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Λ_s-NONUNIFORM MULTIRESOLUTION ANALYSIS

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Abstract

Gabardo and Nashed [‘Nonuniform multiresolution analyses and spectral pairs’, *J. Funct. Anal.* **158**(1) (1998), 209–241] have introduced the concept of nonuniform multiresolution analysis (NUMRA), based on the theory of spectral pairs, in which the associated translated set $\Lambda = \{0, r/N\} + 2\mathbb{Z}$ is not necessarily a discrete subgroup of $\mathbb{R}$, and the translation factor is 2N. Here r is an odd integer with $1 \leq r \leq 2N-1$ such that r and N are relatively prime. The nonuniform wavelets associated with NUMRA can be used in signal processing, sampling theory, speech recognition and various other areas, where instead of integer shifts nonuniform shifts are needed. In order to further generalize this useful NUMRA, we consider the set $\widetilde{\Lambda} = \{0, r_1/N, r_2/N, \ldots, r_q/N\} + s\mathbb{Z}$, where s is an even integer, $q \in \mathbb{N}$, r_i is an integer such that $1 \leq r_i \leq sN-1$, $(r_i, N) = 1$ for all i and $N \geq 2$. In this paper, we prove that the concept of NUMRA with the translation set $\widetilde{\Lambda}$ is possible only if $\widetilde{\Lambda}$ is of the form $\{0, r/N\} + s\mathbb{Z}$. Next we introduce Λ_s-nonuniform multiresolution analysis (Λ_s-NUMRA) for which the translation set is $\Lambda_s = \{0, r/N\} + s\mathbb{Z}$ and the dilation factor is sN, where s is an even integer. Also, we characterize the scaling functions associated with Λ_s-NUMRA and we give necessary and sufficient conditions for wavelet filters associated with Λ_s-NUMRA.

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1. Introduction

A multiresolution analysis, with dilation factor 2 [2, 3, 7], is an increasing sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ along with a $\phi \in V_0$ satisfying: $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0, $\cap_{j\in\mathbb{Z}} V_j = \{0\}$, $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$. The function ϕ is called a scaling function. Multiresolution analysis (MRA) is an important tool, which was introduced by Mallat and Meyer, in constructing a wavelet $\psi \in L^2(\mathbb{R})$ such that the collection $\{2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. The concept of MRA has been extended in several

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ways in past years, like generalizing to $L^2(\mathbb{R}^n)$, allowing the subspaces of MRA to be generated by a Riesz basis instead of an orthonormal basis, admitting a finite number of scaling functions and replacing the dilation factor of two by an integer $N \geq 2$. All these concepts were developed such that the translation set is always a subgroup of $\mathbb{R}$. Gabardo and Nashed [4] considered a generalization of Mallat's classical theory of multiresolution analysis to nonuniform multiresolution analysis (NUMRA) based on the theory of spectral pairs, in which the associated translated set $\Lambda = \{0, (r/N)\} + 2\mathbb{Z}$ is not necessarily a discrete subgroup of $\mathbb{R}$. Here r is an odd integer with $1 \leq r \leq 2N - 1$ such that r and N are relatively prime.

DEFINITION 1.1. Let $N \in \mathbb{N}$ and r be an odd integer relatively prime to N such that $1 \leq r \leq 2N - 1$ and $\Lambda = \{0, (r/N)\} + 2\mathbb{Z}$. An associated nonuniform multiresolution analysis (NUMRA) is a collection $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following conditions:

(i) there exists a $\phi \in V_0$, called a scaling function, such that $\{\phi(\cdot - \lambda) : \lambda \in \Lambda\}$ is an orthonormal basis for V_0;
(ii) $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$;
(iii) $f \in V_j$ if and only if $f(2N\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(iv) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;
(v) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.

Note that if $N = 1$, one obtains the standard definition of multiresolution analysis with dyadic dilation 2. In [5], Gabardo and Yu have constructed wavelets associated with nonuniform multiresolution analysis; these wavelets are called nonuniform wavelets. The main activity in signal processing is that of recovering signals from the samples, but the traditional methods assume that the samples are of uniform spacing; recently, much research has been dedicated to the case of nonuniform spacing of the data samples. The problem of signal reconstruction [4, 9] from nonuniformly sampled data arises in many applications, including sampling systems with sampling jitter, the design of irregularly spaced antenna arrays, reconstruction of signals from noisy samples and the processing of geophysical data. Some fundamental results for nonuniform wavelets and wavelet sets related to spectral pairs can be found in [9]. The nonuniform wavelet construction from the NUMRA could be used in signal processing, sampling theory, speech recognition and various other areas, where nonuniform translations are needed. This motivates us to consider the nonuniform translation set $\widetilde{\Lambda} = \{0, r_1/N, r_2/N, \ldots, r_q/N\} + s\mathbb{Z}$, where s is an even integer, $q \in \mathbb{N}$, r_i is an integer such that $1 \leq r_i \leq sN - 1$ and $N \geq 2$ instead of $\Lambda = \{0, (r/N)\} + 2\mathbb{Z}$. Now one can ask, is it possible to define a NUMRA with $\widetilde{\Lambda}$ as a translation set? In this paper we prove that the concept of nonuniform multiresolution analysis with the translation set $\widetilde{\Lambda}$ can be defined only if $\widetilde{\Lambda}$ is of the form $\{0, (r/N)\} + s\mathbb{Z}$. Hence, we replace the translation set $\Lambda = \{0, (r/N)\} + 2\mathbb{Z}$ by $\Lambda_s = \{0, (r/N)\} + s\mathbb{Z}$ and the dilation factor $2N$ by sN, where s is a positive even integer and r is an odd integer, to introduce Λ_s-nonuniform multiresolution analysis with dilation factor sN. For a given $\phi \in L^2(\mathbb{R})$,

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we define

$$V_j = \begin{cases} \overline{\text{span}}\{\phi(\cdot - \lambda) : \lambda \in \Lambda\} & \text{if } j = 0, \\ \{f : f((2N)^{-j} \cdot) \in V_0\} & \text{if } j \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{1-1}$$

We say that a function $\phi \in L^2(\mathbb{R})$ generates a NUMRA if the spaces in (1-1) together with ϕ form a NUMRA. Recently, in [7], the authors have characterized the scaling functions that generate a NUMRA. This also motivates us to look for a characterization for scaling functions associated with Λ_s-nonuniform multiresolution analysis. This characterization is useful for the construction of wavelets associated with a Λ_s-NUMRA.

2. NUMRA with translation set Λ_s

In this section, we consider a set $\widetilde{\Lambda} = \{r_0/N, r_1/N, r_2/N, \ldots, r_q/N\} + s\mathbb{Z}$, where $r_0 = 0$, $q, r_i, N \in \mathbb{N}, (r_i, N) = 1, i = 1, 2, \ldots, q$ and $s \in 2\mathbb{N}$. We prove that the nonuniform multiresolution analysis with the translation set $\widetilde{\Lambda}$ can be defined only if $\widetilde{\Lambda}$ is of the form $\{0, (r/N)\} + s\mathbb{Z}$, where $1 \leq r \leq sN - 1$ is an odd integer such that r and N are relatively prime. We denote the dilation and translation operators on $L^2(\mathbb{R})$ by

$$(D^j f)(x) = (sN)^{j/2} f((sN)^j x) \quad \text{and} \quad (T_\lambda f)(x) = f(x - \lambda),$$

respectively. For $\phi \in L^2(\mathbb{R})$, the following theorem gives a characterization for the translation system $\{T_\lambda \phi : \lambda \in \widetilde{\Lambda}\}$ to be an orthonormal system in $L^2(\mathbb{R})$.

THEOREM 2.1. *For $\phi \in L^2(\mathbb{R}), \{T_\lambda \phi : \lambda \in \widetilde{\Lambda}\}$ is an orthonormal system if and only if*

$$\sum_{p=0}^{sN-1} e^{2\pi i(p/s)((r_i - r_j)/N)} w\Big(\xi + \frac{p}{s}\Big) = s\delta_{ij} \quad a.e.\ \xi \in \mathbb{R},$$

where $w(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + kN)|^2$, $i, j \in \{0, 1, 2, \ldots, q\}$.

PROOF. Suppose that $\{T_\lambda \phi : \lambda \in \widetilde{\Lambda}\}$ is an orthonormal system. Then

$$\begin{aligned} \delta_{\lambda\sigma} &= \langle T_\lambda \phi, T_\sigma \phi \rangle \\ &= \langle \widehat{T_\lambda \phi}, \widehat{T_\sigma \phi} \rangle \\ &= \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2 e^{-2\pi i \xi(\lambda - \sigma)}\, d\xi. \end{aligned}$$

Suppose that $\lambda = (r_j/N) + sm$ and $\sigma = (r_i/N) + sn$, $i, j \in \{0, 1, \ldots, q\}$, $m, n \in \mathbb{Z}$. Then

$$\begin{aligned} \delta_{ij}\delta_{mn} &= \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2 e^{-2\pi i \xi(sm + (r_j/N))} e^{2\pi i \xi(sn + (r_i/N))}\, d\xi \\ &= \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2 e^{-2\pi i \xi s(m-n)} e^{-2\pi i \xi((r_j - r_i)/N)}\, d\xi \end{aligned}$$

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$$= \sum_{k\in\mathbb{Z}} \int_{kN}^{(k+1)N} |\widehat{\phi}(\xi)|^2 e^{-2\pi i\xi s(m-n)} e^{-2\pi i\xi((r_j-r_i)/N)}\, d\xi$$

$$= \int_0^N \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\eta+kN)|^2 e^{-2\pi i\eta s(m-n)} e^{-2\pi i\eta((r_j-r_i)/N)}\, d\eta$$

$$= \sum_{p=0}^{sN-1} \int_{p/s}^{(p/s)+(1/s)} w(\eta) e^{-2\pi i\eta s(m-n)} e^{-2\pi i\eta((r_j-r_i)/N)}\, d\eta$$

$$= \int_0^{1/s} \sum_{p=0}^{sN-1} w\Big(\xi+\frac{p}{s}\Big) e^{-2\pi i\xi s(m-n)} e^{-2\pi i(\xi+(p/s))((r_j-r_i)/N)}\, d\xi$$

$$= \int_0^{1/s} e^{-2\pi i\xi((r_j-r_i)/N)} \sum_{p=0}^{sN-1} e^{-2\pi i(p/s)((r_j-r_i)/N)} w\Big(\xi+\frac{p}{s}\Big) e^{-2\pi i\xi s(m-n)}\, d\xi.$$

Since $\{\sqrt{s}e^{-2\pi i\xi sl} : l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([0, 1/s))$,

$$e^{-2\pi i\xi((r_j-r_i)/N)} \sum_{p=0}^{sN-1} e^{-2\pi i(p/s)((r_j-r_i)/N)} w\Big(\xi+\frac{p}{s}\Big) = s\delta_{ij} \quad \text{a.e. } \xi \in \mathbb{R}.$$

Thus,

$$\sum_{p=0}^{sN-1} e^{-2\pi i(p/s)((r_j-r_i)/N)} w\Big(\xi+\frac{p}{s}\Big) = s\delta_{ij} \quad \text{a.e. } \xi \in \mathbb{R}$$

and so

$$\sum_{p=0}^{sN-1} e^{2\pi i(p/s)((r_i-r_j)/N)} w\Big(\xi+\frac{p}{s}\Big) = s\delta_{ij} \quad \text{a.e. } \xi \in \mathbb{R}.$$

The converse of the proof follows by just retracing the steps. □

THEOREM 2.2. *If $\{T_\lambda\phi : \lambda \in \widetilde{\Lambda}\}$ is an orthonormal system in $L^2(\mathbb{R})$, then $\widetilde{\Lambda} = \{0, (r/N)\} + s\mathbb{Z}$ and $1 \le r \le sN-1$ is an odd integer.*

PROOF. Suppose that $\{T_\lambda\phi : \lambda \in \widetilde{\Lambda}\}$ is an orthonormal system. Then, from Theorem 2.1,

$$\sum_{p=0}^{sN-1} \sum_{k\in\mathbb{Z}} e^{2\pi i(p/s)((r_i-r_j)/N)} \Big|\widehat{\phi}\Big(\xi+\frac{p}{s}+kN\Big)\Big|^2 = s\delta_{ij}. \qquad (2\text{-}1)$$

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Now let $C_p(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + (p/s) + kN)|^2, 0 \le p \le sN - 1$. Then, by (2-1), for $i = j$,

$$\begin{aligned} s &= \sum_{p=0}^{sN-1} C_p(\xi) \\ &= \sum_{p=0}^{(sN/2)-1} C_p(\xi) + \sum_{p=(sN/2)}^{sN-1} C_p(\xi) \\ &= \sum_{p=0}^{(sN/2)-1} (C_p(\xi) + C_{(sN/2)+p}(\xi)). \end{aligned} \tag{2-2}$$

Again from (2-1), for $i \neq j$,

$$\begin{aligned} 0 &= \sum_{p=0}^{sN-1} e^{2\pi i(p/s)((r_i-r_j)/N)} C_p(\xi) \\ &= \sum_{p=0}^{(sN/2)-1} e^{2\pi i(p/s)((r_i-r_j)/N)} C_p(\xi) + \sum_{p=(sN/2)}^{sN-1} e^{2\pi i(p/s)((r_i-r_j)/N)} C_p(\xi) \\ &= \sum_{p=0}^{(sN/2)-1} e^{2\pi i(p/s)((r_i-r_j)/N)} (C_p(\xi) + e^{\pi i(r_i-r_j)} C_{(sN/2)+p}(\xi)). \end{aligned}$$

If $r_i - r_j$ is even, then

$$\sum_{p=0}^{(sN/2)-1} e^{2\pi i(p/s)((r_i-r_j)/N)} (C_p(\xi) + C_{(sN/2)+p}(\xi)) = 0.$$

Since $\{e^{2\pi i(p/s)((r_i-r_j)/N)} : p \in \{0, 1, \ldots, (sN/2) - 1\}\}$ is linearly independent in $L^2(\mathbb{Z}_{sN/2})$, $C_p(\xi) + C_{(sN/2)+p}(\xi) = 0$ for almost every (a.e.) $\xi \in \mathbb{R}$ and for all $p \in \{0, 1, \ldots, (sN/2) - 1\}$, that is a contradiction to (2-2). Thus, $r_i - r_j$ is an odd integer for all $i \neq j$ and hence $r_i = r_i - r_0$ is an odd integer for all $i = 1, 2, \ldots, q$. Using the fact that r_i and $r_i - r_j, i \neq j, i = 1, 2, \ldots, q, j = 0, 1, \ldots, q$ are odd integers, we can observe that the cardinality of $\{r_1, r_2, \ldots, r_q\}$ is 1. Suppose not. Then there exist r_m and $r_n, m \neq n$, such that r_m, r_n and $r_m - r_n$ are odd integers, which is not possible. Thus, the cardinality of $\{r_1, r_2, \ldots, r_q\}$ is 1. Hence, $\widetilde{\Lambda} = \{0, (r/N)\} + s\mathbb{Z}$. □

DEFINITION 2.3. Let $N \in \mathbb{N}$, s be an even positive integer and $1 \le r \le sN - 1$ be an odd integer relatively prime to N. An associated Λ_s-nonuniform multiresolution analysis (Λ_s-NUMRA) with dilation factor sN is a collection $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following conditions:

(i) there exists a $\phi \in V_0$, called a scaling function, such that $\{T_\lambda \phi : \lambda \in \Lambda_s\}$ is an orthonormal basis for V_0, where $\Lambda_s = \{0, (r/N)\} + s\mathbb{Z}$;
(ii) $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$;
(iii) $f \in V_j$ if and only if $f(sN\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

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(iv) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;
(v) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

When $s = 2$, it gives NUMRA and for $N = 1$ and $s = 2$ it gives classical MRA. The following theorem gives an equivalent condition for a system $\{T_\lambda \phi : \lambda \in \Lambda_s\}$ to be an orthonormal system in $L^2(\mathbb{R})$.

THEOREM 2.4. *Let $\phi \in L^2(\mathbb{R})$, $s \in 2\mathbb{N}$ and $1 \le r \le sN - 1$ be an odd integer. Then $\{T_\lambda \phi : \lambda \in \Lambda_s\}$ is an orthonormal system in $L^2(\mathbb{R})$ if and only if*

$$\sum_{p=0}^{(sN/2)-1} \sum_{k\in\mathbb{Z}} \left|\widehat{\phi}\Big(\xi + \frac{p}{s} + kN\Big)\right|^2 = \frac{s}{2} \quad a.e.\ \xi \in \mathbb{R}. \tag{2-3}$$

PROOF. Suppose that $\{T_\lambda \phi : \lambda \in \Lambda_s\}$ is an orthonormal system. Then, from Theorem 2.1,

$$\sum_{p=0}^{sN-1} \sum_{k\in\mathbb{Z}} \left|\widehat{\phi}\Big(\xi + \frac{p}{s} + kN\Big)\right|^2 = s \quad \text{a.e. } \xi \in \mathbb{R} \tag{2-4}$$

and

$$\sum_{p=0}^{sN-1} \sum_{k\in\mathbb{Z}} e^{2\pi i(p/s)(r/N)} \left|\widehat{\phi}\Big(\xi + \frac{p}{s} + kN\Big)\right|^2 = 0 \quad \text{a.e. } \xi \in \mathbb{R}. \tag{2-5}$$

Let $C_j(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + (j/s) + kN)|^2, 0 \le j \le sN - 1$. Then $C_j(\xi) = C_{sN+j}(\xi)$ for $0 \le j \le sN - 1$. From (2-4),

$$\begin{aligned} s = \sum_{p=0}^{sN-1} C_p(\xi) &= \sum_{p=0}^{(sN/2)-1} C_p(\xi) + \sum_{p=(sN/2)}^{sN-1} C_p(\xi) \\ &= \sum_{p=0}^{(sN/2)-1} [C_p(\xi) + C_{(sN/2)+p}(\xi)]. \end{aligned} \tag{2-6}$$

From (2-5),

$$\begin{aligned} 0 &= \sum_{p=0}^{sN-1} e^{2\pi i(p/s)(r/N)} C_p(\xi) \\ &= \sum_{p=0}^{(sN/2)-1} e^{2\pi i(p/s)(r/N)} C_p(\xi) + \sum_{p=(sN/2)}^{sN-1} e^{2\pi i(p/s)(r/N)} C_p(\xi) \\ &= \sum_{p=0}^{(sN/2)-1} e^{2\pi i(p/s)(r/N)} C_p(\xi) + e^{2\pi i(r/sN)((sN/2)+p)} C_{(sN/2)+p}(\xi) \\ &= \sum_{p=0}^{(sN/2)-1} e^{2\pi i(p/s)(r/N)} [C_p(\xi) + e^{\pi i r} C_{(sN/2)+p}(\xi)]. \end{aligned}$$

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Since r is an odd integer, say $r = 2l + 1$,

$$\sum_{p=0}^{(sN/2)-1} e^{2\pi i(p/s)2l}[C_p(\xi) - C_{(sN/2)+p}(\xi)]e^{2\pi i(p/sN)} = 0. \tag{2-7}$$

Observe that the left-hand side of (2-7) is a discrete Fourier series on the group $\mathbb{Z}/((sN/2)\mathbb{Z})$ and so

$$C_p(\xi) = C_{(sN/2)+p}(\xi).$$

Hence, from (2-6),

$$\sum_{p=0}^{(sN/2)-1} \sum_{k\in\mathbb{Z}} \left|\widehat{\phi}\left(\xi + \frac{p}{s} + kN\right)\right|^2 = \frac{s}{2} \quad \text{a.e. } \xi \in \mathbb{R}.$$

By retracing the steps, we obtain the converse part and hence the proof is complete. □

3. Characterization of scaling functions for Λ_s-NUMRA

In this section, we characterize the functions $\phi \in L^2(\mathbb{R})$ that generate Λ_s-nonuniform multiresolution analysis. For $\phi \in L^2(\mathbb{R})$, define

$$V_j = \begin{cases} \overline{\text{span}}\{\phi(\cdot - \lambda) : \lambda \in \Lambda_s\} & \text{if } j = 0, \\ \{f : f((sN)^{-j} \cdot) \in V_0\} & \text{if } j \neq 0. \end{cases} \tag{3-1}$$

Then it is clear that $f \in V_j$ if and only if $f(sN\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$. One can observe that, if $\{T_\lambda\phi : \lambda \in \Lambda_s\}$ is an orthonormal basis for V_0, then $\{D^jT_\lambda\phi : \lambda \in \Lambda_s\}$ is an orthonormal basis for $V_j, j \in \mathbb{Z}$.

DEFINITION 3.1. A function $\phi \in L^2(\mathbb{R})$ is said to generate a Λ_s-NUMRA if the spaces V_j defined in (3-1) together with ϕ form a Λ_s-NUMRA.

THEOREM 3.2. *Let $\phi \in L^2(\mathbb{R})$ be such that $\{T_\lambda\phi : \lambda \in \Lambda_s\}$ is an orthonormal system and the spaces V_j be as defined in (3-1). Then $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$ if and only if there exist $(1/s)$-periodic functions m_0^1 and m_0^2 such that*

$$\widehat{\phi}(sN\xi) = (m_0^1(\xi) + m_0^2(\xi)e^{-2\pi i\xi(r/N)})\widehat{\phi}(\xi) \quad \textit{a.e. } \xi \in \mathbb{R}.$$

PROOF. Suppose that $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$. Then, by (3-1), it is clear that $f \in V_j$ if and only if $f(sN\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$. Since $\phi \in V_0 \subseteq V_1$, $\phi(\cdot/sN) \in V_0$. As $\{T_\lambda\phi : \lambda \in \Lambda_s\}$ is an orthonormal basis for V_0,

$$\phi\left(\frac{x}{sN}\right) = \sum_{\lambda\in\Lambda_s} a_\lambda\phi(x - \lambda).$$

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Taking the Fourier transform on both sides,

$$\begin{aligned}\widehat{\phi}(sN\xi) &= \frac{1}{sN}\sum_{\lambda\in\Lambda_s} a_\lambda \widehat{\phi}(\xi)e^{-2\pi i\xi\lambda}\\ &= \Big(\frac{1}{sN}\sum_{\lambda\in s\mathbb{Z}} a_\lambda e^{-2\pi i\xi\lambda} + \frac{1}{sN}\sum_{\lambda\in(r/N)+s\mathbb{Z}} a_\lambda e^{-2\pi i\xi\lambda}\Big)\widehat{\phi}(\xi)\\ &= \Big(\frac{1}{sN}\sum_{l\in\mathbb{Z}} a_{sl} e^{-2\pi i\xi sl} + \frac{1}{sN}\sum_{l\in\mathbb{Z}} a_{(r/N)+sl} e^{-2\pi i\xi((r/N)+sl)}\Big)\widehat{\phi}(\xi).\end{aligned}$$

Thus,

$$\widehat{\phi}(sN\xi) = (m_0^1(\xi) + m_0^2(\xi)e^{-2\pi i\xi(r/N)})\widehat{\phi}(\xi),$$

where $m_0^1(\xi) = (1/sN)\sum_{l\in\mathbb{Z}} a_{sl}e^{-2\pi i\xi sl}$ and $m_0^2(\xi) = (1/sN)\sum_{l\in\mathbb{Z}} a_{(r/N)+sl}e^{-2\pi i\xi sl}$ are $(1/s)$-periodic functions in $L^2([0,1/s))$. By reversing the steps above, we get the converse part. □

THEOREM 3.3. *Let $\phi \in L^2(\mathbb{R})$ be such that $\{\phi(\cdot - \lambda) : \lambda \in \Lambda_s\}$ is an orthonormal system and the V_j are the spaces as defined in (3-1). Then $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ if and only if $\lim_{j\to\infty} |\widehat{\phi}((sN)^{-j}\xi)|^2 = c,\ c > 0$, a.e. $\xi \in [-(1/2s),(1/2s))$.*

PROOF. Assume that $\lim_{j\to\infty} |\widehat{\phi}((sN)^{-j}\xi)|^2 = c,\ \ c > 0$. Let $f \in (\overline{\cup_{j\in\mathbb{Z}} V_j})^\perp$ and P_j be the orthogonal projection onto $V_j, j \in \mathbb{Z}$. For $\epsilon > 0$, there exists a $g \in L^2(\mathbb{R})$ with $\widehat{g} \in C_c(\mathbb{R})$ such that

$$\|f - g\| \le \epsilon. \tag{3-2}$$

Also note that $\|P_j g\| = \|P_j(g - f)\| \le \epsilon$ for all $j \in \mathbb{Z}$.

Let us take $\Lambda_s = \{r_0/N, r_1/N\} + s\mathbb{Z}$, where $r_0 = 0$ and $r_1 = r$, for simplicity. For $j \in \mathbb{Z}$,

$$\begin{aligned}\|P_j g\|^2 &= \sum_{\lambda\in\Lambda_s} |\langle g, D^j T_\lambda\phi\rangle|^2\\ &= \sum_{\lambda\in\Lambda_s} |\langle \widehat{g}, \widehat{D^j T_\lambda\phi}\rangle|^2\\ &= \sum_{\lambda\in\Lambda_s}\Big|\int_{\mathbb{R}} (sN)^{-(j/2)}\widehat{g}(\xi)e^{2\pi i\xi\lambda(sN)^{-j}}\overline{\widehat{\phi}\Big(\frac{\xi}{(sN)^j}\Big)}\,d\xi\Big|^2\\ &= \sum_{m=0}^{1}\sum_{l\in\mathbb{Z}}\Big|\int_{\mathbb{R}} (sN)^{-(j/2)}\widehat{g}(\xi)e^{2\pi i\xi((r_m/N)+sl)(sN)^{-j}}\overline{\widehat{\phi}\Big(\frac{\xi}{(sN)^j}\Big)}\,d\xi\Big|^2.\end{aligned}$$

Next, by changing the variable $\xi = (sN)^j\eta$,

$$\|P_j g\|^2 = (sN)^j\sum_{m=0}^{1}\sum_{l\in\mathbb{Z}}\Big|\int_{\mathbb{R}} \widehat{g}((sN)^j\eta)e^{2\pi i\eta(r_m/N)}e^{2\pi i\eta sl}\overline{\widehat{\phi}(\eta)}\,d\eta\Big|^2.$$

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Since $\widehat{g}$ has compact support, we can choose j large enough such that supp $(\widehat{g}((sN)^j\cdot)) \subset [(-1/2s), (1/2s)]$. As $\{\sqrt{s}e^{2\pi i\xi sl} : l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([-1/2s, 1/2s))$,

$$\begin{aligned}\|P_j g\|^2 &= \frac{(sN)^j}{s}\sum_{m=0}^{1}\sum_{l\in\mathbb{Z}}\bigg|\int_{-1/2s}^{1/2s}\widehat{g}((sN)^j\xi)e^{2\pi i\xi(r_m/N)}\sqrt{s}e^{2\pi i\xi sl}\overline{\widehat{\phi}(\xi)}\,d\xi\bigg|^2\\ &= \frac{(sN)^j}{s}\sum_{m=0}^{1}\int_{-1/2s}^{1/2s}|\widehat{g}((sN)^j\xi)\widehat{\phi}(\xi)|^2\,d\xi\\ &= \frac{2(sN)^j}{s}\int_{\mathbb{R}}|\widehat{g}((sN)^j\xi)\widehat{\phi}(\xi)|^2\,d\xi.\end{aligned}$$

By change of variable,

$$\begin{aligned}\|P_j g\|^2 &= \frac{2}{s}\int_{\mathbb{R}}|\widehat{g}(\xi)\widehat{\phi}((sN)^{-j}\xi)|^2\,d\xi\\ &= \frac{2}{s}\int_{-1/2s}^{1/2s}|\widehat{g}(\xi)\widehat{\phi}((sN)^{-j}\xi)|^2\,d\xi.\end{aligned}$$

Since $\lim_{j\to\infty}|\widehat{\phi}((sN)^{-j}\xi)|^2 = c$ a.e. $\xi \in [-(1/2s), (1/2s))$,

$$\begin{aligned}\lim_{j\to\infty}\|P_j g\|^2 &= \lim_{j\to\infty}\frac{2}{s}\int_{-1/2s}^{1/2s}|\widehat{g}(\xi)\widehat{\phi}((sN)^{-j}\xi)|^2\,d\xi\\ &= \frac{2c}{s}\int_{\mathbb{R}}|\widehat{g}(\xi)|^2\,d\xi.\end{aligned}$$

Note that $\|P_j g\|^2 \le \epsilon^2$ for all j and hence

$$\|\widehat{g}\|^2 = \|g\|^2 \le \frac{s\epsilon^2}{2c}.$$

Since $\|f\| \le \epsilon + \|g\|$, we have $\|f\| \le \epsilon + \epsilon\sqrt{s/2c}$, which proves that $f = 0$ and hence $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

Conversely, suppose that $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$. Then $\|f - P_j f\|^2 \to 0$ as $j \to \infty$ for all $f \in L^2(\mathbb{R})$. If $f \in L^2(\mathbb{R})$ is such that $\widehat{f}(\xi) = \chi_{[-(1/2s),1/2s)}(\xi)$, then $\|f\|^2 = \|\widehat{f}\|^2 = 1/s$. Thus $\|P_j f\|^2 \to \|f\|^2 = 1/s$ as $j \to \infty$. Now

$$\begin{aligned}\|P_j f\|^2 &= \sum_{\lambda\in\Lambda_s}|\langle f, D^jT_\lambda\phi\rangle|^2\\ &= \sum_{\lambda\in\Lambda_s}|\langle \widehat{f}, \widehat{D^jT_\lambda\phi}\rangle|^2\\ &= \sum_{\lambda\in\Lambda_s}\bigg|\int_{\mathbb{R}}\widehat{f}(\xi)(sN)^{-(j/2)}e^{2\pi i\xi\lambda(sN)^{-j}}\overline{\widehat{\phi}\Big(\frac{\xi}{(sN)^j}\Big)}\,d\xi\bigg|^2\end{aligned}$$

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$$= \sum_{\lambda\in\Lambda_s} \left| \int_{-1/2s}^{1/2s} (sN)^{-(j/2)} e^{2\pi i\xi\lambda(sN)^{-j}} \overline{\widehat{\phi}\Big(\frac{\xi}{(sN)^j}\Big)}\, d\xi \right|^2$$

$$= \sum_{m=0}^{1} \sum_{l\in\mathbb{Z}} \left| \int_{-1/2s}^{1/2s} (sN)^{-(j/2)} e^{2\pi i\xi((r_m/N)+sl)(sN)^{-j}} \overline{\widehat{\phi}\Big(\frac{\xi}{(sN)^j}\Big)}\, d\xi \right|^2.$$

By changing the variable $\xi = (sN)^j\eta$,

$$\|P_j f\|^2 = (sN)^j \sum_{m=0}^{1} \sum_{l\in\mathbb{Z}} \left| \int_{-(1/2s(sN)^j)}^{1/2s(sN)^j} e^{2\pi i\eta(r_m/N)} e^{2\pi i\eta sl} \overline{\widehat{\phi}(\eta)}\, d\eta \right|^2.$$

For $j \geq 0$, $\Omega_j := [-(1/(2s(sN)^j)), (1/(2s(sN)^j))] \subseteq [-(1/2s), (1/2s)]$. Hence,

$$\|P_j f\|^2 = (sN)^j \sum_{m=0}^{1} \sum_{l\in\mathbb{Z}} \left| \int_{-1/2s}^{1/2s} \chi_{\Omega_j}(\xi) e^{2\pi i\xi(r_m/N)} e^{2\pi i\xi sl} \overline{\widehat{\phi}(\xi)}\, d\xi \right|^2.$$

Since $\{\sqrt{s}e^{2\pi i\xi sl} : l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([-(1/2s), (1/2s)))$,

$$\|P_j f\|^2 = \frac{(sN)^j}{s} \sum_{m=0}^{1} \int_{-1/2s}^{1/2s} \left| \chi_{\Omega_j}(\xi) \overline{\widehat{\phi}(\xi)} \right|^2 d\xi$$

$$= \frac{2}{s} \int_{-1/2s}^{1/2s} |\widehat{\phi}((sN)^{-j}\xi)|^2\, d\xi.$$

Taking the limit as $j \to \infty$,

$$\frac{1}{s} = \lim_{j\to\infty} \frac{2}{s} \int_{-1/2s}^{1/2s} |\widehat{\phi}((sN)^{-j}\xi)|^2\, d\xi$$

and hence

$$\lim_{j\to\infty} \int_{-1/2s}^{1/2s} |\widehat{\phi}((sN)^{-j}\xi)|^2\, d\xi = \frac{1}{2}.$$

From (2-3), and since $\int_{-1/2s}^{1/2s} ((s/2) - \lim_{j\to\infty} |\widehat{\phi}((sN)^{-j}\xi)|^2)\, d\xi = 0$,

$$\lim_{j\to\infty} |\widehat{\phi}((sN)^{-j}\xi)|^2 = \frac{s}{2} \quad \text{a.e. } \xi \in \Big[-\frac{1}{2s}, \frac{1}{2s}\Big).$$ □

THEOREM 3.4. *Let $\phi \in L^2(\mathbb{R})$ and the V_j be as defined in (3-1). Then ϕ generates Λ_s-nonuniform multiresolution analysis with dilation factor sN if and only if:*

(1) $\sum_{p=0}^{(sN/2)-1} \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\xi + (p/s) + kN)|^2 = s/2$ *a.e.* $\xi \in \mathbb{R}$*;*

(2) $\lim_{j\to\infty} |\widehat{\phi}((sN)^{-j}\xi)|^2 = c, \quad c > 0,$ *a.e.* $\xi \in [-(1/2s), 1/2s)$*;*

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(3) *there exist* $(1/s)$*-periodic functions* m_0^1 *and* m_0^2 *such that*

$$\widehat{\phi}(sN\xi) = (m_0^1(\xi) + m_0^2(\xi)e^{-2\pi i\xi(r/N)})\widehat{\phi}(\xi).$$

The proof follows from Theorems 2.4, 3.2 and 3.3.

Next we construct an example of a Λ_s-NUMRA with dilation factor sN using some results of spectral pairs [4].

DEFINITION 3.5. Let A be a measurable subset of $\mathbb{R}$ and $|A|$ the Lebesgue measure of A such that $0 < |A| < \infty$ and $\Lambda \subset \mathbb{R}$ is a discrete set. Then the pair (A, Λ) is said to be a spectral pair if the collection $\{|A|^{-1/2}e^{2\pi i\xi\lambda}\chi_A(\xi) : \lambda \in \Lambda\}$ is a complete orthonormal system for L_A^2, where L_A^2 is the subspace of $L^2(\mathbb{R})$ that consists of functions vanishing outside of A almost everywhere.

PROPOSITION 3.6 [4]. *Let* V_0 *be a closed subspace of* $L^2(\mathbb{R})$ *and suppose that there exist* $\phi \in V_0$ *and a discrete set* $\Lambda \subset \mathbb{R}$ *such that* $\{T_\lambda\phi : \lambda \in \Lambda\}$ *is an orthonormal basis for* V_0*. Then, given a measurable set* $A \subset \mathbb{R}$ *with* $0 < |A| < \infty$*, the mapping* $F : V_0 \to L_A^2$ *defined by* $F(T_\lambda\phi) = |A|^{-1/2}e^{2\pi i\xi\lambda}\chi_A(\xi)$ *is unitary if and only if the collection* $\{|A|^{-1/2}e^{2\pi i\xi\lambda}\chi_A(\xi) : \lambda \in \Lambda\}$ *is an orthonormal basis for* L_A^2.

The above proposition says that (A, Λ) is a spectral pair if and only if the mapping $F : V_0 \to L_A^2$ defined by $F(T_\lambda\phi) = |A|^{-1/2}e^{2\pi i\xi\lambda}\chi_A(\xi)$ is unitary, where V_0 is the space defined as in Proposition 3.6. The following result gives an equivalent condition for a spectral pair.

THEOREM 3.7. *Let* $\Lambda \subset \mathbb{R}$ *be discrete and* $A \subseteq \mathbb{R}$ *be measurable such that* $0 < |A| < \infty$*. Then* (A, Λ) *is a spectral pair if and only if*

$$\sum_{\lambda\in\Lambda} |\widehat{\chi_A}(\xi - \lambda)|^2 = |A|^2 \quad \forall \xi \in \mathbb{R}.$$

For the proof, we refer to [4].

Let r be a positive odd integer and $s = 2r$. Let $\Lambda = \Lambda_s = \{0, (r/N)\} + s\mathbb{Z}$ and $A = [0, 1/s) \cup [N/s, (N+1)/s)$. To show that (A, Λ) is a spectral pair, we have to prove that

$$\sum_{\lambda\in\Lambda} |\widehat{\chi_A}(\xi - \lambda)|^2 = |A|^2 \quad \forall \xi \in \mathbb{R}.$$

That is, we have to prove that

$$\sum_{n\in\mathbb{Z}} |\widehat{\chi_A}(\xi - sn)|^2 + \sum_{n\in\mathbb{Z}} \left|\widehat{\chi_A}\left(\xi - \frac{r}{N} - sn\right)\right|^2 = |A|^2 \quad \forall \xi \in \mathbb{R}. \tag{3-3}$$

If $g = \chi_A * \widetilde{\chi_A}$, where $\widetilde{\chi_A}(x) = \chi_A(-x)$, then $\widehat{g} = |\widehat{\chi_A}|^2 \in L^1(\mathbb{R})$.

Hence, by the Fourier inversion theorem, if $\widehat{g} \in L^1(\mathbb{R})$ and $b > 0$, then

$$\mathcal{F}^{-1}\left(\sum_{n\in\mathbb{Z}} \widehat{g}(\xi - bn)\right) = \frac{1}{b}\sum_{n\in\mathbb{Z}} g\left(\frac{n}{b}\right)\delta_{n/b}(\xi).$$

Using the above fact, we see that (3-3) is equivalent to

$$\frac{1}{s}\sum_{n\in\mathbb{Z}}(1+e^{2\pi i(r/N)(n/s)})(\chi_A*\widetilde{\chi_A})\Big(\frac{n}{s}\Big)\delta_{n/s}=|A|^2\delta_0.$$

Since $s=2r$,

$$\frac{1}{s}\sum_{n\in\mathbb{Z}}(1+e^{\pi i(n/N)})(\chi_A*\widetilde{\chi_A})\Big(\frac{n}{s}\Big)\delta_{n/s}=|A|^2\delta_0. \tag{3-4}$$

Thus, we have to prove (3-4) for $A=[0,1/s)\cup[N/s,(N+1)/s)$. Note that

$$(\chi_A*\widetilde{\chi_A})(n/s)=|A\cap(A+(n/s))|=0$$

for all $n\neq 0,N,-N$. Next, for $n=N,-N$, we have $1+e^{\pi i(n/N)}=0$. Thus,

$$\frac{1}{s}\sum_{n\in\mathbb{Z}}(1+e^{\pi i(n/N)})(\chi_A*\widetilde{\chi_A})\Big(\frac{n}{s}\Big)\delta_{n/s}=\frac{2}{s}|A|\delta_0=|A|^2\delta_0.$$

Hence, (A,Λ_s) is a spectral pair.

Example 1. Let $N\in\mathbb{N}$ and $1\leq r\leq sN-1$ be an odd integer such that r and N are relatively prime, and $s=2r$. Now define $\phi\in L^2(\mathbb{R})$ by $\widehat{\phi}(\xi)=\chi_B(\xi)$, where $B=[-(1/2s),1/2s)\cup[(2N-1)/2s,(2N+1)/2s)$. Since $B=A-(1/2s)$, (B,Λ_s) is a spectral pair. Let

$$V_0=\overline{\operatorname{span}}\{\phi(\cdot-\lambda):\lambda\in\Lambda_s\},$$

where $\Lambda_s=\{0,(r/N)\}+s\mathbb{Z}$, and define the spaces V_j as in (3-1). Since the Fourier transform of $\phi(\cdot-\lambda)$ is $e^{-2\pi i\xi\lambda}\chi_B(\xi)$ and (B,Λ_s) is a spectral pair, $\{\phi(x-\lambda):\lambda\in\Lambda_s\}$ is an orthonormal system in $L^2(\mathbb{R})$. Then, from Theorem 2.4,

$$\sum_{p=0}^{(sN/2)-1}\sum_{k\in\mathbb{Z}}\Big|\widehat{\phi}\Big(\xi+\frac{p}{s}+kN\Big)\Big|^2=\frac{s}{2}\quad\text{a.e. } \xi\in\mathbb{R}.$$

Now $\lim_{j\to\infty}|\widehat{\phi}((sN)^{-j}\xi)|^2=\lim_{j\to\infty}|\chi_B((sN)^{-j}\xi)|^2=\lim_{j\to\infty}|\chi_{(sN)^jB}(\xi)|^2=1$ for all $\xi\in[-(1/2s),(1/2s))$. Next we find the $(1/s)$-periodic functions m_0^1 and m_0^2 such that

$$\widehat{\phi}(sN\xi)=(m_0^1(\xi)+m_0^2(\xi)e^{-2\pi i\xi(r/N)})\widehat{\phi}(\xi).$$

To compute m_0^1 and m_0^2, we use the relation

$$\begin{aligned}\widehat{\phi}(sN\xi)&=\chi_B(sN\xi)=\chi_C(\xi)\\&=(m_0^1(\xi)+m_0^2(\xi)e^{-2\pi i\xi(r/N)})\chi_B(\xi),\end{aligned}$$

where $C=[-(1/2s^2N),1/2s^2N)\cup[(2N-1)/2s^2N,(2N+1)/2s^2N)$. Fixing $\xi\in[-(1/2s^2N),1/2s^2N)$,

$$m_0^1(\xi)+m_0^2(\xi)e^{-2\pi i\xi(r/N)}=\chi_C(\xi)$$

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on the interval $[-(1/2s^2N), 1/2s^2N)$. Note that

$$\chi_c\Big(\xi + \frac{1}{s^2}\Big) = \chi_c(\xi),\ \xi \in \Big[-\frac{1}{2s^2N}, \frac{1}{2s^2N}\Big).$$

Hence, one can choose $m_0^1(\xi) = \chi_c(\xi)$ and $m_0^2(\xi) = 0$ so that m_0^1 and m_0^2 are $(1/s)$-periodic functions such that $m_0^1(\xi) + m_0^2(\xi)e^{-2\pi i\xi(r/N)} = \chi_c(\xi)$. Hence, from Theorem 3.4, $\{V_j, \phi\}_{j\in\mathbb{Z}}$ is a Λ_s-NUMRA.

4. Condition for Λ_s-NUMRA wavelet filters

Suppose that the closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ form a Λ_s-NUMRA with scaling function $\phi \in L^2(\mathbb{R})$. Then there is a function $m_0(\xi)$ of the form

$$m_0(\xi) = m_0^1(\xi) + e^{-2\pi i\xi r/N} m_0^2(\xi),$$

where m_0^1, m_0^2 are $(1/s)$-periodic functions in $L^2([0, 1/s))$ such that

$$\widehat{\phi}(sN\xi) = m_0(\xi)\widehat{\phi}(\xi).$$

Now define the functions $\psi_k \in L^2(\mathbb{R}), k = 1, 2, \ldots, sN-1$, by

$$\widehat{\psi_k}(sN\xi) = m_k(\xi)\widehat{\phi}(\xi), \tag{4-1}$$

where

$$m_k(\xi) = m_k^1(\xi) + e^{-2\pi i\xi r/N} m_k^2(\xi), \quad k = 1, 2, \ldots, sN-1. \tag{4-2}$$

The functions m_k^1 and $m_k^2, k = 1, 2, \ldots, sN-1$, are called Λ_s-NUMRA wavelet filters. In this section, we prove the conditions on $m_k, k = 1, \ldots, sN-1$ such that $\{(sN)^{j/2}\psi_k((sN)^j \cdot -\lambda) : k = 1, 2, \ldots, sN-1, j \in \mathbb{Z}, \lambda \in \Lambda_s\}$ is an orthonormal basis for $L^2(\mathbb{R})$; we call the set of functions $\{\psi_k : k = 1, 2, \ldots, sN-1\}$ as Λ_s-NUMRA wavelets. Let us take $\psi_0 = \phi$ for notational simplicity. The following proposition gives the equivalent conditions for the system $\{\psi_k(\cdot - \lambda) : k = 0, 1, \ldots, sN-1, \lambda \in \Lambda_s\}$ to be an orthonormal set in $L^2(\mathbb{R})$. When $s = 2$, these equivalent conditions coincide with the conditions in [4, 5, 9]. Now we prove the following results using similar arguments as in [4].

PROPOSITION 4.1. *Let $\psi_k \in L^2(\mathbb{R}), k = 0, 1, \ldots, sN-1$. Then the system*

$$\{\psi_k(\cdot - \lambda) : k = 0, 1, \ldots, sN-1, \lambda \in \Lambda_s\}$$

is orthonormal if and only if

$$\sum_{p=0}^{sN-1} w_{kl}\Big(\xi + \frac{p}{s}\Big) = s\delta_{kl} \tag{4-3}$$

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and

$$\sum_{p=0}^{sN-1} \alpha^p w_{kl}\Big(\xi + \frac{p}{s}\Big) = 0, \tag{4-4}$$

where $\alpha = e^{-2\pi i r/sN}$ *and*

$$w_{kl}(\xi) = \sum_{j\in\mathbb{Z}} \widehat{\psi}_k(\xi + jN)\overline{\widehat{\psi}_l(\xi + jN)}, \quad \xi \in \mathbb{R}. \tag{4-5}$$

PROOF. Suppose that $\{\psi_k(\cdot - \lambda) : k = 0, 1, \ldots, sN - 1, \lambda \in \Lambda_s\}$ is an orthonormal system. Then

$$\begin{aligned}
\delta_{kl}\delta_{\lambda\sigma} &= \int_{\mathbb{R}} \psi_k(x-\lambda)\overline{\psi_l(x-\sigma)}\,dx \\
&= \int_{\mathbb{R}} \widehat{\psi}_k(\xi)\overline{\widehat{\psi}_l(\xi)}e^{-2\pi i\xi(\lambda-\sigma)}\,d\xi \\
&= \sum_{j\in\mathbb{Z}} \int_0^N \widehat{\psi}_k(\xi - jN)\overline{\widehat{\psi}_l(\xi - jN)}e^{-2\pi i(\xi-jN)(\lambda-\sigma)}\,d\xi \\
&= \int_0^N \sum_{j\in\mathbb{Z}} \widehat{\psi}_k(\xi - jN)\overline{\widehat{\psi}_l(\xi - jN)}e^{-2\pi i\xi(\lambda-\sigma)}\,d\xi \\
&= \int_0^N w_{kl}(\xi)e^{-2\pi i\xi(\lambda-\sigma)}\,d\xi \\
&= \int_0^{1/s} \sum_{p=0}^{sN-1} w_{kl}\Big(\xi + \frac{p}{s}\Big)e^{-2\pi i\xi(\lambda-\sigma)}e^{-2\pi i(p/s)(\lambda-\sigma)}\,d\xi.
\end{aligned}$$

Letting $\lambda = sm$ and $\sigma = sn, m, n \in \mathbb{Z}$,

$$\delta_{mn}\delta_{kl} = \int_0^{1/s} \sum_{p=0}^{sN-1} w_{kl}\Big(\xi + \frac{p}{s}\Big)e^{-2\pi i\xi s(m-n)}\,d\xi.$$

Since $\{\sqrt{s}e^{-2\pi i\xi sl} : l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([0, 1/s))$,

$$\sum_{p=0}^{sN-1} w_{kl}\Big(\xi + \frac{p}{s}\Big) = s\delta_{kl}.$$

Letting $\lambda = (r/N) + sm$ and $\sigma = sn, m, n \in \mathbb{Z}$,

$$0 = \int_0^{1/s} e^{-2\pi i\xi s(m-n)}e^{-2\pi i\xi r/N} \sum_{p=0}^{sN-1} w_{kl}\Big(\xi + \frac{p}{s}\Big)e^{-2\pi i p(r/sN)}\,d\xi.$$

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Since $\{\sqrt{s}e^{-2\pi i\xi sl} : l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([0, 1/s))$,

$$\sum_{p=0}^{sN-1} \alpha^p w_{kl}\Big(\xi + \frac{p}{s}\Big) = 0,$$

where $\alpha = e^{-2\pi ir/sN}$. By reversing the steps above, we get the converse part. □

Now define an $2sN \times 2sN$ matrix $U(\xi)$ with entries $U_{pq}(\xi), 0 \le p, q \le 2sN - 1$, defined by

$$U_{pq}(\xi) = \begin{cases} m_q^1\Big(\xi + \dfrac{p}{s^2N}\Big) & 0 \le p \le sN-1,\ \ 0 \le q \le sN-1, \\ m_q^2\Big(\xi + \dfrac{p-sN}{s^2N}\Big) & sN \le p \le 2sN-1,\ \ 0 \le q \le sN-1, \\ \alpha^p m_{q-sN}^1\Big(\xi + \dfrac{p}{s^2N}\Big) & 0 \le p \le sN-1,\ \ sN \le q \le 2sN-1, \\ \alpha^p m_{q-sN}^2\Big(\xi + \dfrac{p-sN}{s^2N}\Big) & sN \le p \le 2sN-1,\ \ sN \le q \le 2sN-1. \end{cases}$$

The following proposition characterizes when $\{T_\lambda\psi_k : k = 0, 1, \ldots, sN-1, \lambda \in \Lambda_s\}$ is an orthonormal system in terms of the matrix $U(\xi)$.

PROPOSITION 4.2. *Let $\psi_k \in L^2(\mathbb{R}), k = 0, 1, \ldots, sN-1$. Then the system*

$$\{T_\lambda\psi_k : k = 0, 1, \ldots, sN-1, \lambda \in \Lambda_s\}$$

is orthonormal if and only if the matrix $U(\xi)$ is unitary a.e. $\xi \in \mathbb{R}$.

PROOF. Using (4-1) and (4-5),

$$w_{kl}(sN\xi) = \sum_{j\in\mathbb{Z}} m_k\Big(\xi + \frac{j}{s}\Big)\overline{m_l\Big(\xi + \frac{j}{s}\Big)}\Big|\widehat{\phi}\Big(\xi + \frac{j}{s}\Big)\Big|^2.$$

By the $(1/s)$-periodicity of m_k^1, m_k^2, m_l^1 and m_l^2,

$$\begin{aligned} w_{kl}(sN\xi) = (m_k^1(\xi)\overline{m_l^1(\xi)} + m_k^2(\xi)\overline{m_l^2(\xi)}) &\sum_{j\in\mathbb{Z}} \Big|\widehat{\phi}\Big(\xi + \frac{j}{s}\Big)\Big|^2 \\ + m_k^1(\xi)\overline{m_l^2(\xi)} &\sum_{j\in\mathbb{Z}} e^{2\pi i(\xi+(j/s))r/N}\Big|\widehat{\phi}\Big(\xi + \frac{j}{s}\Big)\Big|^2 \\ + m_k^2(\xi)\overline{m_l^1(\xi)} &\sum_{j\in\mathbb{Z}} e^{-2\pi i(\xi+(j/s))r/N}\Big|\widehat{\phi}\Big(\xi + \frac{j}{s}\Big)\Big|^2. \end{aligned}$$

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By writing $j = p + (sN)q$, $p \in \{0, 1, \ldots, sN - 1\}$, $q \in \mathbb{Z}$,

$$w_{kl}(sN\xi) = (m_k^1(\xi)\overline{m_l^1(\xi)} + m_k^2(\xi)\overline{m_l^2(\xi)}) \sum_{p=0}^{sN-1} w_{00}\Big(\xi + \frac{p}{s}\Big)$$
$$+ m_k^1(\xi)\overline{m_l^2(\xi)} \sum_{p=0}^{sN-1} \alpha^{-p} w_{00}\Big(\xi + \frac{p}{s}\Big)$$
$$+ m_k^2(\xi)\overline{m_l^1(\xi)} \sum_{p=0}^{sN-1} \alpha^{p} w_{00}\Big(\xi + \frac{p}{s}\Big).$$

Next, using (4-3) and (4-4),

$$w_{kl}(sN\xi) = s(m_k^1(\xi)\overline{m_l^1(\xi)} + m_k^2(\xi)\overline{m_l^2(\xi)}). \tag{4-6}$$

From (4-3) and (4-6),

$$\sum_{p=0}^{sN-1} \Big(m_k^1\Big(\xi + \frac{p}{s^2N}\Big)\overline{m_l^1\Big(\xi + \frac{p}{s^2N}\Big)} + m_k^2\Big(\xi + \frac{p}{s^2N}\Big)\overline{m_l^2\Big(\xi + \frac{p}{s^2N}\Big)}\Big) = \delta_{kl}, \tag{4-7}$$

$0 \le k, l \le sN - 1$. From (4-4) and (4-6),

$$\sum_{p=0}^{sN-1} \alpha^p \Big(m_k^1\Big(\xi + \frac{p}{s^2N}\Big)\overline{m_l^1\Big(\xi + \frac{p}{s^2N}\Big)} + m_k^2\Big(\xi + \frac{p}{s^2N}\Big)\overline{m_l^2\Big(\xi + \frac{p}{s^2N}\Big)}\Big) = 0, \tag{4-8}$$

$0 \le k, l \le sN - 1$.

Conditions (4-7) and (4-8) together are equivalent to saying that the matrix $U(\xi)$ is unitary a.e. $\xi \in \mathbb{R}$. It is easy to observe that $\{T_\lambda \psi_k : k = 0, 1, \ldots, sN - 1, \lambda \in \Lambda_s\}$ is an orthonormal system if and only if $U(\xi)$ is unitary a.e. $\xi \in \mathbb{R}$. □

The solvability of the system (4-7) and (4-8), given in [4] for the case $s = 2$, can be extended for any even number s. The following theorem generalizes the result of Gabardo and Nashed [4, Lemma 3.2].

THEOREM 4.3. *If $s = 2r$ and $\{T_\lambda \psi_k : k = 0, 1, \ldots, sN - 1, \lambda \in \Lambda_s\}$ is an orthonormal system, then $\{T_\lambda \psi_k : k = 0, 1, \ldots, sN - 1, \lambda \in \Lambda_s\}$ is complete in V_1.*

PROOF. Suppose that $\{T_\lambda \psi_k : k = 0, 1, \ldots, sN - 1, \lambda \in \Lambda_s\}$ is an orthonormal system. Then, from Proposition 4.2, $U(\xi)$ is unitary for almost every $\xi \in \mathbb{R}$. From condition (iii) of Definition 2.3, it is enough to prove that

$$\Big\{\frac{1}{sN}\psi_k\Big(\frac{x}{sN} - \lambda\Big) : \lambda \in \Lambda_s, k = 0, 1, \ldots, sN - 1\Big\}$$

is complete in V_0. For any $f \in V_0$, there exists $m_f(\xi) = \sum_{\lambda\in\Lambda_s} b_\lambda e^{-2\pi i\lambda\xi}$, where $\sum_{\lambda\in\Lambda_s} |b_\lambda|^2 < \infty$, such that $\widehat{f}(\xi) = m_f(\xi)\widehat{\phi}(\xi)$. Let us define a map $V_0 \longrightarrow L_A^2$, where

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$A = [0, 1/s) \cup [N/s, (N+1)/s)$, by $f \mapsto m_f \chi_A$. Then, by Proposition 3.6, the above map is unitary. Hence, it is enough to prove that

$$S := \{e^{-2\pi i (sN)\xi\lambda} m_k(\xi)\chi_A(\xi) : \lambda \in \Lambda_s, k = 0, 1, \ldots, sN-1\}$$

is complete in L_A^2. Let $g \in L_A^2$ be such that $g \perp S$. Since $\{e^{2\pi i\lambda\xi}\chi_A(\xi) : \lambda \in \Lambda_s\}$ is an orthonormal basis for L_A^2,

$$\begin{aligned} g(\xi) &= \sum_{\lambda\in\Lambda_s} a_\lambda e^{-2\pi i\lambda\xi}\chi_A(\xi) \\ &= (g_1(\xi) + g_2(\xi)e^{-2\pi i\xi\frac{r}{N}})\chi_A(\xi), \end{aligned} \tag{4-9}$$

where $g_1(\xi) = \sum_{l\in\mathbb{Z}} a_{sl}e^{-2\pi isl\xi}$ and $g_2(\xi) = \sum_{l\in\mathbb{Z}} a_{(r/N)+sl}e^{-2\pi isl\xi}$ are $(1/s)$-periodic functions. As $g \perp S$, for any $\lambda \in \Lambda_s$ and any $k \in \{0, 1, \ldots, sN-1\}$,

$$\begin{aligned} 0 &= \int_0^{1/s} e^{-2\pi isN\xi\lambda} m_k(\xi)\overline{g(\xi)}\,d\xi + \int_{N/s}^{(N+1)/s} e^{-2\pi isN\xi\lambda} m_k(\xi)\overline{g(\xi)}\,d\xi \\ &= \int_0^{1/s} e^{-2\pi isN\xi\lambda}\Big\{m_k(\xi)\overline{g(\xi)} + m_k\Big(\xi + \frac{N}{s}\Big)\overline{g\Big(\xi + \frac{N}{s}\Big)}\Big\}\,d\xi. \end{aligned}$$

Using (4-2) and (4-9),

$$\begin{aligned} \int_0^{1s} e^{-2\pi i\xi sN\lambda}[2\{m_k^1(\xi)\overline{g_1(\xi)} + m_k^2(\xi)\overline{g_2(\xi)}\} + e^{2\pi i\xi(r/N)}m_k^1(\xi)\overline{g_2(\xi)}(1 + e^{2\pi i(r/s)}) \\ + e^{-2\pi i\xi(r/N)}m_k^2(\xi)\overline{g_1(\xi)}(1 + e^{-2\pi i(r/s)})]\,d\xi = 0. \end{aligned}$$

Since $s = 2r$ and r is an odd integer,

$$\int_0^{(1/s)} e^{-2\pi i\xi sN\lambda}\{m_k^1(\xi)\overline{g_1(\xi)} + m_k^2(\xi)\overline{g_2(\xi)}\}\,d\xi = 0. \tag{4-10}$$

Now taking $h_k(\xi) = m_k^1(\xi)\overline{g_1(\xi)} + m_k^2(\xi)\overline{g_2(\xi)}$ and letting $\lambda = sm, m \in \mathbb{Z}$,

$$\sum_{j=0}^{sN-1}\int_{(j/s^2N)}^{((j+1)/s^2N)} h_k(\xi)e^{-2\pi i\xi s^2Nm}\,d\xi = 0.$$

Next, by changing the variable,

$$\int_0^{1/s^2N}\sum_{j=0}^{sN-1} h_k(\xi + (j/s^2N))e^{-2\pi i\xi s^2Nm}\,d\xi = 0.$$

Since $\{e^{-2\pi i\xi s^2Nm} : m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([0, 1/s^2N))$,

$$\sum_{j=0}^{sN-1} h_k\Big(\xi + \frac{j}{s^2N}\Big) = 0 \quad \text{a.e. } \xi \in \mathbb{R}. \tag{4-11}$$

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Similarly, by letting $\lambda = sm + (r/N)$ and using (4-10),

$$\sum_{j=0}^{sN-1} \alpha^j h_k\Big(\xi + \frac{j}{s^2 N}\Big) = 0 \quad \text{a.e. } \xi \in \mathbb{R}. \tag{4-12}$$

Consider the vector $u(\xi) \in \mathbb{C}^{2sN}$ defined by

$$u_k(\xi) = \begin{cases} g_1\Big(\xi + \dfrac{k}{sN}\Big) & k = 0, 1, \ldots, sN-1, \\ g_1\Big(\xi + \dfrac{k - sN}{2sN}\Big) & k = sN, sN+1, \ldots, 2sN-1. \end{cases}$$

Thus, from (4-11) and (4-12),

$$U^*(\xi)u(\xi) = 0 \quad \text{a.e. } \xi \in \mathbb{R}.$$

As $U(\xi)$ is unitary for almost every $\xi \in \mathbb{R}$, we get $u(\xi) = 0$ and hence $g = 0$. This proves completeness. □

Assume that $s = 2r$ and let $W_0 = V_0^{\perp}$ in V_1. Then $V_1 = V_0 \oplus W_0$. Writing $V_{j+1} = V_j \oplus W_j$, where W_j is the orthogonal complement of V_j in V_{j+1}, we obtain from (ii), (iv) and (v) of Definition 2.3 that

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j.$$

Choose $m_k^1, m_k^2, k = 1, 2, \ldots, sN-1$, that satisfy (4-7) and (4-8) and define

$$\psi_k, k = 1, 2, \ldots, sN-1,$$

as in (4-1). Then, by Proposition 4.2 and Theorem 4.3,

$$\{T_\lambda \psi_k : k = 0, 1, \ldots, sN-1, \lambda \in \Lambda_s\}$$

is an orthonormal basis for V_1. Since $\{T_\lambda \psi_0 : \lambda \in \Lambda_s\}$ is an orthonormal basis for V_0, $\{T_\lambda \psi_k : k = 1, \ldots, sN-1, \lambda \in \Lambda_s\}$ is an orthonormal basis for W_0. Also, it is easy to observe that, for every $j \in \mathbb{Z}$, $\{(sN)^{j/2}\psi_k((sN)^j \cdot - \lambda) : k = 1, \ldots, sN-1, \lambda \in \Lambda_s\}$ is an orthonormal basis for W_j. Hence,

$$\{(sN)^{j/2}\psi_k((sN)^j \cdot - \lambda) : k = 1, \ldots, sN-1, j \in \mathbb{Z}, \lambda \in \Lambda_s\}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

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