GLOBAL SUPERVENIENCE IN INQUISITIVE MODAL LOGIC

IVANO CIARDELLI

Department of Philosophy, Sociology, Pedagogy, and Applied Psychology,
Università di Padova

Abstract. The notion of *global supervenience* captures the idea that the overall distribution of certain properties in the world is fixed by the overall distribution of certain other properties. A formal implementation of this idea in constant-domain Kripke models is as follows: predicates $Q_1, \dots, Q_m$ globally supervene on predicates $P_1, \dots, P_n$ in world w if two successors of w cannot differ with respect to the extensions of the Q_i without also differing with respect to the extensions of the P_i. Equivalently: relative to the successors of w, the extensions of the Q_i are functionally determined by the extensions of the P_i. In this paper, we study this notion of global supervenience, achieving three things. First, we prove that claims of global supervenience cannot be expressed in standard modal predicate logic. Second, we prove that they can be expressed naturally in an inquisitive extension of modal predicate logic, where they are captured as strict conditionals involving questions; as we show, this also sheds light on the logical features of global supervenience, which are tightly related to the logical properties of strict conditionals and questions. Third, by making crucial use of the notion of *coherence*, we prove that the relevant system of inquisitive modal logic is compact and has a recursively enumerable set of validities; these properties are non-trivial, since in this logic a strict conditional expresses a second-order quantification over sets of successors.

§1. Introduction.

Many important debates in analytic philosophy revolve around claims of *supervenience*. Among the different notions of supervenience considered in the literature, an especially natural one is the notion of *global supervenience* [25]. The idea is that a class of properties $\mathcal{B}$ globally supervenes on a class of properties $\mathcal{A}$ if the overall distribution of the $\mathcal{B}$-properties in the world is fully determined (metaphysically, nomically, or in some other way) by the overall distribution of the $\mathcal{A}$-properties. Although the focus is traditionally on properties, an insightful example of global supervenience can be given if we allow ourselves to consider binary relations: the relation *being a grandparent of* globally supervenes on the relation *being a parent of*, as the overall distribution of parent-of relations in the world fully determines the overall distribution of grandparent-of relations (but not the other way around).

In this paper, we shall be interested in global supervenience from a logical perspective. Thus, instead of focusing directly on properties and relations, we will focus on their linguistic counterparts: predicates. In the setting of constant-domain Kripke models, a natural notion of global supervenience among predicates can be defined: predicates $Q_1, \dots, Q_m$ globally supervene on predicates $P_1, \dots, P_n$ at a world w if two worlds that

Received: May 2, 2024.
2020 *Mathematics Subject Classification*: Primary 03B45.
Key words and phrases: supervenience, inquisitive logic, strict conditional, functional dependence.

doi:10.1017/S175502032500005X

are possible from the point of view of w cannot differ in the extensions of the former predicates without also differing in the extensions of the latter. As a special case, a single predicate Q globally supervenes on a single predicate P at w in case two successors of w cannot differ on the extension of Q without also differing on the extension of P.

This modal notion is a very natural one, both from a philosophical and from a purely mathematical point of view. It thus seems relevant to ask whether global supervenience claims can be expressed in some system of modal logic, so that we can formally regiment and assess inferences involving such claims. Such inferences can be diverse and non-trivial, as illustrated by the following examples:

(1) Necessarily, x is a grandparent of y just in case x is a parent of a parent of y.
Therefore, the grandparent-of relation globally supervenes on the parent-of relation.

(2) For every x, the property *being a sibling of* x globally supervenes on the properties *being a brother of* x and *being a sister of* x.
Therefore, the *sibling-of* relation globally supervenes on the *brother-of* and *sister-of* relations.

Or, moving now to argument schemata for simplicity:

(3) P globally supervenes on Q and R.
P does not globally supervene on R.
Therefore, Q does not globally supervene on R.

(4) Q globally supervenes on P.
It is contingent whether there are any Q.
Therefore, it is not necessarily the case that every object is P.

The contribution of this paper is threefold. First, we show that global supervenience claims cannot be regimented in standard modal predicate logic QML, the extension of first-order predicate logic by modalities $\Box$ and $\Diamond$. In fact, even the most basic example of a global supervenience claim, namely, the claim that a single predicate Q globally supervenes on a single predicate P, cannot be expressed by a QML-formula.

Second, we show that global supervenience claims can be regimented naturally in an extension of modal predicate logic based on inquisitive semantics [12]. The language of this system includes formulas corresponding to questions; for instance, if P is a unary predicate, we have a formula $\forall x?Px$ representing the question *which objects are* P. The modal operator $\Box$ is allowed to apply to arbitrary formulas in the language. In this logic, global supervenience claims can be expressed as strict conditionals having as their antecedent the question about the extension of the subvenient predicates, and as their consequent the question about the extension of the supervenient predicates. Thus, e.g., the global supervenience of Q on P will be expressed by the strict conditional

$$\Box(\forall x?Px \to \forall x?Qx)$$

having the question *which objects are* P as its antecedent, and the question *which objects are* Q as its consequent. We will discuss how this regimentation sheds light on the logical properties of global supervenience claims, which can be traced back to the logical properties of strict conditionals and those of the relevant questions.

Third, we will prove some key meta-theoretic results about our inquisitive modal predicate logic. In particular, we will show that the logic is recursively enumerable and compact; this is non-trivial, since the semantics of implication in inquisitive logic involves a second-order quantification over sets of worlds, which could in principle lead to a logic with essentially second-order features. The proof strategy combines ideas from Meißner & Otto [30] and from Ciardelli & Grilletti [10], making crucial use of the technical notion of *coherence* [26] to give a translation of our modal logic into classical two-sorted first-order logic. Essentially, we will show that the second-order quantification over sets of worlds introduced by implication can be traded in each particular formula for a sequence of first-order quantifications over single worlds, the length of which depends on the particular formula at hand.

The paper may be seen as contributing to three different lines of work: work on the logical analysis of supervenience; work on inquisitive modal logic; and work on logics of dependence.

From the point of view of the literature on the logical analysis of supervenience, what this paper contributes is a modal logic capable of regimenting inferences involving global supervenience. In previous work, Lloyd Humberstone investigated some general properties of supervenience relations [21, 22] and connected a version of individual supervenience to a version of definability [23]. More recently, Goranko & Kuusisto [16] as well as Fan [14] (building again on Humberstone [24]) introduced modal logics of supervenience, but they focused on supervenience between the truth-values of propositions. This kind of supervenience may be regarded as a special case of the general notion of global supervenience considered in this paper (since a proposition may be regarded as a relation with arity 0). But this special case is *very* special, because propositions may only have two extensions at a world (truth and falsity), which allows one to list all the possible ways in which the supervenience may be realized (each "way" corresponding to a particular truth-function) and thereby to treat the logic of supervenience in a purely combinatorial fashion; this is a non-starter in the case of supervenience between relations of arity $n \geq 1$, for which the number of possible extensions at a world is not fixed *a priori*, and is possibly infinite.

From the viewpoint of the literature on inquisitive modal logic, the main finding is an important, and perhaps surprising, difference between the propositional setting and the predicate logic setting. In the propositional setting, the possibility of applying the modality $\Box$ to inquisitive formulas, while interesting for various reasons (e.g., it allows a unified account of knowledge ascriptions: see [7, 11]), does not increase the expressive power of the logic. Indeed, it was proved by Ciardelli [7] that in that setting, every modal formula of the form $\Box\varphi$, where φ is allowed to contain inquisitive operators, is equivalent to some Boolean combination of standard modal logic formulas: for instance, the modal formula $\Box ?p$, expressing the fact that all successors agree on the truth-value of p, is equivalent to the formula $\Box p \vee \Box\neg p$ of standard modal logic. Our results in this paper imply that things are different when we add the modality $\Box$ to inquisitive *predicate* logic: in particular, the strict conditional $\Box(\forall x ?Px \to \forall x ?Qx)$ is not equivalent to any formula in standard modal predicate logic. Thus, in this setting allowing $\Box$ to apply to inquisitive formulas leads to a more expressive modal logic—one that can regiment claims over and above those expressible in standard modal predicate logic.

Finally, from the point of view of the recent literature on the logic of dependence (see, among others, [3, 15, 36]), this paper considers a salient but under-explored variety of functional dependence: while most existing work focuses on functional dependence between the values of variables, our focus is on functional dependence between the extensions of predicates. We explore this notion guided by the idea, discussed in detail by Ciardelli [6], that dependence is intimately related to question entailment, and thus naturally regimented in a logical framework equipped with the resources to express questions.

The paper is structured as follows: in Section 2 we introduce the target notion of global supervenience and discuss some of its basic features; in Section 3 we prove that global supervenience claims are not expressible in standard modal predicate logic; in Sections 4 and 5 we introduce an inquisitive modal predicate logic and show that in this logic, global supervenience claims are expressible as strict implications among questions; we also discuss how this analysis sheds light on the logical features of global supervenience; in Section 6 we prove that our inquisitive modal logic is compact and has a recursively enumerable set of validities; Section 7 concludes the paper, outlining several directions for future work.

§2. Global supervenience. Supervenience claims are at the heart of many debates in various areas of philosophy, from metaphysics to philosophy of science, philosophy of mind, and metaethics. The general idea behind the notion of supervenience is as follows: given two classes of properties $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{B}$ supervenes on $\mathcal{A}$ if there cannot be a difference with respect to $\mathcal{B}$-properties without a corresponding difference in $\mathcal{A}$-properties. This idea can be made more precise in different ways. One approach focuses on individuals: on this understanding, a supervenience relation holds when two *individuals* cannot differ in their $\mathcal{B}$-properties without also differing in their $\mathcal{A}$-properties; this leads to various notions of *individual* supervenience (*weak* or *strong*, depending on whether we compare the two individuals in the same possible world, or across different worlds). Another approach focuses on worlds as a whole: on this more *global* understanding, supervenience holds when two *worlds* cannot differ in the overall distribution of the $\mathcal{B}$-properties without also differing in the overall distribution of the $\mathcal{A}$-properties. For an equivalent formulation, let us call two worlds $\mathcal{A}$- (or $\mathcal{B}$-) *indiscernible* when they do not differ in the distribution of the $\mathcal{A}$ (or $\mathcal{B}$) properties. Then we can use the phrasing originally used by Kim [25]: $\mathcal{B}$ globally supervenes on $\mathcal{A}$ in case any two possible worlds that are $\mathcal{A}$-indiscernible are also $\mathcal{B}$-indiscernible.

This second, global notion of supervenience can in turn be further specified in different ways. Different formalizations have been discussed in the literature (see [4, 27, 29, 34, 35]), and it has been debated which of them, if any, best captures certain supervenience theses. The difficulty with giving a precise definition stems from the fact that, when different worlds come with different domains of individuals, it is not obvious what it means for two worlds to be $\mathcal{A}$- (or $\mathcal{B}$-) indiscernible; as Leuenberger [27] emphasizes, this depends crucially on how individuals are identified across worlds. While this issue is important, if our intention is to study the modal logic of global supervenience, it seems advisable to start by setting aside the complications involved with cross-world identification, focusing first on the case in which the domain of individuals is simply fixed across worlds.[1] In this setting, there is an obvious way to cash out the idea of indiscernibility: two worlds are $\mathcal{A}$-indiscernible if they agree on

[1] Additionally, note that the idea of global supervenience is meaningful not just when the relevant notion of possibility is metaphysical, but for many other notions of possibility as well (epistemic, deontic, historical, etc.) For many applications, the domain of individuals can indeed be held fixed. For instance, talking about the students in a given class, the property *being the tallest* globally supervenes on properties of the form *having height x*; intuitively,

the extension of all the $\mathcal{A}$-properties (and similarly for $\mathcal{B}$).[2] As a result, the claim that $\mathcal{B}$ globally supervenes on $\mathcal{A}$ means that whenever two possible worlds agree on the extension of all $\mathcal{A}$-properties, they also agree on the extension of all $\mathcal{B}$-properties.

This notion of supervenience can be made mathematically precise in the setting of constant-domain Kripke models. Let us recall the relevant definition and fix some notation.

Definition 2.1 (Constant-domain Kripke models). A constant-domain Kripke model is a tuple $M = \langle W, D, R, I \rangle$ where $W \neq \emptyset$ is the universe of possible worlds, $D \neq \emptyset$ the domain of individuals, $R \subseteq W \times W$ the accessibility relation, which determines for each $w \in W$ the set of worlds $R[w] = \{v \in W \mid wRv\}$ which are possible relative to w, and I an interpretation function which, relative to each $w \in W$, assigns to each n-ary predicate symbol P in the language a set of n-tuples $I_w(P) \subseteq D^n$. For readability, we write P_w for $I_w(P)$.[3]

Relative to a Kripke model, a unary predicate P expresses a property, viewed as a function that maps each world w to a corresponding extension P_w. We could thus define a notion of global supervenience as a relation between sets of unary predicates. In fact, however, it is natural to give a more general definition, which applies to predicates of arbitrary arity (cf. [28]). For instance, as mentioned in the introduction, we would want to say that the binary predicate *being a grandparent of* globally supervenes on the binary predicate *being a parent of*, since the extension of the latter fully determines the extension of the former: once we fix who is a parent of whom, that determines who is a grandparent of whom. We will thus define global supervenience as a relation between sets of arbitrary predicates.[4]

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this claim only requires us to consider situations in which the same students have different heights—it need not involve situations in which the set of students is different.

[2] Interestingly, this is also the understanding that Kim seems to have had in mind in the original paper where the term *global supervenience* is introduced [25]. Using the class of all psychological properties as an example, he writes:

> to say that two worlds are psychologically indiscernible is to say that for every psychological property $\mathcal{P}$ and every individual x, x has $\mathcal{P}$ in one just in case x has $\mathcal{P}$ in the other.

Note that the quantification over individuals is not relativized to a particular world, and so a fixed domain of individual seems to be presupposed. It is clear from the context that this example is only intended to illustrate the general idea that, for an arbitrary class of properties $\mathcal{A}$:

> to say that two worlds are $\mathcal{A}$-indiscernible is to say that for every property $\mathcal{P}$ in $\mathcal{A}$ and every individual x, x has $\mathcal{P}$ in one world just in case x has $\mathcal{P}$ in the other.

This is logically equivalent to saying that two worlds are $\mathcal{A}$-indiscernible if they agree on the extension of each $\mathcal{A}$-property.

[3] For simplicity, we assume throughout the paper that our language contains only predicates as non-logical symbols. Everything we say generalizes to the case in which the language contains function symbols, in which case the interpretation function will have to interpret these symbols as well.

[4] The definition can be further extended to the case in which $\mathcal{A}$ and $\mathcal{B}$ are sets of open formulas $\alpha(x_1, \dots, x_n)$ of modal predicate logic, as such formulas, like predicates, also define n-ary

 https://doi.org/10.1017/S175502032500005X

Definition 2.2 (Global supervenience). Let $\mathcal{A}, \mathcal{B}$ be sets of predicate symbols. We represent the claim that $\mathcal{B}$ globally supervenes on $\mathcal{A}$ by the notation:

$$\mathcal{A} \rightsquigarrow \mathcal{B}.$$

We refer to the predicates in $\mathcal{B}$ as the *supervenient* predicates, and to the predicates in $\mathcal{A}$ as the *subvenient* predicates. For readability, we drop the brackets and write $P_1, \dots, P_n \rightsquigarrow Q_1, \dots, Q_m$ instead of $\{P_1, \dots, P_n\} \rightsquigarrow \{Q_1, \dots, Q_m\}$.

Relative to a world w in a constant-domain Kripke model $M = \langle W, D, R, I \rangle$, the supervenience claim $\mathcal{A} \rightsquigarrow \mathcal{B}$ is true if any two successors that agree on the extension of all $\mathcal{A}$-predicates also agree on the extension of all $\mathcal{B}$-predicates. In symbols:

$$\begin{aligned} w \models \mathcal{A} \rightsquigarrow \mathcal{B} \iff &\forall v, u \in R[w] : \\ &(P_v = P_u \text{ for all } P \in \mathcal{A}) \text{ implies } (Q_v = Q_u \text{ for all } Q \in \mathcal{B}). \end{aligned}$$

The special case of the claim $P \rightsquigarrow Q$, involving just one subvenient predicate P and one supervenient predicate Q, will play a prominent role below. Therefore, we spell out its truth-conditions explicitly:

$$w \models P \rightsquigarrow Q \iff \forall v, u \in R[w] : P_v = P_u \text{ implies } Q_v = Q_u$$

Let us illustrate the previous definition by means of an example.

Example 2.3. Suppose our language contains three unary predicates, P, Q, R. Let $\mathbb{N}$ be the set of natural numbers, and let E, O be the sets of even and odd numbers respectively. Consider a model where the domain of individuals is $\mathbb{N}$, the universe is $W = \{v_X \mid X \subseteq \mathbb{N}\}$, the accessibility relation is $R = W \times W$, and the interpretation function is given by:

$$P_{v_X} = X \qquad Q_{v_X} = X \cap E \qquad R_{v_X} = X \cap O.$$

As the reader is invited to check, at any world w in this model, we have:

- Q and R both globally supervene on P: $w \models P \rightsquigarrow Q$ and $w \models P \rightsquigarrow R$;
- P does not globally supervene on either Q or R: $w \not\models Q \rightsquigarrow P$ and $w \not\models R \rightsquigarrow P$;
- P does globally supervene on both Q and R: $w \models Q, R \rightsquigarrow P$.

Global supervenience and functional dependency. From a logical point of view, global supervenience is a form of functional dependency. For instance, $P \rightsquigarrow Q$ means that, relative to the successors of the evaluation world, the extension of Q is functionally determined by the extension of P. More precisely, we have:

$$w \models P \rightsquigarrow Q \iff \text{there exists a function } f \text{ such that } \forall v \in R[w] : Q_v = f(P_v).$$

More generally, if $\mathcal{A}$ and $\mathcal{B}$ are sets of predicates, $\mathcal{A} \rightsquigarrow \mathcal{B}$ means that, relative to the successors of the evaluation world, the extensions of the $\mathcal{B}$-predicates are functionally determined by the (joint) extensions of all the $\mathcal{A}$-predicates. To make this precise, let us define the extension of $\mathcal{A}$ at a world w (notation: $\mathcal{A}_w$) to be the function mapping each predicate $P \in \mathcal{A}$ to the corresponding extension P_w, and similarly for $\mathcal{B}$. Then we have:

$$w \models \mathcal{A} \rightsquigarrow \mathcal{B} \iff \text{there exists a function } f \text{ such that } \forall v \in R[w] : \mathcal{B}_v = f(\mathcal{A}_v).$$

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relations. We focus on predicate symbols for simplicity, but everything we will say generalizes straightforwardly to open formulas.

The fact that global supervenience is a form of functional dependency is reflected by the fact that it satisfies Armstrong's famous axioms for functional dependency [2]. Defining logical entailment ($\Phi \models \psi$), validity ($\models \varphi$), and equivalence ($\varphi \equiv \psi$) in the obvious way with respect to all constant-domain Kripke models, Armstrong's axioms correspond to the following logical facts:

- Reflexivity: if $\mathcal{B} \subseteq \mathcal{A}$ then $\models (\mathcal{A} \rightsquigarrow \mathcal{B})$;
- Augmentation: $(\mathcal{A} \rightsquigarrow \mathcal{B}) \models (\mathcal{A} \cup \mathcal{C} \rightsquigarrow \mathcal{B} \cup \mathcal{C})$ for any $\mathcal{C}$;
- Transitivity: $(\mathcal{A} \rightsquigarrow \mathcal{B}), (\mathcal{B} \rightsquigarrow \mathcal{C}) \models (\mathcal{A} \rightsquigarrow \mathcal{C})$.

Global supervenience and definability. Global supervenience is also tightly connected to the notion of definability. Recall the standard notion of definability in first-order logic: a first-order theory Γ (a set of first-order sentences) defines an k-ary predicate Q in terms of predicates $P_1, \dots, P_n$ iff there is a formula $\varphi(\overline{x})$ with k free variables, in the language including only $P_1, \dots, P_n$ and identity, such that $\Gamma \models \forall \overline{x}(Q\overline{x} \leftrightarrow \varphi(\overline{x}))$. Beth's theorem states that Γ defines Q in terms of $P_1, \dots, P_n$ iff any two models of Γ that have the same domain and assign the same extension to $P_1, \dots, P_n$ also assign the same extension to Q. It then is only a small step to show the following connection between definability and global supervenience.[5]

PROPOSITION 2.4 (Global supervenience and definability). *Let Γ be a set of (non-modal) first-order sentences and let $\Box\Gamma = \{\Box\gamma \mid \gamma \in \Gamma\}$. The following are equivalent:*

1. *Γ defines Q in terms of $P_1, \dots, P_n$;*
2. *$\Box\Gamma \models (P_1, \dots, P_n \rightsquigarrow Q)$.*

Proof. Note that with every world w of a constant domain Kripke model $M = \langle W, D, R, I \rangle$ we can associate a corresponding first-order structure $\mathcal{M}_w = \langle D, I_w \rangle$. For a (non-modal) sentence γ of first-order logic, truth at w in M according to standard Kripke semantics simply coincides with truth in the structure $\mathcal{M}_w$.

1⇒2 Suppose Γ defines Q from $P_1, \dots, P_n$. Consider a constant-domain Kripke model and a world w that satisfies $\Box\Gamma$. We claim that $w \models (P_1, \dots, P_n \rightsquigarrow Q)$. To see this, consider two successors $v, u \in R[w]$ that agree on the extensions of $P_1, \dots, P_n$. Since w satisfies $\Box\Gamma$, both v and u satisfy Γ; hence, the associated first-order structures $\mathcal{M}_v$ and $\mathcal{M}_u$ are two models of Γ that assign the same extensions to $P_1, \dots, P_n$; by Beth's theorem, these structures must assign the same extension to Q, which means that the worlds v and u assign the same extension to Q.

1⇒2 Suppose Γ does not define Q from $P_1, \dots, P_n$. By Beth's theorem there are two models of Γ, $\mathcal{M}_1 = \langle D, \mathcal{I}_1 \rangle$ and $\mathcal{M}_2 = \langle D, \mathcal{I}_2 \rangle$, which assign the same extension to each P_i but a different extension to Q.
We can then consider the Kripke model with two worlds w_1, w_2, constant domain D, total accessibility relation, and interpretation defined so that $I_{w_i} = \mathcal{I}_i$. Since both $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy Γ, both worlds w_1 and w_2 satisfy Γ in M, and therefore both also satisfy $\Box\Gamma$. Since w_1 and w_2 agree on $P_1, \dots, P_n$ but disagree on Q, both worlds falsify $P_1, \dots, P_n \rightsquigarrow Q$. Hence, either of these worlds provides a counterexample to the entailment $\Box\Gamma \models (Q_1, \dots, Q_n \rightsquigarrow P)$. □

[5] In a similar spirit, a connection between a notion of individual supervenience and a more demanding notion of definability is discussed by Humberstone [23].

As an illustration of the connection we just established, consider the inference in Example (1) from the introduction, repeated below:

(5) Necessarily, to be a grandparent is to be a parent of a parent.
So, the grandparent-of relation globally supervenes on the parent-of relation.

This argument can be regimented as follows:

$$\Box\forall x\forall y(Gxy \leftrightarrow \exists z(Pxz \wedge Pzy)) \;\therefore\; P \rightsquigarrow G.$$

Since the first-order formula $\forall x\forall y(Gxy \leftrightarrow \exists z(Pxz \wedge Pzy))$ defines G in terms of P, Proposition 2.4 ensures that the inference is logically valid.

Decomposing global supervenience? The global supervenience claims of the form $P_1, \dots, P_n \rightsquigarrow Q_1, \dots, Q_m$ that we introduced in this section express interesting modal propositions. It is then natural to ask whether these claims can be expressed in terms of the unary modalities $\Box$ and $\Diamond$, along with first-order quantifiers and connectives. If so, the logical properties of global supervenience claims could be analyzed in terms of the familiar properties of the logical primitives involved in the definition. In the following sections we will see that the answer to this question is negative in the context of standard modal predicate logic, but positive in the context of *inquisitive* modal predicate logic.

§3. Global supervenience is not definable in standard modal predicate logic. Consider standard modal predicate logic QML, i.e., the extension of predicate logic by modal operators $\Box$ and $\Diamond$, interpreted over constant-domain Kripke models in the usual way. In this section we show that no formula of QML expresses the claim that predicates $Q_1, \dots, Q_m$ globally supervene on predicates $P_1, \dots, P_n$. In fact, already the claim that a single unary predicate Q globally supervenes on a single unary predicate P is not expressible in QML.

THEOREM 3.1. *Let P, Q be two unary predicates. No sentence α of* QML *has the same truth-conditions as $P \rightsquigarrow Q$.*

In order to prove this theorem, we first recall the notion of bisimilarity for QML, which is a natural combination of the standard notion of bisimulation for propositional modal logic with the notion of back-and-forth equivalence for first-order predicate logic.[6]

DEFINITION 3.2 (Bisimilarity for QML) (see [37]). Let $M_1 = \langle W_1, D_1, R_1, I_1\rangle$ and $M_2 = \langle W_2, D_2, R_2, I_2\rangle$ be two constant-domain Kripke models. Let D_1^* and D_2^* be the sets of finite sequences of elements from D_1 and D_2, respectively. A relation $Z \subseteq (W_1 \times D_1^*) \times (W_2 \times D_2^*)$ is called a *bisimulation* if whenever $(w, \overline{a})Z(v, \overline{b})$ holds, the tuples $\overline{a} = (a_1, \dots, a_n)$ and $\overline{b} = (b_1, \dots, b_n)$ have the same length and the following conditions hold:

- Atomic: for all atomic formulas $\varphi(x_1, \dots, x_n)$ with free variables in $\{x_1, \dots, x_n\}$,
$$M_1, w \models \varphi(a_1, \dots, a_n) \iff M_2, v \models \varphi(b_1, \dots, b_n);$$
- $\Diamond$-forth: for every $w' \in R_1[w]$ there is a $v' \in R_2[v]$ such that $(w', \overline{a})Z(v', \overline{b})$;

[6] Essentially the same notion in the setting of intuitionistic Kripke models is studied in [31].

- $\Diamond$-back: for every $v' \in R_2[v]$ there is a $w' \in R_1[w]$ such that $(w', \overline{a})Z(v', \overline{b})$;
- $\exists$-forth: for every $a_{n+1} \in D_1$ there is a $b_{n+1} \in D_2$ such that $(w, \overline{a}a_{n+1})Z(v, \overline{b}b_{n+1})$;
- $\exists$-back: for every $b_{n+1} \in D_2$ there is a $a_{n+1} \in D_1$ such that $(w, \overline{a}a_{n+1})Z(v, \overline{b}b_{n+1})$.

We say that two worlds $w \in W_1$ and $v \in W_2$ are bisimilar (notation: $M_1, w \sim M_2, v$) if there is a bisimulation Z with $(w, \varepsilon)Z(v, \varepsilon)$, where ε is the empty sequence.

It is straightforward to show that bisimilarity implies QML-equivalence, i.e., that two bisimilar worlds satisfy the same QML-sentences.

Proposition 3.3 (Zoghifard & Pourmahdian [37]). *If $M_1, w \sim M_2, v$, for every sentence α of* QML *we have $M_1, w \models \alpha \iff M_2, v \models \alpha$.*

Equipped with this background, we are now ready to prove Theorem 3.1.[7]

Proof of Theorem 3.1. We give a model that contains two worlds w_0, w_1 which agree on the truth of all QML-sentences, and yet they disagree about the truth of $P \rightsquigarrow Q$. $\square$

Our model has the set $\mathbb{N}$ of natural numbers as its domain. Let E and O be the sets of even and odd numbers respectively, and consider the following family of subsets of $\mathbb{N}$:

$$\mathcal{X} = \{X \subseteq \mathbb{N} \mid X \cap E \text{ is finite and } X \cap O \text{ is co-finite}\}.$$

The universe of possible worlds of our model includes, in addition to w_0 and w_1, worlds of the form v_{Xi} where $X \in \mathcal{X}$ and $i \in \{0, 1\}$. At world v_{Xi}, the extension of P is X, while the extension of Q is either $\emptyset$ or $\mathbb{N}$ depending on the Boolean value i:

$$P_{v_{Xi}} = X, \qquad Q_{v_{Xi}} = \begin{cases} \mathbb{N} & \text{if } i = 1 \\ \emptyset & \text{if } i = 0. \end{cases}$$

At worlds w_0 and w_1, the extension of both predicates is empty.[8]

Next, we define a function $\tau : \mathcal{X} \to \{0, 1\}$ as follows, where $\#(X \cap E)$ denotes the cardinality of the set $X \cap E$:

$$\tau(X) = \begin{cases} 0 & \text{if } \#(X \cap E) \text{ is even} \\ 1 & \text{if } \#(X \cap E) \text{ is odd.} \end{cases}$$

Note that the function is well-defined: for $X \in \mathcal{X}$, the intersection $X \cap E$ is finite by definition of $\mathcal{X}$, and so the cardinality $\#(X \cap E)$ is a natural number, either even or odd.

Finally, the accessibility relation of our model is defined as follows:

- $R[w_0] = \{v_{Xi} \mid X \in \mathcal{X} \text{ and } i \in \{0, 1\}\}$;
- $R[w_1] = \{v_{Xi} \mid X \in \mathcal{X} \text{ and } i = \tau(X)\}$;
- $R[v] = \emptyset$ for any world v distinct from w_0, w_1.

The basic idea behind the model we just defined is illustrated visually in Figure 1.

We have $w_1 \models P \rightsquigarrow Q$: suppose v_{Xi} and v_{Yj} are successors of w_1 that assign the same extension to P; then $X = P_{v_{Xi}} = P_{v_{Yi}} = Y$, and so by the definition of $R[w_1]$ we

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7 The idea for the proof of this theorem was developed in collaboration with Gianluca Grilletti.

8 For simplicity, we proceed under the assumption that P and Q are the only predicates in the language, along with identity. If this is not the case, it suffices to assume that the extension of other predicates is empty at every world in our model; the argument below then generalizes straightforwardly.

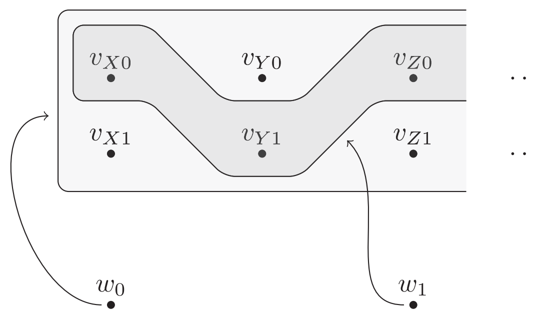

Figure 1. Basic idea behind the model used in the proof of Theorem 3.1. The two shaded areas represent the sets of successors of the worlds w_0 and w_1. The key aspect of the model is that, for each set $X \in \mathcal{X}$, both worlds v_{X0} and v_{X1} are successors of w_0, while only one of them is a successor of w_1.

have $i = \tau(X) = \tau(Y) = j$, which implies that the extension of Q is the same in v_{Xi} as in v_{Yj}.

By contrast, $w_0 \not\models P \rightsquigarrow Q$: indeed, for an arbitrary set $X \in \mathcal{X}$, the worlds v_{X0} and v_{X1} are both successors of w_0, and they assign the same extension X to P, but they disagree on the extension of Q.

It remains to be shown that w_0 and w_1 satisfy the same sentences of QML. Given Proposition 3.3, it suffices to show that w_0 and w_1 are bisimilar. For this, we define a relation Z which consists of the following pairs:

- all pairs of the form $((w_0, \overline{a}), (w_1, \overline{a}))$;
- all pairs of the form $((v_{Xi}, \overline{a}), (v_{Yj}, \overline{b}))$ such that, if n is the size of $\overline{a}$ and $\overline{b}$, the following three conditions hold:
    1. $i = j$;
    2. for all $k \leq n : a_k \in X \iff b_k \in Y$;
    3. for all $k, h \leq n : (a_k = a_h) \iff (b_k = b_h)$.

We are going to show that Z so defined is a bisimulation. We need to show that for each pair, all the five conditions in the definition of a bisimulation are satisfied. Consider first a pair of the form $((w_0, \overline{a}), (w_1, \overline{a}))$, where $\overline{a} = (a_1, \dots, a_n)$. For such a pair, the only condition that is not straightforward to verify is $\Diamond$-forth (we leave it to the reader to check the other conditions).

- $\Diamond$-forth. Take any $v_{Xi} \in R[w_0]$. If $i = \tau(X)$ then we have $v_{Xi} \in R[w_1]$ and obviously $(v_{Xi}, \overline{a}) Z (v_{Xi}, \overline{a})$. So we may suppose $i \neq \tau(X)$. In this case, let e be an even number which is not in X and which is distinct from each element a_k for $k \leq n$. We know such an e exists, since X contains only finitely many even numbers. Now let $Y = X \cup \{e\}$. Then $\#(Y \cap E) = \#(X \cap E) + 1$, and since $i \neq \tau(X)$ it follows that $i = \tau(Y)$. This means that $v_{Yi} \in R[w_1]$, and we have that $(v_{Xi}, \overline{a}) Z (v_{Yi}, \overline{a})$: the first and third condition of the definition of Z are obviously satisfied; as for the second condition, the fact that $a_k \in X \iff a_k \in Y$ is guaranteed by the fact that X and Y only differ on the individual e, which is distinct from each of the a_k.

Next, consider a pair of the form $((v_{Xi}\overline{a}), (v_{Yj}, \overline{b}))$ in Z. In this case, the $\Diamond$-forth and $\Diamond$-back conditions are trivial since both worlds v_{Xi} and v_{Yj} have no successors. For the other three conditions, we reason as follows.

- Atomic condition. The atomic formulas we need to consider are of three forms: (i) Qx_k for $k \leq n$; (ii) Px_k for $k \leq n$; (iii) $(x_k = x_h)$ for $h, k \leq n$. Agreement with respect to these formulas is guaranteed precisely by the three conditions 1-3 in the definition of Z for such pairs. More specifically:
  - by condition 1 we have $i = j$, and therefore the extension of Q is either empty in both worlds, or the entire domain in both worlds; this guarantees that $v_{Xi} \models Qa_k \iff v_{Yj} \models Qb_k$;
  - by condition 2 we have for every $k \leq n$ that $a_k \in X \iff b_k \in Y$, which means that $v_{Xi} \models Pa_k \iff v_{Yj} \models Pb_k$;
  - by condition 3 we have for every $k, h \leq n$ that $(a_k = a_h) \iff (b_k = b_h)$, which guarantees agreement about identity atoms $(x_k = x_h)$.
- $\exists$-forth. Consider any object $a_{n+1} \in D$. If $a_{n+1} = a_k$ for some $k \leq n$ we may take $b_{n+1} = b_k$ and it is straightforward to check that $(v_{Xi}, \overline{a}a_{n+1})Z(v_{Yj}, \overline{b}b_{n+1})$. If on the other hand a_{n+1} is distinct from each a_k for $k \leq n$, we may take b_{n+1} to be any number distinct from each b_k for $k \leq n$, with the condition that $b_{n+1} \in Y \iff a_{n+1} \in X$. Since both Y and its complement $\mathbb{N} - Y$ are infinite, picking such a b_{n+1} is always possible. It is then easy to verify that $(v_{Xi}, \overline{a}a_{n+1})Z(v_{Xj}, \overline{b}b_{n+1})$.
- $\exists$-back. The reasoning is analogous to the one for $\exists$-forth.

Thus, Z is indeed a bisimulation. Since $(w_0, \varepsilon)Z(w_1, \varepsilon)$ (where ε is the empty sequence), the worlds w_0 and w_1 are bisimilar, and so by Proposition 3.3, w_0 and w_1 satisfy the same sentences of QML. This completes the proof of the theorem. □

The proof we just saw can be generalized straightforwardly to show the following result.

Theorem 3.4. *For any predicate symbols $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_m$ (with $n, m \geq 1$), there is no sentence α of* QML *equivalent to $(P_1, \ldots, P_n \rightsquigarrow Q_1, \ldots, Q_m)$.*

Proof sketch. We define a model M in the same way as above, except that we assign the following extensions to the predicates relative to a world v_{Xi}:

- for $j \leq n$, $(P_j)_{v_{Xi}} = X^k$ where k is the arity of P_j
- for $j \leq m$, $(Q_j)_{v_{Xi}} = \begin{cases} \mathbb{N}^k & \text{if } i = 1 \\ \emptyset & \text{if } i = 0 \end{cases}$ where k is the arity of Q_j

Crucially, we still have that two worlds v_{Xi} and v_{Yj} agree on the extension of $P_1, \ldots, P_n$ iff $X = Y$, and they agree on the extension of $Q_1, \ldots, Q_m$ iff $i = j$. Thus, the supervenience claim $P_1, \ldots, P_n \rightsquigarrow Q_1, \ldots, Q_m$ still amounts to the claim that for two successors v_{Xi} and v_{Yj}, $X = Y$ implies $i = j$. This is true at w_1 but false at w_0. The rest of the proof then proceeds as above, with obvious adjustments. □

The discussion in this section shows that standard modal predicate logic QML is not sufficiently expressive to regiment global supervenience claims. We are now going to see that a simple inquisitive extension of QML does provide us with the resources to express these claims in a logically perspicuous way.

§4. Adding questions to modal predicate logic. In this section we introduce an inquisitive extension of modal predicate logic, denoted $\mathsf{InqQML}^-_\Box$, obtained by adding the modality $\Box$ to a fragment of inquisitive predicate logic.[9]

Syntax. As usual, the definition starts with a signature Σ. For simplicity, we focus on the case in which Σ is a *relational* signature, i.e., consists only of a set of predicate symbols, each with an associated arity; however, our discussion extends naturally to the case in which Σ contains also individual constants and function symbols (see [9], for the details in the setting without modalities).

The language of $\mathsf{InqQML}^-_\Box$ is given by the following definition:

$$\varphi \ := \ Px_1 \dots x_n \mid x_1 = x_2 \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x \varphi \mid \Box\varphi \mid \varphi \vee\!\!\!\vee \varphi,$$

where P is an n-ary predicate symbol in Σ and x or x_i stand for first-order variables.

The operator $\vee\!\!\!\vee$, called *inquisitive disjunction*, is regarded as a question-forming operator. Thus, for instance, the formula $Px \vee\!\!\!\vee \neg Px$ is interpreted intuitively as the question *whether or not x is P*. In order to express such yes/no questions more succinctly, it is useful to introduce an inquisitive operator '?', defined as follows:

$$?\varphi := \varphi \vee\!\!\!\vee \neg\varphi.$$

The $\vee\!\!\!\vee$-free fragment of the language can be identified with the language of standard modal predicate logic QML, with a particular choice of primitives. The other logical operators, namely, negation, classical disjunction, and the existential quantifier, can then be defined as follows:

$$\neg\varphi := \varphi \to \bot \qquad \varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi) \qquad \exists x\varphi := \neg\forall x \neg\varphi.$$

Below, we will also consider $\mathsf{InqQML}^?_\Box$, a fragment of $\mathsf{InqQML}^-_\Box$ where the only inquisitive operator is '?'. More explicitly, the syntax of this fragment is given by:

$$\varphi \ := \ Px_1 \dots x_n \mid x_1 = x_2 \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x\varphi \mid \Box\varphi \mid ?\varphi.$$

As we will see, while $\mathsf{InqQML}^?_\Box$ is much less expressive than $\mathsf{InqQML}^-_\Box$, it already includes the resources needed to express global supervenience claims.

Semantics. Models for $\mathsf{InqQML}^-_\Box$ are standard constant-domain Kripke models, as given by Definition 2.1. However, following the basic idea of inquisitive semantics, the interpretation of formulas is not given by a recursive definition of truth at a possible world; instead, it is given by a definition of a relation of *support* relative to an *information state*, where an information state is defined as a set of worlds $s \subseteq W$.

Definition 4.1 (Semantics of $\mathsf{InqQML}^-_\Box$). Let $M = \langle W, D, R, I\rangle$ be a constant-domain Kripke model. The relation $s \models_g \varphi$ of support between an information state $s \subseteq W$ and a formula φ of $\mathsf{InqQML}^-_\Box$ relative to an assignment $g : \mathrm{Var} \to D$ is defined inductively by the following clauses:[10]

[9] Full inquisitive predicate logic also contains an inquisitive existential quantifier $\exists\!\!\!\exists$, which we leave out of consideration here; this omission is the reason for the superscript '–' in the notation $\mathsf{InqQML}^-_\Box$. For an introduction to inquisitive predicate logic, the reader is referred to [9]; for a study of the properties of $\Box$ in the context of inquisitive *propositional* logic, see [7].

[10] In inquisitive predicate logic, a more general treatment of identity is typically considered [9]. We stick with the present clause for simplicity, as the motivations for the generalization are largely orthogonal to our present concerns.

- $M, s \models_g Rx_1, \dots, x_n \iff$ for all $w \in s : \langle g(x_1), \dots, g(x_n)\rangle \in R_w$;
- $M, s \models_g (x_1 = x_2) \iff s = \emptyset$ or $g(x_1) = g(x_2)$;
- $M, s \models_g \bot \iff s = \emptyset$;
- $M, s \models_g \varphi \wedge \psi \iff M, s \models_g \varphi$ and $M, s \models_g \psi$;
- $M, s \models_g \varphi \vee\!\!\!\vee \psi \iff M, s \models_g \varphi$ or $M, s \models_g \psi$;
- $M, s \models_g \varphi \to \psi \iff \forall t \subseteq s : M, t \models_g \varphi$ implies $M, t \models_g \psi$;
- $M, s \models_g \forall x\varphi \iff$ for all $d \in D : M, s \models_{g[x\mapsto d]} \varphi$;
- $M, s \models_g \Box\varphi \iff$ for all $w \in s : M, R[w] \models_g \varphi$.

As usual, $g[x \mapsto d]$ is the assignment that maps x to d and agrees with g on other variables. When the model M is clear from the context, we suppress reference to it.

We will come back to the clause for $\Box$ below; all the other clauses are the standard ones from inquisitive predicate logic (see [9] for discussion).

As customary in inquisitive logic, the relation of support has two basic features:

- Persistency: if $t \subseteq s$ and $M, s \models_g \varphi$, then $M, t \models_g \varphi$;
- Empty state property: $M, \emptyset \models_g \varphi$ for every formula φ.

As usual, the interpretation of a formula φ only depends on the values that g assigns to the free variables in φ; in particular, if φ is a sentence, its interpretation does not depend on the assignment, and we may omit reference to it.

Entailment is defined in the obvious way: a set of formulas Φ entails ψ (notation: $\Phi \models \psi$) if relative to every model and assignment, every state that supports all formulas in Φ also supports ψ. We say that φ and ψ are equivalent ($\varphi \equiv \psi$) if they entail each other, i.e., if they are supported by the same states in every model. We say that ψ is *valid* (notation: $\models \psi$) if it is entailed by the empty set of premises—in other words, if it is supported by every state in every model relative to every assignment. We say that ψ is *consistent* if it does not entail $\bot$—i.e., if it is supported by some non-empty state in some model relative to some assignment.[11]

Although the basic notion in inquisitive logic is support at an information state, a notion of truth at a world is retrieved in the following way.

Definition 4.2 (Truth at a world). φ is true at a world $w \in W$ relative to assignment g, denoted $M, w \models_g \varphi$, if φ is supported by the singleton state $\{w\}$. In symbols: $M, w \models_g \varphi \iff M, \{w\} \models_g \varphi$.

For certain formulas φ, support at a state s boils down to truth at each world in s. If this is the case, the semantics of φ is fully determined by its truth conditions; we then say that φ is *truth-conditional*.

Definition 4.3 (Truth-conditionality). A formula φ is truth-conditional if for every model M, state s and assignment g:

$$M, s \models_g \varphi \iff \text{for all } w \in s : M, w \models_g \varphi.$$

We can define a syntactic fragment of our language that contains only and, up to equivalence, all truth-conditional formulas. This fragment consists of *declaratives*, defined as follows.

[11] The restriction to *nonempty* states is crucial, since in any model, the empty state supports any formula, including $\bot$.

DEFINITION 4.4 (Declaratives). A formula of $\mathsf{InqQML}^{-}_{\Box}$ is a declarative if every occurrence of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ (and, thus, of '?') is within the scope of a modality $\Box$.

Thus, for instance, $\Box?Px$ is a declarative, while $?\Box Px$ is not. We can then show the following fact (the proof is left to the reader; essentially the same result for propositional modal logic is proved in Ciardelli [7, Corollary 6.3.11]).

PROPOSITION 4.5. *Every declarative is truth-conditional. Moreover, every truth-conditional formula in* $\mathsf{InqQML}^{-}_{\Box}$ *is equivalent to some declarative.*

Note that the declarative fragment includes all formulas of standard modal predicate logic, QML, as such formulas do not contain any occurrence of the inquisitive operators. This means that, in particular, all formulas of QML are truth-conditional, i.e., their semantics is completely determined by their truth conditions. Moreover, using the previous proposition, it is easy to show that these truth conditions are just the familiar ones given by Kripke semantics. This means that $\mathsf{InqQML}^{-}_{\Box}$ is in a precise sense a conservative extension of QML: for all formulas of QML, the results of our semantics are essentially equivalent to those of standard Kripke semantics, and entailment among standard modal formulas coincides with entailment in QML.

Now let us come back to the semantics of modal formulas $\Box\varphi$ in $\mathsf{InqQML}^{-}_{\Box}$. Proposition 4.5 ensures that such formulas are always truth-conditional, regardless of the argument φ. Thus, in order to understand their semantics, it suffices to consider their truth conditions, which are as follows:

$$M, w \models_g \Box\varphi \iff M, R[w] \models_g \varphi$$

In words: $\Box\varphi$ is true at a world w iff φ is supported by the set of successors of w. If φ is truth-conditional (and, in particular, if φ is a QML-formula) this further boils down to φ being true at each successor of w—and thus to the familiar clause for $\Box$ in Kripke semantics. However, below we will be especially interested in the case in which the argument φ is *not* truth-conditional; in that case, the condition that φ be supported at $R[w]$ does not simply boil down to φ being true at each world in $R[w]$.

As an example of a formula that is not truth-conditional, take $\forall x?Px$, where P is a unary predicate. Keeping in mind that $?Px := Px \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg Px$ and that standard formulas are truth-conditional with the usual truth-conditions, we have:

$$\begin{aligned}
s \models \forall x?Px &\iff \forall d \in D : s \models_{[x\mapsto d]} ?Px \\
&\iff \forall d \in D : (s \models_{[x\mapsto d]} Px) \text{ or } (s \models_{[x\mapsto d]} \neg Px) \\
&\iff \forall d \in D : (\forall w \in s : d \in P_w) \text{ or } (\forall w \in s : d \notin P_w) \\
&\iff \forall d \in D\ \forall w, v \in s : (d \in P_w \iff d \in P_v) \\
&\iff \forall w, v \in s\ \forall d \in D : (d \in P_w \iff d \in P_v) \\
&\iff \forall w, v \in s : P_w = P_v.
\end{aligned}$$

That is, the sentence $\forall x?Px$ is supported at a state s iff all worlds in s agree on the extension of P. Intuitively, this formula may be seen as regimenting the question *which objects are* P, which asks for a specification of the extension of predicate P.

As we will now show, global supervenience claims can be expressed in $\mathsf{InqQML}^{-}_{\Box}$ as strict conditionals involving such questions.

§5. Global supervenience in inquisitive modal logic.

Expressing global supervenience. We saw in Section 3 that the claim that a predicate Q globally supervenes on a predicate P is not expressible in QML. We will now show that, by contrast, this claim *is* expressible in $\mathsf{InqQML}_{\Box}^{-}$, by means of the following sentence:

$$\Box(\forall x?Px \rightarrow \forall x?Qx).$$

Note that this is a strict conditional having the question *which objects are P* ($\forall x?Px$) as its antecedent, and the question *which objects are Q* ($\forall x?Qx$) as its consequent. Intuitively, the formula may be read as: relative to the set of successors of the world of evaluation, settling which objects are P implies settling which objects are Q. This is exactly what the global supervenience claim amounts to.

PROPOSITION 5.1. *Let P, Q be unary predicates. For every constant-domain Kripke model M and world* w *we have*:

$$M, w \models \Box(\forall x?Px \rightarrow \forall x?Qx) \iff M, w \models P \rightsquigarrow Q.$$

Proof. If s is an information state, let us say that "P is constant in s" in case P has the same extension in every world in s, i.e., $P_v = P_u$ for all $v, u \in s$. Recall that we have: $s \models \forall x?Px \iff P$ is constant in s (and similarly for Q). We then have the following chain of equivalences:

$$\begin{aligned} w \models \Box(\forall x?Px \rightarrow \forall x?Qx) &\iff R[w] \models \forall x?Px \rightarrow \forall x?Qx \\ &\iff \forall s \subseteq R[w] : s \models \forall x?Px \text{ implies } s \models \forall x?Qx \\ &\iff \forall s \subseteq R[w] : (P \text{ constant in } s) \text{ implies } (Q \text{ constant in } s) \\ &\iff \forall v, u \in R[w] : (P_v = P_u) \text{ implies } (Q_v = Q_u) \\ &\iff w \models P \rightsquigarrow Q. \end{aligned}$$

For the crucial equivalence between the third and the fourth line we may argue as follows. Suppose the condition on the third line holds, i.e., every state $s \subseteq R[w]$ in which P is constant is one in which Q is constant. Consider two successors $v, u \in R[w]$: if $P_v = P_u$, this means that P is constant in the state $\{v, u\} \subseteq R[w]$; therefore Q is also constant in this state, which means that $Q_v = Q_u$. Hence, the fourth line holds.

Suppose now the condition on the third line does not hold, i.e., there is a state $s \subseteq R[w]$ on which P is constant but Q is not. Since Q is not constant in s, there are two worlds $v, u \in s$ with $Q_v \neq Q_u$. And since P is constant in s, we have $P_v = P_u$. So, there are two successors of P which agree on P but not on Q, which means that the fourth line does not hold. □

In case we want to express the global supervenience of several (not necessarily unary) predicates $Q_1, \dots, Q_m$ on several other predicates $P_1, \dots, P_n$, the strategy generalizes straightforwardly: it suffices to conjoin all questions about the extensions of the P_i in the antecedent, and all questions about the extensions of the Q_i in the consequent.

PROPOSITION 5.2. *Let $P_1, \dots, P_n, Q_1, \dots, Q_m$ be arbitrary predicate symbols. The strict conditional*

$$\Box\Big(\bigwedge_{i=1}^{n} \forall \overline{x}_i ?P_i \overline{x}_i \rightarrow \bigwedge_{i=1}^{m} \forall \overline{y}_i ?Q_i \overline{y}_i\Big),$$

where $\overline{x}_i$ and $\overline{y}_j$ denote sequences of variables whose size matches the arity of the corresponding predicate P_i or Q_j, has the same truth-conditions as the supervenience claim $P_1, \dots, P_n \rightsquigarrow Q_1, \dots, Q_m$.

The proof is a straightforward adaptation of the one given for the case $n = m = 1$.[12]

In sum, global supervenience claims can be expressed in the modal logic $\mathsf{InqQML}_{\Box}^{-}$ as strict conditionals having as their antecedents the questions about the extensions of the subvenient predicates, and as their consequents the questions about the extensions of the supervenient predicates. Note that in the formulas expressing global supervenience, inquisitive disjunction occurs only via the operator '?'. Therefore, these formulas in fact belong to the fragment $\mathsf{InqQML}_{\Box}^{?}$. This is interesting since we will see in Section 6 that formulas of $\mathsf{InqQML}_{\Box}^{?}$ have some special semantic properties.

Finally, it is worth pointing out that Ciardelli [8] argued in detail for a general analysis of dependence claims as strict conditionals involving questions. Global supervenience is a special kind of dependence, and accordingly, our analysis fits this general pattern.[13]

Expressive power considerations. As we mentioned in the introduction, in the propositional setting, questions in the scope of $\Box$ do not add to the expressive power of the system: any formula of the form $\Box\varphi$, where φ possibly contains inquisitive operators, is equivalent to a disjunction of modal formulas $\Box\alpha$ where α does not contain inquisitive operators—and, therefore, to a formula of standard modal logic (see [7, Corollary 6.3.11]). The results we have seen in this paper imply that the same is not true in the predicate logic setting.

COROLLARY 5.3. *In $\mathsf{InqQML}_{\Box}^{-}$, there are formulas of the form $\Box\varphi$ which are not equivalent to any formula of standard modal predicate logic* QML. *In particular, the formula $\Box(\forall x?Px \to \forall x?Qx)$ is not equivalent to any* QML*-formula.*

Proof. By Proposition 5.1, the given formula is true at a world iff Q globally supervenes on P. By Theorem 3.1, no formula of QML has these truth conditions. □

Thus, in the domain of predicate logic, generalizing the modality $\Box$ to inquisitive arguments leads to a more expressive modal logic, one in which we may regiment interesting modal claims that cannot be expressed in standard modal logic.

The logic of global supervenience. To conclude this section, we now illustrate how analyzing global supervenience in terms of inquisitive strict conditionals sheds light on the logical properties of this notion, allowing us to trace them back to logical properties of strict conditionals and questions. An extensive discussion of the validities of inquisitive predicate logic would take us too far afield (see [9] for a survey). Let us just recall that declaratives obey classical predicate logic, while formulas involving

[12] The strategy further generalizes in the obvious way to express the global supervenience of arbitrary open formulas $\beta_1(\overline{y}_1), \dots, \beta_m(\overline{y}_m)$ on other open formulas $\alpha_1(\overline{x}_1), \dots, \alpha_n(\overline{x}_n)$. For instance, the fact that the property being either R or S globally supervenes on the property of being both P and Q is expressed by the formula:

$$\Box(\forall x?(Px \wedge Qx) \ \to \ \forall x?(Rx \vee Sx)).$$

[13] Crucially, however, expressing supervenience claims requires the resources of modal *predicate* logic. Previous work in inquisitive modal logic, including the cited paper, has focused on the propositional case.

inquisitive vocabulary obey intuitionistic logic, with $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ in the role of intuitionistic disjunction, plus some additional principles. In particular, the constant domain assumption leads to the validity of the equivalence

$$\forall x(\varphi(x) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi) \equiv \forall x\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi,$$

where x does not occur free in ψ (this equivalence is familiar from constant-domain intuitionistic logic, cf. [17]).

As for the modality $\Box$, the following proposition states some of its key properties (we leave the straightforward verification to the reader; cf. [7, Chapter 6], for analogous results in propositional modal logic).

PROPOSITION 5.4. *For any formulas* φ, ψ *of* $\mathsf{InqQML}^{-}_{\Box}$ *and set of formulas* Φ, *the following hold.*

- $\to$*-distributivity*: $\Box(\varphi \to \psi) \models \Box\varphi \to \Box\psi$;
- $\wedge$*-distributivity*: $\Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$;
- $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$*-pseudo-distributivity*: $\Box(\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi) \equiv \Box\varphi \vee \Box\psi$;
- $\forall$*-distributivity*: $\Box\forall x\varphi \equiv \forall x\Box\varphi$;
- *Monotonicity*: $\Phi \models \psi$ *implies* $\Box\Phi \models \Box\psi$, *where* $\Box\Phi = \{\Box\varphi \mid \varphi \in \Phi\}$.

The third item in the list captures the interaction of $\Box$ with inquisitive disjunction: an inquisitive disjunction under $\Box$ matches a classical disjunction over $\Box$. The other items are familiar from standard constant-domain modal logic.[14] Note that in the statement of Monotonicity, Φ may be empty, which gives Necessitation as a special case: $\models \psi$ implies $\models \Box\psi$.

Let us now examine how several properties of global supervenience, and interesting inferences involving this notion, can be analyzed in the light of our inquisitive regimentation of supervenience claims.

EXAMPLE 5.5 (Armstrong's axioms). We saw in Section 2 that some important properties of supervenience are captured by Armstrong's axioms for functional dependence. Under our analysis, these properties emerge as special cases of familiar facts about the logic of strict implication.[15] For readability, we write $\vec{P}$ and $\vec{Q}$ for sequences of predicates $P_1, \dots, P_n$ and $Q_1, \dots, Q_m$.

- Reflexivity: $\models (\vec{P}, \vec{Q} \rightsquigarrow \vec{P})$.
  Given our regimentation, this amounts to the validity of the formula

  $$\Box(\varphi \wedge \psi \to \varphi),$$

  where φ is the conjunction of all the questions about the extensions of the P_i (i.e., $\forall \overline{x}_i ?P_i\overline{x}_i$) and ψ is the conjunction of all the questions about the

[14] Note that these items imply that $\Box$ can be pushed through all other logical operators of our language, except for implication, where the entailment holds only in one direction. In view of this, it is not surprising that a witness for Corollary 5.3, i.e., a formula $\Box\varphi$ which cannot be reduced to a formula of QML, has the form $\Box(\chi \to \psi)$.

[15] The analogy between Armstrong's axioms and the properties of implication has beed noticed multiple times in the literature (see, a.o., [13, 32, 33]), though an explanation of this analogy has been largely missing. An exception is the work of Abramsky & Väänänen [1] who, analogously to us, regiment certain dependence claims in terms of an implication operator very much related to our $\to$.

extensions of the Q_i. Since every formula of the form $\varphi \wedge \psi \to \varphi$ is a logical validity, so is the above formula by Necessitation.

- Augmentation: $(\vec{P} \rightsquigarrow \vec{Q}) \models (\vec{P}, \vec{R} \rightsquigarrow \vec{Q}, \vec{R})$.
  Given our regimentation, this amounts to the entailment

$$\Box(\varphi \to \psi) \models \Box(\varphi \wedge \chi \to \psi \wedge \chi),$$

  where φ and ψ are as in the previous item and χ the conjunction of the questions about the R_i. Since the entailment $\varphi \to \psi \models \varphi \wedge \chi \to \psi \wedge \chi$ is valid, the above entailment is valid by the monotonicity of $\Box$.
- Transitivity: $(\vec{P} \rightsquigarrow \vec{Q}), (\vec{Q} \rightsquigarrow \vec{R}) \models (\vec{P} \rightsquigarrow \vec{R})$.
  Given our regimentation, this amounts to the entailment

$$\Box(\varphi \to \psi), \Box(\psi \to \chi) \models \Box(\varphi \to \chi),$$

  where φ, ψ, and χ are as in the previous item. The validity of the entailment follows immediately from the transitivity of $\to$ and the monotonicity of $\Box$.

Example 5.6. For another example of inference that relies only on the properties of strict conditionals, consider the reasoning in (3) from the introduction, repeated below along with its formalization in $\mathsf{InqQML}_{\Box}^{-}$.

(6)	P globally supervenes on Q and R.	$\Box(\forall x?Qx \wedge \forall x?Rx \to \forall x?Px)$
	P does not globally supervene on Q.	$\neg\Box(\forall x?Qx \to \forall x?Px)$
	Therefore, R does not globally supervene on Q.	$\therefore \neg\Box(\forall x?Qx \to \forall x?Rx)$

To see that this inference is valid, note that by the basic properties of conjunction and implication we have for any formulas φ, ψ, χ

$$\varphi \wedge \psi \to \chi,\ \varphi \to \psi \models \varphi \to \chi$$

whence by the monotonicity of $\Box$ we have

$$\Box(\varphi \wedge \psi \to \chi),\ \Box(\varphi \to \psi) \models \Box(\varphi \to \chi)$$

and then by classical reasoning with declaratives:

$$\Box(\varphi \wedge \psi \to \chi),\ \neg\Box(\varphi \to \chi) \models \neg\Box(\varphi \to \psi).$$

Clearly, the validity of the above inference is an instance of this general fact.

The inferences we considered so far are valid based only on the properties of strict conditionals. We now turn to a couple of examples where the logical properties of questions also play a central role.

Example 5.7. Consider the inference in (4) from the introduction, repeated below along with its formalization:

(7)	Q globally supervenes on P.	$\Box(\forall x?Px \to \forall x?Qx)$
	It is contingent whether there are any Q.	$\neg\Box\exists xQx \wedge \neg\Box\neg\exists xQx$
	Therefore, it is not necessarily the case that every object is P.	$\therefore \neg\Box\forall xPx$

To see that this inference is valid, we may start from the following (inquisitive) instance of modus ponens:

$$\forall x?Px,\ \forall x?Px \to \forall x?Qx \models \forall x?Qx.$$

By intuitionistic reasoning we have the entailment $\forall xPx \models \forall x?Px$ (intuitively: the information that all objects are P settles the question which objects are P); by intuitionistic reasoning and the constant domain principle we have $\forall x?Qx \models ?\exists xQx$ (intuitively: the information which objects are Q settles the question whether there are any Q).[16] Putting these facts together, we have:

$$\forall xPx,\ \forall x?Px \to \forall x?Qx\ \models\ ?\exists xQx.$$

By the monotonicity of $\Box$, this implies:

$$\Box\forall xPx, \Box(\forall x?Px \to \forall x?Qx) \models \Box?\exists xQx.$$

Recall that $?\exists xQx$ abbreviates $\exists xQx \mathbin{\vee\!\!\!\vee} \neg\exists xQx$. By $\mathbin{\vee\!\!\!\vee}$-pseudo-distributivity of $\Box$ over this formula, the above entailment can be rewritten as

$$\Box\forall xPx, \Box(\forall x?Px \to \forall x?Qx) \models \Box\exists xQx \vee \Box\neg\exists xQx$$

and finally, by classical reasoning with declaratives, this is equivalent to

$$\Box(\forall x?Px \to \forall x?Qx), \neg\Box\exists xQx \wedge \neg\Box\neg\exists xQx \models \neg\Box\forall xPx$$

which amounts to the validity of the argument in 5.7. Thus, the validity of this argument can be traced back to certain facts about the logic of questions, together with the monotonicity of $\Box$ and its pseudo-distributivity over $\mathbin{\vee\!\!\!\vee}$.

EXAMPLE 5.8. Next, consider the inference in (2) from the introduction, repeated below:

(8) For every person x, the property *being a sibling of x* globally supevenes on the properties *being a brother of x* and *being a sister of x*.
Therefore, the *sibling-of* relation globally supervenes on the *brother-of* and *sister-of* relations.

Let B, S, G be three binary predicates standing respectively for 'brother of', 'sister of', 'sibling of'. Given our regimentation, this inference has the following form:

$$\forall x\Box(\forall y?Bxy \wedge \forall y?Sxy \to \forall y?Gxy)\ \therefore\ \Box(\forall x\forall y?Bxy \wedge \forall x\forall y?Sxy \to \forall x\forall y?Gxy).$$

To see that this inference is valid, it suffices to note that for any formulas φ, ψ, χ, the following holds:

$$\forall x(\varphi \wedge \psi \to \chi) \models \forall x\varphi \wedge \forall x\psi \to \forall x\chi.$$

From this, by the monotonicity of $\Box$ and the commutation of $\Box$ with $\forall$ we obtain:

$$\forall x\Box(\varphi \wedge \psi \to \chi) \models \Box(\forall x\varphi \wedge \forall x\psi \to \forall x\chi).$$

The validity of the above inference is an instance of this general scheme.

Finally, let us consider again the connection between definability and global supervenience established by Proposition 2.4, in light of our inquisitive analysis.

EXAMPLE 5.9 (Global supervenience and definability). It is immediate to check (and a well-known fact, see [5]) that definability is an instance of question entailment, in

16 For the interested reader, here are the details. Since $Qx \models \exists xQx$, by standard disjunctive reasoning we have $Qx \mathbin{\vee\!\!\!\vee} \neg Qx \models \exists xQx \mathbin{\vee\!\!\!\vee} \neg Qx$, whence we have $\forall x(Qx \mathbin{\vee\!\!\!\vee} \neg Qx) \models \forall x(\exists xQx \mathbin{\vee\!\!\!\vee} \neg Qx)$. By definition of ? the premise is just $\forall x?Qx$, and by the constant domain principle the conclusion entails $\exists xQx \mathbin{\vee\!\!\!\vee} \forall x\neg Qx$. The latter is equivalent to $\exists xQx \mathbin{\vee\!\!\!\vee} \neg\exists xQx$ (since $\forall x\neg Qx \equiv \neg\exists xQx$ as declaratives obey classical logic), which is nothing but $?\exists xQx$. Cf. the natural proof of a similar entailment in Ciardelli [9, ex. 6.2.2].

the following sense. If Γ is a set of standard (non-inquisitive) first-order formulas, the following are equivalent:[17]

- Γ defines Q in terms of $P_1, \dots, P_n$;
- $\Gamma, \forall \overline{x}_1 ?P_1\overline{x}_1, \dots, \forall \overline{x}_1 ?P_1\overline{x}_1 \models \forall \overline{y} ?Q\overline{y}$.

By the properties of implication in inquisitive logic, the latter is equivalent to:

- $\Gamma \models \forall \overline{x}_1 ?P_1\overline{x}_1 \wedge \cdots \wedge \forall \overline{x}_1 ?P_1\overline{x}_1 \to \forall \overline{y} ?Q\overline{y}$.

By the monotonicity of $\Box$, this is equivalent to the following entailment, whose right-hand side is the formula expressing the global supervenience $P_1, \dots, P_n \rightsquigarrow Q$:

- $\Box\Gamma \models \Box(\forall \overline{x}_1 ?P_1\overline{x}_1 \wedge \cdots \wedge \forall \overline{x}_1 ?P_1\overline{x}_1 \to \forall \overline{y} ?Q\overline{y})$.

In this way, the connection between definability and global supervenience expressed by Proposition 2.4 can be traced back—via the logic of the strict conditional—to a more fundamental connection between definability and the logic of questions.

§6. Meta-theoretic results. In the previous section we saw that moving from standard modal predicate logic to an inquisitive extension leads to an enhanced expressive power. One may wonder if this greater expressive power leads to a difference in the key meta-theoretic properties of the logic. In particular, one may ask the following questions for our logic $\mathsf{InqQML}_{\Box}^{-}$.

- **Effectivity**
  Is the set of validities recursively enumerable?
- **Compactness**
  Is it generally the case that if Φ entails ψ, some finite subset of Φ entails ψ?

The answer to these questions is not obvious. After all, the semantics of implication involves a quantification over subsets, and thus introduces a second-order element into the semantics, which may in principle lead to a loss of effectivity or compactness. Nevertheless, we will show that the above questions both have a positive answer.[18] The proof strategy combines ideas recently pioneered by Meißner & Otto [30] and by Ciardelli & Grilletti [10], making crucial use of the notion of *coherence* (first introduced by [26], in the setting of team semantics).

6.1. Finite coherence. For $n \in \mathbb{N}$, we call a formula φ of $\mathsf{InqQML}_{\Box}^{-}$ *n-coherent* if, in order to check whether φ is supported by a state s, it suffices to check if it is supported by every *n-small* subset of s, where a subset counts as n-small if its cardinality is at

[17] ten Cate & Shan [5] relied on this connection to axiomatize a predecessor of inquisitive logic, the *logic of interrogation* [19]. This logic contains questions of the form $?\overline{x}\alpha(\overline{x})$, where α is a standard first-order formula, which have the same semantics as formulas $\forall \overline{x} ?\alpha(x)$ in inquisive logic. However, the logic of interrogation does not allow questions to be embedded under logical operators, and so the analysis of supervenience we are proposing would not be possible in that framework.

[18] As mentioned in Footnote 9, standard inquisitive first-order logic includes, in addition to inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, also an inquisitive existential quantifier $\exists$. It is an open question whether the answer to the above questions remains positive if we add this quantifier to $\mathsf{InqQML}_{\Box}^{-}$. Indeed, the above questions are open for the (non-modal) inquisitive first-order logic InqBQ. See Ciardelli [9] for discussion.

most n. More formally φ is n-coherent if for any model M, state s, and assignment g we have:

$$M, s \models_g \varphi \iff \forall t \subseteq s \text{ with } \#t \leq n : M, t \models_g \varphi.$$

We say that φ is *finitely coherent* if it is n-coherent for some $n \in \mathbb{N}$. Note that the left-to-right direction of the above biconditional holds for every formula φ by persistency, and so the condition really amounts to the right-to-left direction. Also, note that 1-coherence is nothing but the notion of *truth-conditionality* defined above. Moreover, it will be useful to remark the following fact explicitly.

REMARK 6.1. *If φ is* n-*coherent, then φ is* m-*coherent for all $m > n$.*

Following Ciardelli & Grilletti [10], we show that every formula φ of $\mathsf{InqQML}_{\Box}^{-}$ is n_φ-coherent for a number n_φ that can be computed from the syntax of φ.[19]

DEFINITION 6.2. We assign to each formula φ of $\mathsf{InqQML}_{\Box}^{-}$ a number n_φ as follows:

- $n_p = 1$ if α is an atom or $\bot$
- $n_{\varphi \wedge \psi} = \max\{n_\varphi, n_\psi\}$
- $n_{\varphi \to \psi} = n_\psi$
- $n_{\varphi \vee \psi} = n_\varphi + n_\psi$
- $n_{\forall x \varphi} = n_\varphi$
- $n_{\Box \varphi} = 1$.

PROPOSITION 6.3. *For every formula φ of* $\mathsf{InqQML}_{\Box}^{-}$, *$\varphi$ is n_φ-coherent.*

Proof. Essentially the same as that of Ciardelli & Grilletti [10, Proposition 5.3]. The only new case is the one for modal formulas $\Box\varphi$, which is immediate. □

Thus, in particular, every formula of $\mathsf{InqQML}_{\Box}^{-}$ is finitely coherent. For the language $\mathsf{InqQML}_{\Box}^{?}$, where the only inquisitive operator is '?', we can prove something stronger.

PROPOSITION 6.4. *For every formula φ of* $\mathsf{InqQML}_{\Box}^{?}$, *$\varphi$ is* 2-*coherent.*

Proof. The proof is by induction on φ. If φ is atomic, $\bot$, or a modal formula $\Box\psi$, then φ is truth-conditional (i.e., 1-coherent), and thus also 2-coherent by Remark 6.1. It remains to be shown that if φ and ψ are 2-coherent, so are $\varphi \wedge \psi$, $\varphi \to \psi$, $\forall x\varphi$, and $?\varphi$. We only spell out the case for $?\varphi$, since the other cases are straightforward.

Suppose φ is 2-coherent. We claim that $?\varphi$ is 2-coherent as well. To see this, take an arbitrary model M, state s, and assignment g, and suppose $M, s \not\models_g ?\varphi$: we must show that there is a $t \subseteq s$ with $\#t \leq 2$ such that $M, t \not\models_g \varphi$. We distinguish two cases.

- Case 1: for some $w \in s$, $w \not\models_g \varphi$. In this case let $w_- \in s$ be a world with $w_- \not\models_g \varphi$. There must also be a world $w_+ \in s$ with $w_+ \models_g \varphi$, otherwise by the semantics of negation we would have $s \models_g \neg\varphi$, and so also $s \models_g ?\varphi$.
  Now consider the substate $t = \{w_+, w_-\} \subseteq s$. We cannot have $t \models_g \varphi$, otherwise by persistency we would have $w_- \models_g \varphi$, contrary to assumption; similarly, we cannot have $t \models_g \neg\varphi$, otherwise we would have $w_+ \not\models_g \varphi$. Thus, we have $t \not\models_g \varphi$ and $t \not\models_g \neg\varphi$, which means that $t \not\models_g ?\varphi$, and clearly $\#t \leq 2$.
- Case 2: for all $w \in s$, $w \models_g \varphi$. Since $s \not\models_g \varphi$ and φ is 2-coherent, there is some $t \subseteq s$ with $\#t \leq 2$ and $t \not\models_g \varphi$. Note in particular that this means that $t \neq \emptyset$ (since the empty state supports every formula), and thus there is a world

[19] This does not hold, in general, for formulas including the inquisitive existential quantifier $\exists$. See Ciardelli & Grilletti [10].

$w_0 \in t$. We also have that $t \not\models_g \neg\varphi$: for if we had $t \models_g \neg\varphi$, by the semantics of negation we would have $w_0 \not\models_g \varphi$, contradicting the assumption that φ is true at all worlds in s. Thus, we have $t \not\models_g \varphi$ and $t \not\models_g \neg\varphi$, whence $t \not\models_g ?\varphi$.

In either case, we have found that there is a substate t of s of cardinality at most 2 with $t \not\models_g ?\varphi$, which is what we had to show to prove that $?\varphi$ is 2-coherent. □

6.2. Translation to classical first-order logic. In the previous section, we have shown that for a formula φ of $\mathsf{InqQML}^-_\Box$, it is possible to compute a number n_φ for which φ is coherent. We will now show, building on ideas from Meißner & Otto [30] and Ciardelli & Grilletti [10], that this makes it possible to give a translation from $\mathsf{InqQML}^-_\Box$ to two-sorted first-order predicate logic. The existence of this translation will then allow us to give a positive answer to the key meta-theoretic questions discussed at the beginning of this section.

First, we associate to a signature Σ a corresponding signature Σ^* over two sorts: $\mathtt{w}$, for worlds, and $\mathtt{e}$, for individuals. For each n-ary predicate symbol P in Σ, the signature Σ^* contains a corresponding predicate symbol P^* of arity $n+1$, where the first argument is of sort $\mathtt{w}$ and the remaining arguments of sort $\mathtt{e}$. In addition, Σ^* contains a binary predicate R^*, both arguments of which are of type $\mathtt{w}$.

Next, to each constant-domain Kripke model $M = \langle W, D, R, I\rangle$ for Σ we associate a corresponding two-sorted relational structure $M^* = \langle W, D, I^*\rangle$ for the signature Σ^*, where the domain of sort $\mathtt{w}$ is W, the domain of sort $\mathtt{e}$ is D, and the interpretation function I^* is defined as follows:

- $I^*(R^*) = R$;
- $I^*(P^*) = \{\langle w, d_1, \dots, d_n\rangle \in W \times D^n \mid \langle d_1, \dots, d_n\rangle \in I_w(P)\}$ for an n-ary $P \in \Sigma$.

Note that the map $M \mapsto M^*$ yields a one-to-one correspondence between constant-domain Kripke models for Σ and two-sorted relational structures for Σ^*.

Now let $\mathsf{2FOL}$ denote the language of two-sorted first-order predicate logic over the signature Σ^*. Let us use $\mathsf{w}_0, \mathsf{w}_1, \dots$ as well as $\mathsf{v}_0, \mathsf{v}_1, \dots$ to denote first-order variables of sort $\mathtt{w}$ (worlds), and let us use $\overline{\mathsf{w}}$ and $\overline{\mathsf{v}}$ to denote non-empty sequences of such variables. We write $\overline{\mathsf{v}} \sqsubseteq \overline{\mathsf{w}}$ to mean that $\overline{\mathsf{v}}$ is a subsequence of $\overline{\mathsf{w}}$ in the sense that, if $\overline{\mathsf{w}} = \mathsf{w}_1 \dots \mathsf{w}_n$, then $\overline{\mathsf{v}}$ is a sequence of the form $\mathsf{w}_{i_1} \dots \mathsf{w}_{i_k}$ with $1 \le i_1 < \dots < i_k \le n$ (thus, for example, we have $\mathsf{w}_1\mathsf{w}_3 \sqsubseteq \mathsf{w}_1\mathsf{w}_2\mathsf{w}_3$, but not $\mathsf{w}_3\mathsf{w}_1 \sqsubseteq \mathsf{w}_1\mathsf{w}_2\mathsf{w}_3$).

For each sequence $\overline{\mathsf{w}} = \mathsf{w}_1 \dots \mathsf{w}_n$ of world-variables, we define a corresponding map $\mathrm{tr}_{\overline{\mathsf{w}}}$ from formulas of $\mathsf{InqQML}^-_\Box$ to formulas of $\mathsf{2FOL}$, as follows.

$$
\begin{array}{lcl}
\mathrm{tr}_{\overline{\mathsf{w}}}(Px_1 \dots x_k) & = & \bigwedge_{i=1}^{n} P^*\mathsf{w}_i x_1 \dots x_k \\
\mathrm{tr}_{\overline{\mathsf{w}}}(\bot) & = & \bot \\
\mathrm{tr}_{\overline{\mathsf{w}}}(\varphi \wedge \psi) & = & \mathrm{tr}_{\overline{\mathsf{w}}}(\varphi) \wedge \mathrm{tr}_{\overline{\mathsf{w}}}(\psi) \\
\mathrm{tr}_{\overline{\mathsf{w}}}(\varphi \vee\!\!\!\vee \psi) & = & \mathrm{tr}_{\overline{\mathsf{w}}}(\varphi) \vee \mathrm{tr}_{\overline{\mathsf{w}}}(\psi) \\
\mathrm{tr}_{\overline{\mathsf{w}}}(\varphi \to \psi) & = & \bigwedge_{\overline{\mathsf{v}} \sqsubseteq \overline{\mathsf{w}}}(\mathrm{tr}_{\overline{\mathsf{v}}}(\varphi) \to \mathrm{tr}_{\overline{\mathsf{v}}}(\psi)) \\
\mathrm{tr}_{\overline{\mathsf{w}}}(\forall x\varphi) & = & \forall x\, \mathrm{tr}_{\overline{\mathsf{w}}}(\varphi) \\
\mathrm{tr}_{\overline{\mathsf{w}}}(\Box\varphi) & = & \bigwedge_{i=1}^{n} \forall \mathsf{v}_1 \dots \mathsf{v}_{n_\varphi}((\bigwedge_{j=1}^{n_\varphi} R^*\mathsf{w}_i\mathsf{v}_j) \to \mathrm{tr}_{\overline{\mathsf{v}}}(\varphi)) \quad \text{with } \overline{\mathsf{v}} = \mathsf{v}_1 \dots \mathsf{v}_{n_\varphi}.
\end{array}
$$

Note that the translation of $\Box\varphi$ makes use of the number n_φ given by Definition 6.2. The translations preserve the semantics of $\mathsf{InqQML}^-_\Box$ in the following sense.

PROPOSITION 6.5 (Translating $\mathsf{InqQML}^-_\Box$ semantics). *Let M be a constant-comain Kripke model, g an assignment, and s a finite nonempty state. Now let $\overline{\mathsf{w}}$ be a sequence*

$\mathsf{w}_1 \dots \mathsf{w}_n$ *of world-variables with* $n \geq \#s$, *and let* g^* *be an assignment for the variables of* 2FOL *which agrees with* g *on all variables of sort* e *and such that* $g^*[\{\mathsf{w}_1, \dots, \mathsf{w}_n\}] = s$. *Then for any formula* φ *of* $\mathsf{InqQML}^-_\Box$ *we have*:

$$M, s \models_g \varphi \iff M^* \models_{g^*} tr_{\overline{\mathsf{w}}}(\varphi).$$

Proof. Most of the proof is a tedious but straightforward case-by-case verification: the given translation just states in first-order logic the semantic clauses for $\mathsf{InqQML}^-_\Box$, as given in Definition 4.1. The one exception is given by formulas of the form $\Box\varphi$, whose semantic clause does not directly match their translation. The claim that $s \models_g \Box\varphi$ amounts to the condition:

$$\text{for all } w \in s : R[w] \models_g \varphi,$$

whereas the claim that $M^* \models_{g^*} \mathrm{tr}_{\overline{\mathsf{w}}}(\Box\varphi)$ amounts (via the definition of the translation and the induction hypothesis) to the condition:

$$\text{for all } w \in s, \text{ for all } v_1, \dots, v_{n_\varphi} \in R[w] : \{v_1, \dots, v_{n_\varphi}\} \models_g \varphi.$$

We need to show that these two conditions are equivalent. For this, it suffices to show that for any world w we have

$$R[w] \models_g \varphi \iff \text{for all } v_1, \dots, v_{n_\varphi} \in R[w] : \{v_1, \dots, v_{n_\varphi}\} \models_g \varphi.$$

But note that, as we let the variables $v_1, \dots, v_{n_\varphi}$ range over $R[w]$, the set $\{v_1, \dots, v_{n_\varphi}\}$ ranges over all and only the subsets of $R[w_i]$ of size at most n_φ. Thus the above equivalence amounts to the claim

$$R[w] \models_g \varphi \iff \text{for all } t \subseteq R[w] \text{ with } \#t \leq n_\varphi : t \models_g \varphi$$

and this holds because φ is n_φ-coherent by Proposition 6.3. □

Using the result we just proved, we may show that entailment claims in $\mathsf{InqQML}^-_\Box$ can be translated to entailment claims in 2FOL.

PROPOSITION 6.6 (Translating $\mathsf{InqQML}^-_\Box$-entailments). *Let* $\Phi \cup \{\psi\}$ *be a set of formulas in* $\mathsf{InqQML}^-_\Box$. *For any sequence* $\overline{\mathsf{w}} = \mathsf{w}_1 \dots \mathsf{w}_n$ *of world variables of length* $n \geq n_\psi$, *we have*

$$\Phi \models \psi \iff tr_{\overline{\mathsf{w}}}(\Phi) \models_{\mathsf{2FOL}} tr_{\overline{\mathsf{w}}}(\psi),$$

where $\models$ *on the left-hand side denotes entailment in* $\mathsf{InqQML}^-_\Box$, $\models_{\mathsf{2FOL}}$ *denotes entailment in two-sorted first-order logic, and* $tr_{\overline{\mathsf{w}}}(\Phi) = \{tr_{\overline{\mathsf{w}}}(\varphi) \mid \varphi \in \Phi\}$.

Proof. Suppose $\Phi \not\models \psi$. Then there exists a constant domain Kripke model M, a state t, and an assignment g such that $M, t \models_g \varphi$ for all $\varphi \in \Phi$ but $M, t \not\models_g \psi$. Since ψ is n_ψ-coherent, we may find some $s \subseteq t$ with $\#s \leq n_\psi$ such that $M, s \not\models_g \psi$; by persistency, we also have $M, s \models_g \varphi$ for all $\varphi \in \Phi$. Now let g^* be any assignment on our two-sorted language that matches g on variables of type e and such that $g^*[\{\mathsf{w}_1, \dots, \mathsf{w}_n\}] = s$: crucially, such an assignment exists since $\#s \leq n_\psi \leq n$. By the previous proposition we have $M^* \models_{g^*} \mathrm{tr}_{\overline{\mathsf{w}}}(\varphi)$ for all $\varphi \in \Phi$ (since $M, s \models_g \varphi$) but $M^* \not\models_{g^*} \mathrm{tr}_{\overline{\mathsf{w}}}(\psi)$ (since $M, s \not\models_g \psi$), which shows that $\mathrm{tr}_{\overline{\mathsf{w}}}(\Phi) \not\models_{\mathsf{2FOL}} \mathrm{tr}_{\overline{\mathsf{w}}}(\psi)$.

For the converse, suppose $\mathrm{tr}_{\overline{\mathsf{w}}}(\Phi) \not\models_{\mathsf{2FOL}} \mathrm{tr}_{\overline{\mathsf{w}}}(\psi)$. Then there is a model N of the signature Σ^* and an assignment h such that $N \models_h \mathrm{tr}_{\overline{\mathsf{w}}}(\varphi)$ for all $\varphi \in \Phi$ but $N \not\models_h \mathrm{tr}_{\overline{\mathsf{w}}}(\psi)$. Now take a constant domain Kripke model M such that $M^* = N$ (which exists since the map $M \mapsto M^*$ is a bijection between constant domain Kripke models for the signature Σ and first-order structures for the signature Σ^*), and consider the state $s = \{h(\mathsf{w}_1), \dots, h(\mathsf{w}_n)\}$. Consider the assignment g which is just the restriction of h to variables of sort e. By the previous proposition we have $M, s \models_g \varphi$ for all $\varphi \in \Phi$, but $M, s \not\models_g \psi$. Thus, $\Phi \not\models \psi$. □

Finally, using this result, it is easy to show that $\mathsf{InqQML}^-_\Box$ (and thus also its fragment $\mathsf{InqQML}^?_\Box$) is effective and compact.

THEOREM 6.7 (Effectiveness). *The set of valid formulas in* $\mathsf{InqQML}^-_\Box$ *is r.e.*

Proof. By Proposition 6.6, the task of deciding whether $\varphi \in \mathsf{InqQML}^-_\Box$ is a validity can be (computably) reduced to the task of deciding whether its translation $\mathrm{tr}_{\overline{\mathsf{w}}}(\varphi)$ relative to a set of variables $\overline{\mathsf{w}}$ of size n_φ is a validity of 2FOL. As the latter task is semi-decidable, so is the former. □

THEOREM 6.8 (Compactness). *Let* $\Phi \cup \{\psi\}$ *be a set of formulas in* $\mathsf{InqQML}^-_\Box$. *If* $\Phi \models \psi$, *then* $\Phi_0 \models \psi$ *for some finite* $\Phi_0 \subseteq \Phi$.

Proof. Immediate from Proposition 6.6 and the compactness of first-order logic. □

Note that the compactness property we just proved also implies the following compactness property, obtained as the special case in which $\psi = \bot$: for any set of $\mathsf{InqQML}^-_\Box$-formulas Φ, if every finite subset of Φ is consistent, then Φ is consistent.[20]

We can thus conclude this section with a positive answer to the questions posed at the beginning: while extending QML with inquisitive disjunction increases the expressive power of our logic, this expressive power does not result in a loss of the core meta-theoretic properties of QML.

§7. Further work. We close by outlining some salient directions for future work. First, given our meta-theoretical results on $\mathsf{InqQML}^-_\Box$, it is natural to aim for a complete proof system. The main obstacle in this respect is that, at present, there is no established proof system for InqBQ^-, the non-modal fragment of $\mathsf{InqQML}^-_\Box$. Ciardelli & Grilletti [10] provide a proof system for inquisitive first-order logic InqBQ which is complete relative to the fragment InqBQ^-, but this is not a proof system *for* InqBQ^-, as proofs of validities in InqBQ^- may make use of formulas which are not in InqBQ^-. Provided this obstacle is removed, and a proper proof system for InqBQ^- is established, we conjecture that extending this system with modal axioms or rules capturing (some of) the facts in Proposition 5.4 will lead to a complete system for $\mathsf{InqQML}^-_\Box$.

A second research direction concerns the expressive power of the system $\mathsf{InqQML}^-_\Box$. We have seen in this paper that $\mathsf{InqQML}^-_\Box$ can express properties which are not invariant under the standard notion of first-order bisimulation, such as those expressed by global supervenience claims. It is then natural to ask if the expressive power of $\mathsf{InqQML}^-_\Box$ can be characterized by means of a more demanding simulation game. Besides providing insight into the expressive power of our logic, such a game would furnish a precious

20 In inquisitive logic the version of compactness formulated in terms of entailment is usually stronger than the version formulated in terms of consistency: while the consistency of Φ reduces to the validity of the entailment $\Phi \models \bot$, the validity of an arbitrary entailment $\Phi \models \psi$ does not reduce to the consistency of $\Phi \cup \{\neg\psi\}$. In the setting of $\mathsf{InqQML}^-_\Box$, however, the two versions of compactness are indeed equivalent, since it is easy to show that the validity of $\Phi \models \psi$ reduces to the consistency of $\Box\Phi \cup \{\neg\Box\psi\}$.

tool to show that certain properties are not expressible in it. A starting point for this enterprise may be the Ehrenfeucht–Fraïssé-style game described by Grilletti & Ciardelli [18] for inquisitive first-order logic.

Finally, a third line of research concerns the logical analysis of supervenience. As we mentioned in Section 2, the general idea of supervenience can be made precise in different ways. Besides global supervenience, we also have the notion of individual supervenience: a class of properties $\mathcal{B}$ supervenes on a class of properties $\mathcal{A}$ in this sense if two *individuals* cannot differ with respect to the $\mathcal{B}$-properties without also differing with respect to the $\mathcal{A}$-properties. This notion further branches into *weak* and *strong* individual supervenience, depending on whether we compare individuals within the same possible world, or across possibly different worlds. Thus, for instance, the weak/strong individual supervenience of a single property Q on another property P relative to a world w amounts to the following conditions.

- Weak:
  $\forall d, d' \in D \forall v \in R[w] : (d \in P_v \iff d' \in P_v)$ implies $(d \in Q_w \iff d' \in Q_w)$.
- Strong:
  $\forall d, d' \in D \forall v, v' \in R[w] : (d \in P_v \iff d' \in P_{v'})$ implies $(d \in Q_v \iff d' \in Q_{v'})$.

It is not hard to see that these claims, unlike claims of global supervenience, *are* expressible in QML, by means of the following formulas (the point generalizes to supervenience claims involving more than one supervenient and subvenient property):

- Weak: $\Box(\forall x(Qx \leftrightarrow Px) \vee \forall x(Qx \leftrightarrow \neg Px) \vee \forall x Qx \vee \forall x \neg Qx)$;[21]
- Strong: $\Box\forall x(Qx \leftrightarrow Px) \vee \Box\forall x(Qx \leftrightarrow \neg Px) \vee \Box\forall x Qx \vee \Box\forall x \neg Qx$.

Following the main idea of the present paper, however, it would be natural to pursue an analysis of weak and strong individual supervenience as inquisitive strict conditionals, so as to bring out a common logical core of supervenience claims, facilitating a comparison of the different varieties of supervenience. Such an inquisitive analysis of individual supervenience seems natural; after all, the individual supervenience of Q on P is naturally formulated as a condition involving questions, namely: for an arbitrary object x, the question whether x is Q is fully determined by the question whether x is P (globally across all successors, in the case of strong supervenience, or locally in each individual successor, in the case of weak supervenience). Pursuing this formalization, however, requires the resources to ask questions about arbitrary objects—resources which are not available in the logic $\mathsf{InqQML}_{\Box}^{-}$. To make such questions expressible, we may extend $\mathsf{InqQML}_{\Box}^{-}$ with resources from team semantics [20], and with a quantifier $[x]$ that creates one possibility for each possible value of the variable x, in a way which is familiar from work on dependence logic [36]. We leave the exploration of this extension for future work.

Acknowledgments. I would like to thank Gianluca Grilletti for his contribution in developing the key idea for the proof of Theorem 3.1. In addition, I would like to thank Balder ten Cate, Simone Conti, Valentin Goranko, Ali Hamed, Eugenio Orlandelli, Claudio Ternullo, and Giorgio Venturi for useful discussions, and two anonymous RSL reviewers for precious suggestions.

[21] Or, even more straightforwardly: $\Box\forall x\forall y((Px \leftrightarrow Py) \to (Qx \leftrightarrow Qy))$. The less obvious formulation in the main text has the merit of facilitating the comparison with strong supervenience.

Funding. This project has been financially supported by the Italian Ministry of University and Research (MUR) through a *Levi Montalcini* fellowship, and by the European Research Council (ERC) through the research project *Inquisitive Modal Logic* (InqML, grant agreement number 101116774).

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DEPARTMENT OF PHILOSOPHY
SOCIOLOGY, PEDAGOGY, AND APPLIED PSYCHOLOGY
UNIVERSITÀ DI PADOVA
PADUA, ITALY
E-mail: ivano.ciardelli@unipd.it