# GROUP CHARACTERS AND NORMAL HALL SUBGROUPS

P. X. GALLAGHER

To RICHARD BRAUER on his 60th birthday

## 1. Introduction

Let $G$ be a finite group and let $\psi$ be an (ordinary) irreducible character of a normal subgroup $N$. If $\psi$ extends to a character of $G$ then $\psi$ is invariant under $G$, but the converse is false. In section 3 it is shown that if $\psi$ extends coherently to the intermediate groups $H$ for which $H/N$ is elementary, then $\psi$ extends to $G$. If $N$ is a Hall subgroup, then in order for $\psi$ to extend to $G$ it is sufficient that $\psi$ be invariant under $G$. This leads to a construction of the characters of $G$ from the characters of $N$ and the characters of the subgroups of $G/N$ in this case.

Let $1(H)_G$ denote the character of $G$ induced by the 1-character of the $\pi$-Hall subgroup $H$. If $H$ is normal in $G$ then the degree of each irreducible component of $1(H)_G$ divides the index of $H$. In section 4, the converse is proved in the case in which $G$ is $\pi$-solvable and in the case in which $G$ has a nilpotent $\pi'$-Hall subgroup.

## 2. Notation and Preliminary Results

In what follows, group means finite group. For a character $\psi$ of a subgroup $H$ of a group $G$, $\psi_G$ denotes the character of $G$ induced by $\psi$ (for a definition, see [1]). Induction has the following properties:

(i) If $H \subset K \subset G$, then $(\psi_K)_G = \psi_G$.

(ii) If $\chi$ is a character of $G$, then $(\psi_G, \chi)_G = (\psi, \chi)_H$.

If $\chi_1$ and $\chi_2$ are characters of $G$, then $\chi_2 \in \chi_1$ means that there is a character $\chi_3$ such that $\chi_1 = \chi_2 + \chi_3$. For example, $\psi \in \psi_G | H$ for each character $\psi$ of the subgroup $H$ of $G$.

The 1-character of $G$ is denoted by $1(G)$.

Received November 18, 1961.



https://doi.org/10.1017/S0027763000023849 Published online by Cambridge University Press

The degree of the representation with character $\chi$ is denoted by $f_\chi$. A linear character is a character of degree 1.

If $\phi$ is a matrix representation, then det $\phi$ is a linear character. Equivalent representations have the same determinant, so we may put det $\phi = N(\chi)$, where $\chi$ is the character of $\phi$. For each linear character $\omega$, we have $N(\omega\chi) = \omega^{f_\chi} N(\chi)$.

If $\pi$ is a set of prime numbers and $f$ is a rational integer, then $f$ divides $\pi$ means that each prime factor of $f$ is in $\pi$. A $\pi$-group is a group whose order divides $\pi$. A group is $\pi$-solvable if each of its composition factors is either a $\pi$-group or a $\pi'$-group, where $\pi'$ is the complimentary set of prime numbers. A $\pi$-Hall subgroup of a group $G$ is a $\pi$-subgroup whose index in $G$ divides $\pi'$.

In what follows, character means irreducible character.

We will need the following results due to Clifford [3]:

THEOREM 1. *Let $\psi$ be a character of the normal subgroup $N$ of $G$ and let $T$ be the subgroup fixing $\psi$. Then for each character $\chi \in \psi_G$ there is a unique character $\zeta$ of $T$ such that $\zeta \in \psi_T$ and $\chi \in \zeta_G$. For this $\zeta$, $\zeta | N = a\psi$ for some positive integer $a$, and $\zeta_G = \chi$.*

*Proof*. Given $\chi \in \psi_G$, choose a character $\zeta$ of $T$ such that $\zeta \in \psi_T$ and $\chi \in \zeta_G$. Then $\zeta | N \in \psi_T | N = (T:N)\psi$, so $\zeta | N = a\psi$ for some positive integer $a$.

For $\rho \notin T$, $(\zeta^\rho, \zeta)_N = a_2(\psi^\rho, \psi)_N = 0$. Therefore $(\zeta_G, \zeta_G)_G = (\zeta_G, \zeta)_T = \sum_{\rho \bmod T} (\zeta^\rho, \zeta)_{T \cap T_\rho} (T : T \cap T^\rho)^{-1} = 1$. It follows that $\zeta_G = \chi$.

Since $\chi | T = \zeta + \zeta_1$, where $\psi \notin \zeta_1 | N$, the only character of $T$ with the required properties is $\zeta$.

COROLLARY. *With the same hypotheses, $(G:T)f_\psi$ divides $f_\chi$.*

THEOREM 2. *With the same hypotheses, if $\psi$ extends to a character $\psi_1$ of $T$, then as $\omega$ ranges over the characters of $T/N$, $(\omega\psi_1)_G$ ranges over distinct characters of $G$, and*

$$\psi_G = \sum_\omega f_\omega \cdot (\omega\psi_1)_G \tag{1}$$

*Proof*. Both $\psi_T$ and $\sum f_\omega \cdot \omega\psi_1$ are 0 off $N$ and restrict on $N$ to $(T:N)\psi$. Therefore they are equal. Inducing to $G$, we have (1). From (1) it follows that

$$(T:N) = (\psi_G, \psi)_N = (\psi_G, \psi_G)_G = \sum_{\omega, \omega'} f_\omega f_{\omega'} ((\omega\psi_1)_G, (\omega'\psi_1)_G)_G.$$

The diagonal terms alone contribute at least $\sum f_\omega^2 = (T:N)$, so the $(\omega\psi_1)_G$ are irreducible and distinct.

COROLLARY. *If $N$ is normal in $G$ and $\psi_1$ is a fixed extension of the character $\psi$ of $N$ to $G$, then each extension of $\psi$ to $G$ is uniquely of the form $\omega\psi_1$ where $\omega$ is a linear character of $G/N$.*

THEOREM 3. *Let $N$ be a normal subgroup of prime index $p$ in $G$. Then each invariant character of $N$ extends to a character of $G$.*

*Proof.* Let $\psi$ be an invariant character of $N$. Let $\chi$ be a character in $\psi_G$. By Theorem 1, $\chi | N = a\psi$. For each character $\omega$ of $G/N$, $\omega\chi$ is a character of $G$ and $\omega\chi | N = a\psi$. The characters $\omega\chi$ are distinct. In fact, if $\omega\chi = \omega'\chi$ for $\omega \neq \omega'$, then $\chi = 0$ off $N$, so $a^2 = (\chi, \chi)_N = p(\chi, \chi)_G = p$, a contradiction. Therefore

$$\sum_\omega a \cdot \omega\chi \in \psi_G.$$

Since $(\psi_G, \psi_G)_G = (\psi_G, \psi)_N = p$, this implies that $a = 1$. Thus $\chi$ is an extension of $\psi$ to $G$.

## 3. Extension of Characters from Normal Subgroups

An elementary group is a direct product of a cyclic group with a group of prime-power order. According to a theorem of Brauer [1], each character of a group $G$ may be written in the form $\sum a_\xi \xi_G$ where $\xi$ ranges over all linear characters of all elementary subgroups $H$ of $G$ and the coefficients $a_\xi$ are rational integers. Using Brauer's theorem we will find conditions under which an invariant character of a normal subgroup can be extended to a character of the whole group.

THEOREM 4. *Let $N$ be a normal subgroup of $G$ and let $\psi$ be a character of $N$. Suppose that for each intermediate group $H$ for which $H/N$ is elementary $\psi$ may be extended to a character $\psi(H)$ of $H$ in such a way that*

(i) $\psi(H)^\rho = \psi(H^\rho) \qquad (\rho \in G)$,

(ii) $\psi(H') = \psi(H'') | H' \qquad (H' \subset H'')$.

*Then $\psi$ may be extended to a character of $G$.*

*Proof.* By Brauer's theorem we may write

$$1(G/N) = \sum a_\xi \xi_G \tag{2}$$

https://doi.org/10.1017/S0027763000023849 Published online by Cambridge University Press

where $\xi$ ranges over all linear characters of $H/N$, for all intermediate groups $H$ for which $H/N$ is elementary, and the coefficients $a_\xi$ are rational integers. Put

$$\chi = \sum a_\xi (\xi \cdot \psi(H_\xi))_G, \tag{3}$$

where $H_\xi$ is the group of which $\xi$ is a character. Then $\chi$ is a generalized character of $G$ for which $\chi | N = \psi$.

From (2) and (3) we have

$$1 = (1(G/N),\ 1(G/N))_G = \sum a_\xi a_{\xi'} (\xi_G,\ \xi'_G)_G,$$

and

$$(\chi,\ \chi)_G = \sum a_\xi a_{\xi'} ((\xi\psi(H_\xi))_G,\ (\xi'\psi(H_{\xi'}))_G)_G$$

We shall show that corresponding inner products in these two sums are equal, proving that $(\chi, \chi)_G = 1$ and hence that $\chi$ or $-\chi$ is a character, from which since $\chi | N = \psi$, it will follow that $\chi$ is a character.

For any two characters $\zeta$ and $\zeta'$ of subgroups $H$ and $H'$ of $G$,

$$(\zeta_G,\ \zeta'_G)_G = (\zeta_G,\ \zeta')_{H'} = \sum_{\rho \bmod H} (\zeta^\rho,\ \zeta')_{H' \cap H^\rho} (H' : H' \cap H^\rho)^{-1}.$$

Thus it is enough, in view of (i) and (ii), to show that for each intermediate subgroup $H$ for which $H/N$ is elementary, and each pair of linear characters $\omega$ and $\omega'$ of $H/N$,

$$(\omega\psi(H),\ \omega'\psi(H))_H = (\omega,\ \omega')_H.$$

This is true since by the corollary to Theorem 2, $\omega\psi(H)$ and $\omega'\psi(H)$ are equal only for $\omega = \omega'$.

As an application of Theorem 4, we will prove the following special result which may also be proved using Schur's lemma and factor sets.

THEOREM 5. *Let $N$ be a normal subgroup of $G$ and let $\psi$ be a character of $N$ such that*

(a) *$\psi$ is invariant under $G$,*

(b) *$N(\psi)$ extends to a character $\xi$ of $G$,*

(c) *$f_\psi$ is prime to $(G:N)$.*

*Then there is a unique character $\chi$ of $G$ such that $\chi | N = \psi$ and $N(\chi) = \xi$.*

*Proof.* If $\psi$ extends to a character of $G$, there is a unique extension $\chi$ such that $N(\chi) = \xi$. In fact, all extensions are given by $\omega\chi_1$ where $\chi_1$ is any

https://doi.org/10.1017/S0027763000023849 Published online by Cambridge University Press

one extension and $\omega$ ranges over the linear characters of $G/N$. Since $\xi N(\chi_1)^{-1}$ is a linear character of $G/N$ and $f_\psi$ is prime to $(G:N)$, there is a unique linear character $\omega$ of $G/N$ for which $\omega^{f_\psi} = \xi N(\chi_1)^{-1}$, i.e., for which $N(\omega\chi_1) = \omega^{f_\psi} N(\chi_1) = \xi$.

To prove the existence of an extension, suppose first that $G/N$ is supersolvable. If $G = N$, put $\chi = \psi$. If $G \neq N$, let $K/N$ be a normal subgroup of prime order of $G/N$. Since $\psi$ is invariant under $K$, Theorem 3 shows that $\psi$ has an extension to a character of $K$. By the last paragraph, $\psi$ has a unique extension $\psi_1$ for which $N(\psi_1) = \xi | K$. For each $\rho \in G$, $\psi_1^\rho$ is an extension of $\psi$ to $K$ for which $N(\psi_1^\rho) = \xi | K$. Thus $\psi_1$ is invariant under $G$. By induction, $\psi_1$ extends to a character of $G$.

To complete the proof in the general case, we note that elementary subgroups are supersolvable. Hence there are unique extensions $\psi(H)$ of $\psi$ to the intermediate groups $H$ for which $H/N$ is elementary such that $N(\psi(H)) = \xi | H$. Properties (i) and (ii) of these extensions follow easily from this uniqueness and the invariance of $\xi$ under conjugation.

LEMMA 1. *Let $G = HN$ where $H$ is a subgroup of $G$ and $N$ is a normal subgroup of $G$ and $H \cap N = 1$. Then each linear character of $N$ which is invariant under $G$ extends to a character of $G$.*

*Proof.* Let $\delta$ be an invariant linear character of $N$. Put $\xi(\sigma\tau) = \delta(\tau)$ for $\sigma \in H$, $\tau \in N$. Then $\xi | N = \delta$ and $\xi$ is a linear character of $G$, since for $\sigma_1, \sigma_2 \in H$ and $\tau_1, \tau_2 \in N$,

$$\xi(\sigma_1\tau_1\sigma_2\tau_2) = \xi(\sigma_1\sigma_2\tau_1^{\sigma_2}\tau_2) = \delta(\tau_1^{\sigma_2}\tau_2)$$
$$= \delta(\tau_1^{\sigma_2})\delta(\tau_2) = \delta(\tau_1)\delta(\tau_2) = \xi(\sigma_1\tau_1)\xi(\sigma_2\tau_2).$$

THEOREM 6. *If $N$ is a normal Hall subgroup of $G$, then each invariant character of $N$ extends to a character of $G$.*

*Proof.* Let $\psi$ be an invariant character of $N$. Since $f_\psi$ divides $(N:1)$, $f_\psi$ is prime to $(G:N)$. By a theorem of Schur [5, p. 162], there is a subgroup $H$ such that $G = HN$ and $H \cap N = 1$. Therefore $N(\psi)$, which is also invariant, extends to a linear character of $G$. Consequently $\psi$ extends to a character of $G$, by Theorem 5.

THEOREM 7. *Let $N$ be a normal Hall subgroup of $G$. Pick a character $\psi$*

https://doi.org/10.1017/S0027763000023849 Published online by Cambridge University Press

*of $N$, let $\psi_1$ be an extension of $\psi$ to the group $T$ fixing $\psi$, and let $\omega$ be a character of $T/N$. Then $(\omega\psi_1)_G$ is a character of $G$, and each character of $G$ is obtained in this way.*

*Proof.* By Theorem 6, $\psi_1$ exists. By Theorem 2, each $(\omega\psi_1)_G$ is a character of $G$ and each character $\chi \in \psi_G$ is obtained in this way. For each character $\chi$ of $G$ there is a $\psi$ such that $\chi \in \psi_G$ so $\chi$ is of the form $(\omega\psi_1)_G$ for some $\psi$ and some $\omega$.

## 4. Conditions Implying a Hall Subgroup is Normal

It was proved by N. Ito [4] that if the degree of each character of a solvable group is prime to $p$, then the group has an abelian normal $p$-Sylow subgroup. In this section we prove an analogous result in which the conclusion is "$G$ has a normal $\pi$-Hall subgroup" and the hypothesis "$p \nmid f_\chi$, for all characters $\chi$" is replaced by

$\alpha$: $G$ has a $\pi$-Hall subgroup $H$ and the degree of each character in $1(H)_G$ divides $\pi'$.

LEMMA 2. *If $G$ satisifes $\alpha$ and $N \lhd G$, then $N$ and $G/N$ also satisfy $\alpha$.*

*Proof.* Let $G$ satisfy $\alpha$ relative to the $\pi$-Hall subgroup $H$ and let $N \lhd G$. Then $H \cap N$ is a $\pi$-Hall subgroup of $N$ and $HN/N$ is a $\pi$-Hall subgroup of $G/N$.

Since $\psi \in \psi_G | N$ for each character $\psi$ of $N$, $\psi \in 1(H \cap N)_N$ implies $1(H \cap N) \in \psi | H \cap N \in \psi_G | H \cap N$ and hence

$$(\psi,\ 1(H)_G)_N = (\psi_G,\ 1(H)_G)_G = (\psi_G,\ 1(H))_H = (\psi_G,\ 1(H \cap N))_{H \cap N}(H : H \cap N)^{-1} \neq 0.$$

Therefore for each character $\psi \in 1(H \cap N)_N$, there is a character $\chi \in 1(H)_G$ for which $\psi \in \chi | N$ and hence for which $f_\psi$ divides $f_\chi$. By hypothesis, $f_\chi$ divides $\pi'$, so $f_\psi$ also divides $\pi'$. Thus $N$ satisfies $\alpha$ relative to $H \cap N$.

Since $1(HN) \in 1(H)_{HN}$, we have

$$1(HN)_G \in (1(H)_{HN})_G = 1(H)_G.$$

If $\chi$ is a character of $G/N$ in $1(HN)_G$, then $\chi$ is a character of $G$ in $1(H)_G$. By hypothesis $f_\chi$ divides $\pi'$. Thus $G/N$ satisfies $\alpha$ relative to $HN/N$.

THEOREM 8. *If $G$ is a $\pi$-solvable group which satisfies $\alpha$, then the $\pi$-Hall subgroup $H$ is normal in $G$.*

*Proof.* Let $M$ be a minimal normal subgroup of $G$. Then $G/M$ is also a $\pi$-solvable group which satisfies $\alpha$, so by induction $G/M$ has a normal $\pi$-Hall subgroup $K/M$.

If $M$ is a $\pi$-group, then $K$ is a normal $\pi$-Hall subgroup of $G$.

If $K \neq G$, then $K$ is a $\pi$-solvable group which satisfies $\alpha$, so by induction $K$ has a normal $\pi$-Hall subgroup, which is then a normal $\pi$-Hall subgroup of $G$.

Finally, suppose $M$ is a $\pi'$-group and $K = G$. Then $G = HM$ and $H \cap M = 1$. Consequently, $1(H)_G | M = \sum f_\psi \psi$, where $\psi$ ranges over the characters of $M$. Let $\psi$ be a character of $M$. Then there is a character $\chi \in 1(H)_G$ for which $\psi \in \chi | M$. Let $T$ be the subgroup fixing $\psi$. Then $(G:T)$ divides $f_\chi$. By the hypothesis, $f_\chi$ divides $\pi'$, so $(G:T)$ divides $\pi'$.

Since $(G:T)$ divides $(H:1)$, which divides $\pi$, $(G:T) = 1$. Thus each character $\psi$ of $M$ is invariant under $G$. It follows that $H$, in its action on $M$ by conjugation, preserves the $M$-classes of $M$. By a theorem of Burnside [2, p. 89], $H$ centralizes $M$. Thus $G = H \times M$, so $H \lhd G$.

Lemma 3. *Let $A$ be an abelian group, $B$ a subgroup of $A$, and $\chi$ a possibly reducible character of $A$ such that $\chi = 0$ off $B$. Then $(A:B)$ divides $f_\chi$.*

*Proof.* Let $\chi = \sum a(\xi)\xi$ be the decomposition of $\chi$ into linear characters. For each character $\omega$ of $A/B$, we have $\omega\chi = \chi$, so $a(\omega\xi) = a(\xi)$. Also, $\omega\xi = \omega'\xi$ only for $\omega = \omega'$. Hence

$$\chi = \sum_{\xi}{}' a(\xi) \sum_{\omega} \omega\xi = \sum_{\omega} \omega \sum_{\xi}{}' a(\xi)\xi,$$

where the prime indicates that the summation is only over some $\xi$. It follows that $f_\chi = (A:B)\sum' a(\xi)$.

For a character $\chi$ of a group $G$, denote by $Z(\chi)$ the normal subgroup on which $|\chi| = f_\chi$. Denote by $Z(K)$ the center of a group $K$.

Lemma 4. *Suppose that the group $G$ has a Hall subgroup $K$ and let $\chi$ be a character of $G$ such that $f_\chi$ divides $(K:1)$. Then $(Z(K):Z(K)\cap Z(\chi))$ divides $f_\chi$.*

*Proof.* For $\sigma \in Z(K)$, the centralizer of $\sigma$ contains $K$, so the number of elements in the class of $\sigma$ is prime to $(K:1)$ and hence to $f_\chi$. By a theorem of Burnside [2, p. 322], either $\sigma \in Z(\chi)$ or $\chi(\sigma) = 0$. Thus $\chi | Z(K)$ is 0 off $Z(K) \cap Z(\chi)$, and the result now follows by Lemma 3.

THEOREM 9. *If $G$ satisfies $\alpha$ relative to a $\pi$-Hall subgroup $H$ and $G$ has a nilpotent $\pi'$-Hall subgroup $K$, then $H$ is normal in $G$.*

*Proof.* It is enough by Theorem 8 to prove that $G$ is $\pi$-solvable. Since the hypotheses of Theorem 9 carry over to normal subgroups and factor groups, it is enough to show that if $G$ has no proper normal subgroups, then $K=1$.

If $G$ has no proper normal subgroups, then $Z(\chi)=1$ for all characters $\chi$ of $G$ except $1(G)$. Lemma 4 then shows that $(Z(K):1)$ divides $f_\chi$ for all $\chi \in 1(H)_G$ except $1(G)$. Putting

$$1(H)_G = 1(G) + \sum a_\chi \chi,$$

we then have

$$(K:1) = (G:H) = 1 + \sum a_\chi f_\chi \equiv 1 \text{ mod } (Z(K):1).$$

This implies that $Z(K)=1$. Since $K$ is nilpotent, $K=1$.

## REFERENCES


[1] Brauer, R. and Tate, J.: On the characters of finite groups. Ann. of Math. **62**, 1–7 (1955).

[2] Burnside, W.: Theory of Groups of Finite Order. Second Edition, Cambridge Univ. Press. 1911.

[3] Clifford, A. H.: Representations induced in an invariant subgroup. Ann. of Math. **38**, 533–550 (1937).

[4] Ito, N.: Some studies on group characters. Nagoya Math. Journal **2**, 17–28 (1951).

[5] Zassenhaus, H.: The Theory of Groups. Second Edition, Chelsea Publishing Co. 1958.


*Massachusetts Institute of Technology*
*and Paris, France*