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SETS OF DETERMINATION AND KERNELS OF CERTAIN ASSOCIATED OPERATORS

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Abstract Let m be a measure supported on a relatively closed subset X of the unit disc. If f is a bounded function on the unit circle, let f_m denote the restriction to X of the harmonic extension of f to the unit disc. We characterize those m such that the pre-adjoint of the linear map $f \to f_m$ has a non-trivial kernel.

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1. Introduction

Let D denote the open unit disc in the complex plane, let T be the unit circle and let $\mathrm{d}\theta$ denote the normalized linear measure on T. Let m denote a positive σ-finite measure on D and consider the operator

$$\Gamma : f \to \tilde{f}|_X,$$

where $f \in L^\infty(\mathrm{d}\theta)$, X is the closed support (relative to D) of m and $\tilde{f}|_X$ denotes the restriction to X of the harmonic extension $\tilde{f}$ of f to D. We are interested here in the operator

$$\Lambda : L^1(m) \to L^1(\mathrm{d}\theta)$$

having Γ as its adjoint. Our main concern is to give necessary and sufficient conditions on m in order that Λ has a non-trivial kernel. This was one of several problems studied by Bonsall in [**1**] and our work is motivated by his results.

Let $\phi \in \ker \Lambda$. We assume that ϕ is real valued and can conclude that

$$\int_T \Lambda(\phi)(\zeta) f(\zeta)\,\mathrm{d}\zeta = \int_D \phi(z)\tilde{f}(z)\,\mathrm{d}m = 0$$

for any bounded measurable function f on the unit circle T. Since ϕ is real valued, it is evident that $\phi \in \ker \Lambda$ if and only if

$$\int_D \phi(z)h(z)\,\mathrm{d}m = 0$$

for any $h \in H^\infty(D)$, where $H^\infty(D)$ consists of all bounded analytic functions in D.

The kernel of Λ is also closely connected with sets of determination for bounded harmonic functions. Let U denote an open subset of the plane and let the space of all bounded harmonic functions on U be denoted by $h^\infty(U)$. A subset S of U is called a set of determination for $h^\infty(U)$ if

$$\sup\{|f(z)|,\ z \in S\} = \sup\{|f(z)|,\ z \in U\}$$

for all $f \in h^\infty(U)$. If $H^\infty(U)$ denotes the bounded analytic functions on U, sets of determination for $H^\infty(U)$ are defined in the same way. By the work of Brown *et al.* [**2**] and Bonsall [**1**], it follows that the sets of determination for $H^\infty(D)$ and $h^\infty(D)$ coincide. Hence, this is true if D is replaced by any set U conformally equivalent to the unit disc D. Moreover, $S \subset D$ is a set of determination for $h^\infty(D)$ if and only if it has the following geometric property.

Definition 1.1. Almost all $z \in T$ (with respect to $\mathrm{d}\theta$) is in the closure of a sequence $\{z_n\}$ from S converging non-tangentially to z, meaning that $|z - z_n|/(1 - |z_n|)$ remains bounded as $n \to \infty$.

We shall start by looking at discrete measures m of the form

$$m = \sum m_\nu,$$

where m_ν is the point measure at $z_\nu \in D$, $\nu = 1, 2, 3, \dots$. In this case it is an easy exercise to show that Λ has the form

$$\Lambda : \{\lambda_\nu\} \to \sum \lambda_\nu P_{z_\nu},$$

where $\{\lambda_\nu\} \in l^1$ and P_{z_ν} denotes the Poisson kernel for z_ν considered as an element of $L^1(\mathrm{d}\theta)$. The adjoint map Γ is now given by

$$\Gamma : f \to \{\tilde{f}(z_\nu)\}.$$

Note that Γ is an isometry if and only if $S = \{z_\nu\}$ is a set of determination for $H^\infty(D)$. As observed by Bonsall, it follows in this case from Banach's Closed Range Theorem [**1**] that, for such S, Λ is surjective. But then it is easy to see that $\ker \Lambda$ is non-trivial. Indeed, since $S \setminus \{z_1\}$ is a set of determination if and only if S is a set of determination, it follows that

$$P_{z_1} = \sum_{2}^{\infty} \gamma_\mu P_{z_\mu}$$

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for suitable coefficients $\{\gamma_\mu\} \in l^1$. Hence, the sequence $1, -\gamma_2, -\gamma_3, \dots$ is mapped to 0 by $\varLambda$. The main result in § 2 is that, for discrete measures m, $\ker \varLambda$ is non-empty if and only if S has a subset being a set of determination at some 'local' level inside D.

If m is not discrete, the characterization of when $\varLambda$ has a non-trivial kernel is more complicated. The details are given in § 3. For a given compact set Y, the uniform algebra $R(Y)$ consists of all uniform limits on Y of rational functions with their poles away from Y. We say that $R(Y)$ is a Dirichlet algebra on ∂Y if the real parts of functions in $R(Y)$ are dense in $C(\partial Y)$. We refer the reader to [**5**] for more details about $R(Y)$ and general properties of uniform algebras.

2. Main result

Theorem 2.1. *Let m be a positive measure supported on a countable subset S of the unit disc D. The associated linear operator $\varLambda$ has a non-trivial kernel if and only if there is a simply connected subset U of D such that $S \cap U$ is a set of determination for $H^\infty(U)$.*

Proof. Suppose that $\phi \in L^1(m)$ is non-zero and $\varLambda(\phi) = 0$. Then the measure $\sigma = \phi\, \mathrm{d}m$ annihilates any $f \in H^\infty(D)$.

The simply connected subset U of D postulated in Theorem 2.1 is obtained by following lemma.

Lemma 2.2. *Let $\varUpsilon$ be the collection of all compact subsets Y of $\bar{D}$ which support σ and such that $R(Y)$ is Dirichlet and σ is orthogonal to $R(Y)$. Then if $Y \in \varUpsilon$, we can find $\tilde{Y} \in \varUpsilon$ such that $\tilde{Y} \subset Y$ and $|f(z)| \leqslant \|f\|_{S\cap Y^{\mathrm{o}}}$ for all $z \in \tilde{Y} \cap Y^{\mathrm{o}}$ and $f \in H^\infty(Y^{\mathrm{o}})$.*

To prove Lemma 2.2, let us first remark that, for any $Y \in \varUpsilon$, the measure σ is carried by Y^{o} and is orthogonal to $H^\infty(Y^{\mathrm{o}})$. This follows since any $\zeta \in \partial Y$ is a peak point for $R(Y)$ and since $R(Y)$ is pointwise boundedly dense in $H^\infty(Y^{\mathrm{o}})$ when $R(Y)$ is a Dirichlet algebra. Now let $Y \in \varUpsilon$ and define

$$Z = \{z \in Y^{\mathrm{o}} : |f(z)| \leqslant \|f\|_{S\cap Y^{\mathrm{o}}},\ f \in H^\infty(Y^{\mathrm{o}})\},$$

where, for a set K, $\|f\|_K$ denotes the supremum of $|f|$ over K. Let $\tilde{Y}$ denote the closure of Z. The uniform algebra $R(\tilde{Y})$ consists of all uniform limits on $\tilde{Y}$ of rational functions with their poles away from $\tilde{Y}$. Consider the Banach algebra B consisting of all functions on $\tilde{Y} \cap Y^{\mathrm{o}}$ being uniformly approximable there by functions from $H^\infty(Y^{\mathrm{o}})$. If $a \in Y^{\mathrm{o}} \setminus \tilde{Y}$, there is $h \in H^\infty(Y^{\mathrm{o}})$ such that $h(a) = 1$ and $\|h\|_{S\cap Y^{\mathrm{o}}} < 1$. Then

$$1 - h = (z - a)g$$

with $g \in H^\infty(Y^{\mathrm{o}})$ and hence

$$\frac{1}{z-a} = \frac{g}{1-h} = \sum_{n=0}^{\infty} g h^n,$$

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and we conclude that the function $(z-a)$ is invertible in B. Since σ is carried by $S\cap Y^{\mathrm{o}}$, it follows that

$$\int f\,\mathrm{d}\sigma = 0 \quad \text{for all } f \in R(\tilde{Y}).$$

We must show that $\tilde{Y} \in \Upsilon$. From the way $\tilde{Y}$ was constructed, it follows by the maximum principle that any component V of $C \setminus \tilde{Y}$ with $V \cap Y^{\mathrm{o}} \neq \emptyset$ has the property that ∂V must meet $C \setminus Y^{\mathrm{o}}$. We recall the well-known fact that the analytic capacity of a compact connected set is comparable to its diameter (see, for example, [**5**, Theorem 2.1, p. 199]). Therefore, near any $\zeta \in \partial\tilde{Y} \cap Y^{\mathrm{o}}$ there is, for small r, a compact connected set in $\{z : |z-\zeta| < r\} \cap (C \setminus \tilde{Y})$ with analytic capacity comparable to r.

By the work in [**3**, **5**], a general set $Y \in \Upsilon$, being a Dirichlet algebra, is characterized by the following:

(a) $R(\partial Y) = C(\partial Y)$;

(b) $R(Y)$ is pointwise boundedly dense in $H^{\infty}(Y^{\mathrm{o}})$, and $C \setminus Y^{\mathrm{o}}$ is connected.

Moreover, properties (a) and (b) can be characterized using analytic capacity (see [**5**, Chapter VII] and [**6**]). If ζ is a boundary point of $\tilde{Y}$ not in Y^{o}, it must belong to ∂Y, and we can verify that (a) and (b) hold for $\tilde{Y}$ since they hold for Y. In addition, the interior of $\tilde{Y}$ has a connected complement, since it is obtained from $C \setminus Y^{\mathrm{o}}$ by adding certain connected sets V having non-empty intersection with $C \setminus Y^{\mathrm{o}}$. It follows that $\tilde{Y} \in \Upsilon$.

Now let $\lambda(\Upsilon)$ be the greatest lower bound to area(Y) as Y varies over Υ. Let Y_n denote a sequence from Υ such that area$(Y_n) \to \lambda(\Upsilon)$. By in [**7**, Theorem 3.6] we may assume that $Y_n \subset Y_{n-1}$, and by the same theorem it follows that

$$Y_{\infty} = \bigcap_{n=1}^{\infty} Y_n$$

belongs to Υ. Since the measure σ was assumed to be non-trivial and $R(Y_{\infty})$ is a Dirichlet algebra on ∂Y_{∞}, we must have $\lambda(\Upsilon) > 0$, and the interior Ω of Y_{∞} must be non-empty. Since Y_{∞} has minimal area within Υ, Lemma 2.2 gives that $S\cap\Omega$ is a set of determination for $H^{\infty}(\Omega)$.

Conversely, assume that there is a simply connected set U such that $S \cap U$ is a set of determination for $H^{\infty}(U)$. Let V denote a component of U. If $S \cap V = z_1, z_2, \dots$, there exist coefficients γ_{μ}, $\mu = 2, 3, \dots$, such that

$$h(z_1) = \sum_{2}^{\infty} \gamma_{\mu} h(z_{\mu})$$

for all $h \in H^{\infty}(V)$. This follows from the corresponding result in the unit disc by conformal mapping. Hence, $\sigma = m_{z_1} - \sum_2^{\infty} \gamma_{\mu} m_{z_{\mu}}$ belongs to the kernel of Λ. □

3. General measures

In this section we consider a positive regular Borel measure m on D, which is finite on compact subsets of D, and investigate for which m the linear operator

$$\Lambda : L^1(m) \to L^1(\mathrm{d}\theta)$$

described in § 1 has a non-trivial kernel. The main obstacle when trying to repeat the construction from § 2 comes from the fact that the restriction $m|_X$ of the measure m to the boundary ∂X of a compact subset X of $\bar{D}$ may be non-trivial. As a consequence, the concept 'set of determination' for $H^\infty(\Omega)$ must be generalized to cover sets that partially meet the boundary of Ω. The sets X we encounter when characterizing $\ker \Lambda$ for general m will fortunately have the following nice property: $R(X)$ is a Dirichlet algebra viewed as a function algebra on ∂X. Owing to the fundamental work by Davie [**3**] and Gamelin and Garnett [**6**] on $R(X)$ as a Dirichlet algebra, these more general sets of determination are easily described in terms of harmonic measure and conformal mappings from D to components Ω of the interior X^{o} of X.

We now give the known facts from [**3**,**6**] which we shall need to find $\ker \Lambda$ for general m.

Consider a compact subset X of $\bar{D}$ such that $R(X)$ is a Dirichlet algebra on ∂X. Let U_i denote a component of the interior X^{o} of X.

Lemma 3.1. *Let $\phi : D \to U_i$ be conformal. There is then a subset $S_i \subset \partial D$ such that ϕ has radial limit $\phi^*(\mathrm{e}^{\mathrm{i}\theta}) \in \partial U_i$ for all $\mathrm{e}^{\mathrm{i}\theta} \in S_i$ and such that ϕ^* is one-to-one on S_i and $T \setminus S_i$ has zero linear measure.*

This result is due to Davie [**3**]. We can now define a measure λ_i on ∂U_i by the relation

$$\lambda_i(B) = \mu\{\mathrm{e}^{\mathrm{i}\theta} : \phi^*(\mathrm{e}^{\mathrm{i}\theta}) \in B\},$$

where μ denotes normalized linear measure on T and B is any Borel set. It is not hard to see that λ_i is actually a harmonic measure on ∂U_i representing the point $\phi(0)$. Moreover, if $i \neq j$, λ_i and λ_j are mutually singular. We form the measure

$$\lambda = \sum_i 2^{-i} \lambda_i,$$

where the summation extends over all components U_i of X^{o}. Of course, this definition of λ makes sense for any open plane set, and in the following λ shall have this meaning when U is given. If $f \in L^\infty(\lambda)$, we define

$$\tilde{f}(z) = \int_{\partial U} f \, \mathrm{d}\lambda_z,$$

where $z \in U$ and λ_z is harmonic measure on ∂U_i representing z; here U_i is the component of U containing z. Moreover, since $R(X)$ is a Dirichlet algebra on ∂X, the mapping $f \to \tilde{f}$ is an isometry from $H^\infty(\lambda)$ onto $H^\infty(U)$, where $H^\infty(\lambda)$ denotes the w^* closure of $R(\bar{U})$ in $L^\infty(\lambda)$.

Definition 3.2. Let U denote an open subset of the complex plane with connected complement. Let $B \subset \partial U$ be a Borel set and let $E \subset U$ be relatively closed in U. The pair $\{B, E\}$ is called a set of determination for $h^\infty(U)$ if

$$\|f\|_{L^\infty(\lambda)} = \max\{\|\chi_B f\|_{L^\infty(\lambda)}, \|\tilde{f}\|_E\}$$

for any $f \in L^\infty(\lambda)$, where χ_B denotes the characteristic function of B and $\|g\|_E$ denotes the pointwise supremum of $|g|$ on E.

In connection with the statement and proof of the next theorem it may be appropriate to briefly discuss the case where $U = Y^o$ and $R(Y)$ is a Dirichlet algebra on ∂Y. In the light of the isometric map $f \to \tilde{f}$ mentioned before Definition 3.2, we can define a pair $\{B, E\}$ as a set of determination for $H^\infty(U)$ exactly as in Definition 3.2. It is then clear that $\{B, E\}$ is also a set of determination for $h^\infty(U)$, because if u is a bounded harmonic function in U with a harmonic conjugate $\tilde{u}$, then $e^{u+i\tilde{u}} \in H^\infty(U)$ and from this it is immediate that u satisfies the conditions in Definition 3.2. In this special case (which occurs in Theorem 3.3) we can describe sets of determination for $H^\infty(U)$ in a more geometric way using the notation from Lemma 3.1: $\{B, E\}$ is a set of determination for $H^\infty(U)$ if and only if, for all i, the non-tangential closure of $\phi^{-1}(E \cap U_i)$ contains almost all of $T \setminus ((\phi^*)^{-1}(B \cap \partial U_i))$. For details see the proof of Theorem 3.3.

Suppose now that μ is a measure absolutely continuous with respect to λ, so that $d\mu = h\, d\lambda$. The set of all ζ such that $h(\zeta) \neq 0$ is called a minimal support set for μ. Such a set is, of course, unique up to a set of zero λ-measure.

We can now formulate our main result for general measures.

Theorem 3.3. *Let m be a positive regular Borel measure supported on the open unit disc D. The linear operator $\Lambda : L^1(m) \to L^1(d\theta)$ has a non-trivial kernel if and only if there is a simply connected subset U of D such that $R(\bar{U})$ is a Dirichlet algebra on ∂U and the following holds. If $m|_{\partial U} = \mu + \nu$ with μ absolutely continuous and ν singular with respect to λ, then $\{B, E\}$ is a set of determination for $h^\infty(U)$, where B is a minimal support set for μ and $E \neq \emptyset$ is the closed support of m in U.*

Before proving Theorem 3.3, let us consider a measure m corresponding to a set of determination in the special case where U is the unit disc D and λ is linear measure on ∂D. We assume here that m is a measure on the closed unit disc and that B and E correspond to m as in Theorem 3.3. Let m_D denote the restriction of m to D. We assume that m_D is non-zero. If $\{B, E\}$ is a set of determination for $h^\infty(D)$, it is evident that the mapping

$$f \to \{f\chi_B, \tilde{f}|_E\}$$

is an isometry from $L^\infty(\lambda)$ into $L^\infty(\chi_B\lambda) \times L^\infty(m_D)$.

Moreover, we conclude in this special case that $\{B, E\}$ is a set of determination for $h^\infty(D)$ if and only if the non-tangential closure of E on ∂D contains λ-almost all points of $\partial D \setminus B$. Indeed, if this geometric condition holds, Fatou's Theorem on non-tangential limits gives the isometry, while if the geometric condition fails, the isometric property fails by a standard argument [**2**, **8**]. It is easy to verify that the isometry given above is the adjoint of the mapping

$$T : L^1(\chi_B \lambda) \times L^1(m_D) \to L^1(\lambda)$$

given by

$$T(\phi, \psi) = \phi + \int P_z \psi \, \mathrm{d}m_D(z).$$

Here P_z denotes the Poisson kernel and $\int P_z \psi \, \mathrm{d}m_D(z)$ is the (unique) function in $L^1(\lambda)$ such that the duality relation

$$\left\langle \int P_z \psi \, \mathrm{d}m_D(z), h \right\rangle = \int \psi(z) \tilde{h}(z) \, \mathrm{d}m_D(z)$$

holds with $h \in L^\infty(\lambda)$. We can easily verify that T has a non-trivial kernel. Indeed, since its adjoint T^* is an isometry, it follows in particular that T^* is one-to-one and has closed range. By a theorem of Banach [**4**, p. 488], it follows that T is surjective. Let K denote a compact subset of D with $0 < m_D(K) < \infty$ and let $f_K = T(\chi_K)$. Now replace the measure m by the measure m_0 obtained from m by removing all mass located on K. The operator T_0 associated with m_0 is clearly also surjective, and therefore $f_K = T_0(g)$ for some $g \in L^1(m_0)$. Hence, $g - \chi_K$ belongs to the kernel of T.

We can now give the proof of Theorem 3.3. Let us first show that $\ker \Lambda$ is non-trivial if m has the properties listed in Theorem 3.3. Let $H^\infty(\lambda)$ denote the w^* closure of $R(\bar{U})$ in $L^\infty(\lambda)$. As is well known, since $R(\bar{U})$ is a Dirichlet algebra, the mapping $f \to \tilde{f}$ is an isometry between $H^\infty(\lambda)$ and $H^\infty(U)$. Fix a component U_0 of U. Let $\phi : D \to U_0$ denote a conformal map and let ψ denote its inverse. Then ϕ extends to be defined almost everywhere on ∂D by Fatou's Theorem and this extended function is still denoted by ϕ. Likewise, ψ has a natural extension to ∂U_0 defined almost everywhere with respect to λ in the light of the isometric mapping between $H^\infty(\lambda)$ and $H^\infty(U)$. This extension is also denoted by ψ. Then the composite functions $\phi \circ \psi$ and $\psi \circ \phi$ coincide with the identity almost everywhere with respect to λ and the linear measure on ∂D, respectively. We can conclude that $\{B, E\}$ is a set of determination for $h^\infty(U_0)$ if and only if $\{\phi^{-1}(B), \phi^{-1}(E)\}$ is a set of determination for $h^\infty(D)$. Owing to the geometric characterization of sets of determination mentioned in the remarks following Theorem 3.3, we conclude that even $\{B, E \setminus V\}$ is a set of determination for $h^\infty(U_0)$, provided that V is a set with compact closure inside U_0. As in the special case where $U_0 = D$, we conclude that the mapping

$$T_0 : L^1(\chi_B \lambda) \times L^1(m_{U_0}) \to L^1(\lambda)$$

given by

$$T_0(\phi, \psi)(f) = \int_{\partial U_0} f \phi \, \mathrm{d}\lambda + \int_{U_0} \psi \tilde{f} \, \mathrm{d}m$$

has a non-trivial kernel. If $T_0(\phi, \psi) = 0$, it is clear that also $\Lambda(\phi + \psi) = 0$ (here we consider $\phi + \psi$ as an element of $L^1(m)$), and hence Λ has a non-trivial kernel.

Conversely, let us assume that Λ has a non-trivial kernel. Let $\phi \neq 0$ be a real-valued element of $\ker \Lambda$. This means that the measure $\nu = \phi \, \mathrm{d}m$ is orthogonal to $H^\infty(D)$. We formulate the following result, which is similar to Lemma 2.2.

Lemma 3.4. *Let Υ be the collection of all compact subsets X of $\bar{D}$ which support ν and such that $R(X)$ is Dirichlet and ν is orthogonal to $R(X)$. Then if $X \in \Upsilon$, we can find $\tilde{X} \in \Upsilon$ such that $\tilde{X} \subset X$ and $|f(z)| \leqslant \|f\|_{L^\infty(|\nu|)}$ for all $z \in \tilde{X} \cap X^{\mathrm{o}}$ and $f \in H^\infty(X^{\mathrm{o}})$.*

We now explain the content of Lemma 3.4 in some detail. Given $X \in \Upsilon$, let λ denote the weighted sum of harmonic measures defined after Lemma 3.1. Since compact subsets of ∂X of zero λ-measure are peak interpolation sets for $R(X)$, $\nu|_{\partial X}$ is absolutely continuous with respect to λ. If $f \in H^\infty(X^{\mathrm{o}})$, it has, as explained after Lemma 3.1, 'boundary values' in $H^\infty(\lambda)$ and the meaning of $\|f\|_{L^\infty(|\nu|)}$ should now be clear.

The proof of Lemma 3.4 and the application of it to complete the proof of Theorem 3.3 is so similar to the corresponding argument in § 2 that we omit the details.

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