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A REMARK ON EXACT SIMULATION OF TEMPERED STABLE ORNSTEIN–UHLENBECK PROCESSES

TAKUJI ARAI,* *Keio University*
YUTO IMAI,** *Nishogakusha University*

Abstract

Qu, Dassios, and Zhao (2021) suggested an exact simulation method for tempered stable Ornstein–Uhlenbeck processes, but their algorithms contain some errors. This short note aims to correct their algorithms and conduct some numerical experiments.

Keywords: Monte Carlo simulation; Ornstein–Uhlenbeck process; tempered stable process; inverse Gaussian process

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1. Introduction

A stochastic process $Z = \{Z_t\}_{t\geq 0}$ is said to be a tempered stable (TS) subordinator if it is a driftless subordinator with the Lévy measure $\nu(\mathrm{d}y) = \theta y^{-\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y$, $y > 0$, where $\alpha \in (0, 1)$ and $\beta, \theta \in \mathbb{R}^+$. In this case, we call the distribution of Z_1 a tempered stable distribution with parameters α, β, θ, and denote it by TS (α, β, θ). In addition, a process $X = \{X_t\}_{t\geq 0}$ is said to be a TS-based Ornstein–Uhlenbeck (OU-TS) process if it is a solution to the following stochastic differential equation:

$$\mathrm{d}X_t = -\delta X_t\,\mathrm{d}t + \rho\,\mathrm{d}Z_t, \qquad X_0 > 0, \tag{1}$$

where $\delta > 0$ and $\rho > 0$. For any $t \geq 0$ and $\tau > 0$, we have

$$X_{t+\tau} = \mathrm{e}^{-\delta\tau}X_t + \rho\int_t^{t+\tau}\mathrm{e}^{-\delta(t+\tau-s)}\,\mathrm{d}Z_s.$$

Qu et al. [2] suggested an exact simulation algorithm for $X_{t+\tau}$ given X_t. In addition, they separately gave another algorithm available only for the case of $\alpha = \frac{1}{2}$. Here we correct the two algorithms and introduce the results of some numerical experiments.

2. Mathematical background and algorithms

The infinitesimal operator $\mathcal{A}$ of X is given by

$$\mathcal{A}f(x, t) = \frac{\partial f}{\partial t} - \delta x\frac{\partial f}{\partial x} + \int_0^\infty \{f(x+\rho y, t) - f(x, t)\}\,\nu(\mathrm{d}y),$$

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* Postal address: Department of Economics, Keio University, 2-15-45 Mita, Minato-ku, Tokyo, 108-8345, Japan. Email: arai@econ.keio.ac.jp
** Postal address: Faculty of International Politics and Economics, Nishogakusha University, 6-16 Sanbancho, Chiyoda-ku, Tokyo, 102-8336, Japan. Email: y-imai@nishogakusha-u.ac.jp

where $f\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is differentiable on x and t. We can derive this by applying [1, (6.36)] to the stochastic differential equation (1). [2, (3.2)] gave a representation of $\mathcal{A}$ incorrectly as follows:

$$\mathcal{A}f(x, t) = \frac{\partial f}{\partial t} - \delta x \frac{\partial f}{\partial x} + \rho \int_0^\infty \{f(x+y, t) - f(x, t)\}\, \nu(\mathrm{d}y).$$

Thus, all the subsequent arguments in [2] must be corrected, but this error does not affect the case of $\rho = 1$. Now, we fix $t \geq 0$ and $\tau > 0$, and define a process $Y = \{Y_s\}_{t \leq s \leq t+\tau}$ as

$$Y_s := \exp\left\{-X_s \kappa \mathrm{e}^{\delta s} + \int_0^s \Phi(\rho\kappa \mathrm{e}^{\delta u})\, \mathrm{d}u\right\},$$

where $\kappa \in \mathbb{R}$, and Φ is the Laplace exponent of Z, i.e. $\Phi(u) := \int_0^\infty (1 - \mathrm{e}^{-uy})\, \nu(\mathrm{d}y)$. When $\mathcal{A}f(x, t) = 0$, the process $f(X_t, t)$ is a martingale. From this point of view, we can see that Y is a martingale. For any $\eta \in \mathbb{R}^+$, taking $\kappa = \eta \mathrm{e}^{-\delta(t+\tau)}$, we obtain

$$\mathbb{E}\left[\mathrm{e}^{-\eta X_{t+\tau}} \mid X_t\right] = \exp\left\{-\eta X_t \mathrm{e}^{-\delta\tau} - \int_{\rho\eta \mathrm{e}^{-\delta\tau}}^{\rho\eta} \frac{\Phi(z)}{\delta z}\, \mathrm{d}z\right\}. \tag{2}$$

From the view of [2, (3.7)–(3.9)], (2) implies

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-\eta X_{t+\tau}} \mid X_t] = \exp&\left\{-\eta w X_t - \frac{\rho^\alpha \theta(1-w^\alpha)}{\alpha\delta} \int_0^\infty (1 - \mathrm{e}^{-\eta s}) s^{-\alpha-1} \mathrm{e}^{-(\beta/w\rho)s}\, \mathrm{d}s\right\} \\ &\times \exp\left\{-\frac{\theta\beta^\alpha \Gamma(1-\alpha) D_w}{\alpha\delta} \int_0^\infty (1 - \mathrm{e}^{-\eta s})\right. \\ &\left.\times \int_1^{1/w} \frac{((\beta/\rho)u)^{1-\alpha}}{\Gamma(1-\alpha)} s^{(1-\alpha)-1} \mathrm{e}^{-(\beta/\rho)us} f_V(u)\, \mathrm{d}u\, \mathrm{d}s\right\},\end{aligned} \tag{3}$$

where $w := \mathrm{e}^{-\delta\tau}$, $\Gamma(\,\cdot\,)$ is the Gamma function, and

$$D_w := \int_1^{1/w} (u^{\alpha-1} - u^{-1})\, \mathrm{d}u = \frac{w^{-\alpha} - 1}{\alpha} + \ln w, \qquad f_V(u) := \frac{u^{\alpha-1} - u^{-1}}{D_w}, \qquad u \in [1, 1/w].$$

Equation (3) can be obtained by replacing θ and β in [2, Theorem 3.1] with $\rho^{\alpha-1}\theta$ and β/ρ, respectively. Thus, [2, Algorithm 3.2] can be corrected by replacing all θs and βs appearing there in the same way. As for the correction of [2, Algorithm 3.4], we have only to change the distribution of $\widetilde{IG}$ into

$$\mathrm{IG}\left(\frac{2\rho}{c\delta}(\sqrt{w} - w), \frac{4\rho}{\delta^2}(1 - \sqrt{w})^2\right),$$

where IG (μ, λ) denotes the IG distribution with the mean parameter μ and the rate parameter λ. Furthermore, [2, Proposition 6.1] can be corrected with the same replacements of θ and β as above, but additionally, the function $h(\,\cdot\,)$ needs to be replaced by $h(\,\cdot\,/\rho)$.

Remark 1. Denoting the solution to (1) by X^{ρ, X_0} with emphasis on $\rho > 0$ and the initial value $X_0 > 0$, we have $X^{\rho, X_0} = \rho X^{1, X_0/\rho}$ for any $\rho > 0$ and $X_0 > 0$. Thus, the algorithm with $\rho = 1$ can be generalized to any case of $\rho > 0$.

TABLE 1. OU-TS process with $\alpha = 0.25$.

ρ	$\mathbb{E}[{X_5}^2]$	Corrected algorithm			[2, Algorithm 3.2]		
		Estim	Diff	Error %	Estim	Diff	Error %
0.5	20.6047	20.5968	0.0079	0.0383	21.0192	−0.4145	−2.0117
1	29.8371	29.8275	0.0096	0.0322	29.8636	−0.0265	−0.0888
2	54.7854	54.8371	−0.0517	−0.0944	51.4652	3.3202	6.0604
5	181.4987	181.5756	−0.0769	−0.0424	148.0422	33.4565	18.4335

3. Numerical results

As can be seen in [2, Tables 1 and 2], even using the original algorithms in [2] the errors are kept small enough as long as the means are computed. Thus, we compute the second moments instead and compare the results of the original and corrected algorithms.

Here, we execute simulations for an OU-TS process with $\alpha = 0.25$. For the other parameters, we set $\delta = 0.2$, $\beta = 0.5$, $\theta = 0.25$ and vary the value of ρ as 0.5, 1, 2, 5. We set $X_0 = 10.0$ and simulate $X_{0.5}$; next, we simulate X_1 using the value of $X_{0.5}$, which is repeated until we simulate X_5. We carried out the simulation one million times, and compared their mean square with the second moment $\mathbb{E}[X_5^2]$, where we can calculate $\mathbb{E}[X_5^2]$ by using [2, (3.5)] and (2) as follows:

$$\mathbb{E}[X_5^2] = \left\{ wX_0 + \frac{\rho\theta}{\delta\beta^{1-\alpha}}(1-w)\Gamma(1-\alpha) \right\}^2 + \frac{\rho^2\theta}{2\delta\beta^{2-\alpha}}(1-w^2)\Gamma(2-\alpha),$$

where $w = \mathrm{e}^{-5\delta}$. The algorithms were coded in MATLAB (R2022b).

The simulation results are given in Table 1. Note that "Estim" in the third column represents the mean square of one million simulation results, and "Diff" and "Error" in the fourth and fifth columns are defined as Diff $:=$ Estim $- \mathbb{E}[X_5^2]$ and Error $:= (\text{Diff}/\mathbb{E}[X_5^2]) \times 100$, respectively. The last three columns display the results for the original algorithm, [2, Algorithm 3.2]. As seen in Table 1, the errors of the corrected algorithm are small enough regardless of the value of ρ, but for the original algorithm this is not the case. Furthermore, similar results were obtained for the cases of OU-TS with $\alpha = 0.75$ and OU-IG (inverse Gaussian), but they are not tabulated.

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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