \frac{1}{2}x^{-3}(x^2-\nu^2)^4 \frac{d}{dx}\{f(x)g(x)\} \\ &= x^6 - 4\nu^2x^4 + \nu^2(5\nu^2+4)x^2 - 2\nu^4(\nu^2-1) \end{aligned}$$

is positive.

Now,

$$h'_\nu(x) = 2x[3x^4 - 8\nu^2x^2 + \nu^2(5\nu^2+4)].$$

This can vanish only where

$$3x^2 = 4\nu^2 \pm \nu(\nu^2-12)^{1/2}.$$

Hence, $h_\nu(x) > 0$, $x > \nu$, for $\nu^2 \leq 12$, since $h_\nu(\nu) = 6\nu^2 > 0$, and $h_\nu(x)$ increases for $x > 0$ for these ν.

The Sonin-Butlewski-Pólya Theorem therefore applies when $\nu^2 \leq 12$; it establishes the assertions of the final sentence of our theorem for the case $\nu^2 > 12$.

For $\nu^2 > 12$, $x > 0$, $h_\nu(x)$ has a unique local minimum, say at $x = \mu_\nu$, when $3x^2 = 4\nu^2 + \nu(\nu^2-12)^{1/2}$ Now,

$$27\nu^{-3}h_\nu(\mu_\nu) = -2[\nu^3 - 99\nu + (\nu^2-12)^{3/2}],$$

a quantity which vanishes for $\nu = \nu_0$ and is positive for $2(3^{1/2}) < \nu < \nu_0$.

Thus, $h_\nu(x) > 0$, $x \geqq \nu$, $0 < \nu < \nu_0$, so that the final sentence of the theorem is established in respect to sequence (i).

The final assertion regarding sequence (i) follows from the observations that $\mu_\nu < 2^{1/2}\nu$ and that $h_\nu(2^{1/2}\nu) = 10\nu^4 > 0$, $\nu^2 > 12$ so that $h_\nu(x) > 0$, $x \geq 2^{1/2}\nu$, verifying the increasing character of $f(x)g(x)$ for $x \geq 2^{1/2}\nu$, $\nu^2 > 12$.

There remain only the assertions about the sequence (ii). These follow from the properties of sequence (i) since the Bessel differential equation (2.1) implies

$$J_\nu(j''_{\nu k}) = -\frac{j''_{\nu k}}{(j''_{\nu k})^2 - \nu^2} J'_\nu(j''_{\nu k}).$$

Remembering that $j''_{\nu k} > \nu$, we note that $[j''_{\nu k}/({j''_{\nu k}}^2 - \nu^2)]^2$ decreases as k increases, with ν fixed. Thus $\{{J_\nu}^2(j''_{\nu k})\}$ and hence $\{|J_\nu(j''_{\nu k})|\}$ decrease as the indicated k increases, as a consequence of what has been demonstrated concerning sequence (i).

Remarks.

1. This proof, based on differential equations as it is, covers also the general solution of (2.1) $C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha, 0 \le \alpha < \pi, \nu > 0$, beginning with zeros, $c''_{\nu k}$ of $C''_\nu(x)$ larger than ν.

2. The value λ of k with which the stated monotonicites have been shown to commence depends of course on ν. But it appears to grow rather slowly with ν. For example, $\lambda \le 3$ when $\nu = 30$.

3. Numerical evidence suggests that the sequence (ii), but not sequence (i), begins to decrease already from $k = 1$. When $\nu = \frac{3}{2}$, it turns out that $|J'_\nu(j''_{\nu 1})| < |J'_\nu(j''_{\nu 2})| > |J'_\nu(j''_{\nu 3})| > \cdots > |J'_\nu(j''_{\nu k})| > \cdots$, so that some restriction of the sort imposed in the theorem is not entirely superfluous.

4. The monotonicity of the sequence $\{|C'_\nu(c''_{\nu k})|\}$, or equivalently, that of $\{[C'_\nu(c''_{\nu k})]^2\}$, $k = \lambda, \lambda+2, \ldots, c''_{\nu k} > \nu > 0$, as discussed in Theorem 11.1 (i) and in Remark 1 above, is partially a special case of a theorem due to J. Vosmanský [**10**, Theorem 4.2, p. 63]. He has shown that the sequence $\{[C'_\nu(c''_{\nu k})]^2\}$, is monotonic of order n provided the first terms are dropped so as to begin the sequence with $k = q(n)$. The quantity $q(n)$ is defined to be the least integer q for which

$$c''_{\nu q} > \gamma_{n+2}(\alpha) = \max\left\{\nu\alpha_{n+2}, \left[\nu^2 + \frac{1}{2} + (2\nu^2 + \frac{1}{4})^{1/2}\right]^{1/2}\right\},$$

where α_n is the unique zero in $1 < s < \infty$ of

$$G_n(s) = 3 - \left[\frac{s}{s-1}\right]^{n+1} - \left[\frac{s}{s+1}\right]^{n+1}.$$

He showed also [**10**, Corollary 4.1, p. 63], under the same restriction $k \ge q(n)$, that the following sequences are also monotonic of order n:

$$\{\Delta(c''_{\nu k})^\alpha\}, \quad 0 < \alpha \le 1$$
$$\{(c''_{\nu k})^\alpha\}, \quad \alpha < 0$$
and $$\{\ln(c''_{\nu k+1}/(c''_{\nu k})\}.$$

We thank a Referee for reminding us of these results.

A sequence $\{a_k\}, k = 0, 1, \ldots$, is said to be monotonic of order n if $(-1)^i\Delta^i a_k \ge 0$, $i = 0, 1, \ldots, n$; $k = 0, 1, 2, \ldots$, where $\Delta^0 a_k = a_k$, $\Delta^{i+1} a_k = \Delta^i a_{k+1} - \Delta^i a_k$. In [**10**] strict inequality is established. We hazard the guess that these sequences are even completely monotonic, i.e., that they are monotonic of order n for all positive integers n, beginning perhaps already with $k = 2$.

Simple decrease of a sequence of non-negative terms corresponds to monotonicity of order 1. In this case, Vosmanský's condition becomes $c''_{\nu k} > \gamma_3(\alpha) \geq \alpha_3 \nu$, where $\alpha_3 = 4.830752$ [**10**, p. 58] is the unique zero in $1 < s < \infty$ of

$$G_3(s) = 3 - \left[\frac{s}{s-1}\right]^4 - \left[\frac{s}{s+1}\right]^4.$$

Vosmanský's result, while including as a special case that the sequence $\{|C'_\nu(c''_{\nu k})|\}$ decreases from a certain k on, requires that $c''_{\nu k}$ exceed (at least) $\alpha_3\nu$, whereas in Theorem 11.1 (i) and Remark 1 it suffices to have $c''_{\nu k} \geq 2^{1/2}\nu$.

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