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Legendre quadrature for the discretization of 1D radiating panels

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Research Paper

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Abstract

In [A. Capozzoli, C. Curcio, A. Liseno, MMS, Pizzo Calabro, Italy, 2022], the problem of modeling a source/scatterer using an equivalent radiator has been addressed and an approach has been given and numerically assessed.

Once dimensioned the radiating panel, a practical implementation can be provided by a non-uniform array. The element positions should be chosen so that the array is capable to approximate, with an adequate accuracy, the fields radiated by the equivalent radiator. Here, the array element positioning is performed by exploiting a quadrature rule which takes into account that the singular functions supported on the region of interest associated to the most significant singular values of the radiation operator are related to those supported on the equivalent panel by a radiation integral. The quadrature rule enables also to choose a set of weights which are essential in the definition of the element excitation coefficients from the knowledge of the source distribution on the equivalent panel. For simplicity, a one-dimensional problem with a Legendre quadrature rule is considered. The approach is numerically assessed by checking the capability of the array to radiate, with a satisfactory degree of accuracy, the singular functions associated to the region of interest.

Introduction

Modeling a source or a scatterer using an equivalent radiator is of interest in many applications as antenna synthesis [1], electromagnetic compatibility [2], the design of complex waveform generators [3], computational electromagnetics and inverse scattering [4, 5].

The problem can be framed as an extension of one of the classical equivalence theorems. It can be formulated as that of determining the shape and dimensions of a radiating surface capable to produce, in a targeted region of space Ω, an electromagnetic field as close as possible to that generated by any "primary" radiator/scatterer, contained in a prefixed region of space Γ, according to a prefixed tolerance. In [6], the dimensioning problem has been dealt with as the determination of effective subspaces associated to the two operators linking the radiator/scatterer or the equivalent radiating panel Γ to their respective fields radiated over the region of interest. A solution has been provided by aid of the singular value decomposition (SVD) of the two mentioned operators. In particular, the singular functions of such operators associated to the significant singular values identify the linear subspaces to which the fields radiated by the radiator/scatterer and by the panel belong. The dimensions of the equivalent radiator are determined to reduce, as much as possible, the error by which the field radiated by the equivalent panel approximates the primary one, independently from any allowed primary field itself. In [6], the case of a flat and rectangular Γ and of a flat and rectangular Ω parallel to Γ has been considered. For the sake of simplicity, a scalar problem has been dealt with.

Once dimensioned the radiating panel, a practical implementation is in order. A possibility is to consider a discretization of the equivalent panel based on the use of a non-uniformly spaced array of radiators, a problem already explored in [7]. The element positions should be properly chosen so that the array is capable to approximate, with an adequate degree of accuracy, any field radiated by the equivalent radiator. Following the idea in [7], in this paper, the array element positioning is worked out by exploiting a quadrature rule [8–10] transforming an integration into a summation. In other words, taking into account that the singular functions supported on the region of interest associated to the significant singular values of the radiation operator are related to those supported on the equivalent panel by a radiation integral, the quadrature rule replaces the integral representation of the singular functions into an approximation by a weighted summation. The quadrature rule defines also a set of weights which are essential in the

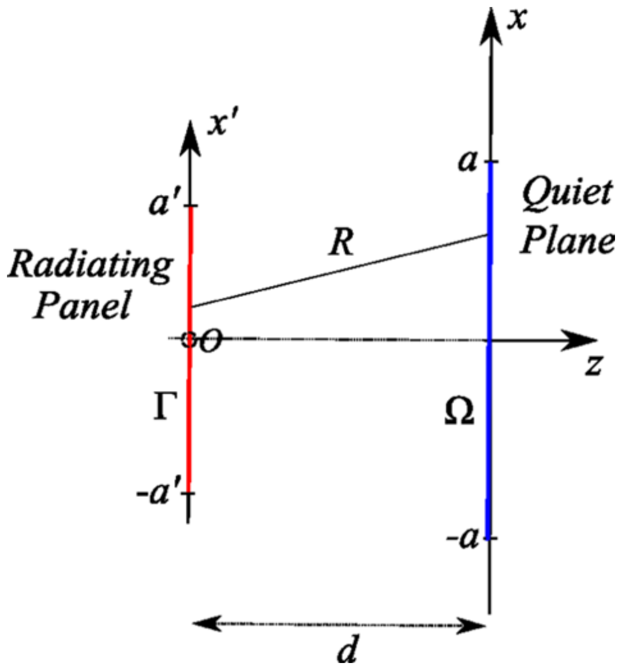

Figure 1. Geometry of the radiating panel.

definition of the element excitation coefficients from the knowledge of the source distribution on Γ.

For the sake of simplicity, in this paper, we consider a one-dimensional problem and a Legendre quadrature rule. The approach is numerically assessed by checking the capability of the array to radiate, within a prefixed tolerance, all the singular functions associated to the region of interest.

The paper is organized as follows. In section "The Radiation Operator and Its SVD," the radiation operator from the equivalent panel to the region of interest is briefly introduced along with its SVD. In "Legendre Quadrature" section, the guidelines for the Legendre quadrature rule are recalled. Section "Panel Discretization" is devoted to the panel discretization by the quadrature-defined array. The performance of the approach is assessed in the following section. Finally, conclusions are drawn and future developments are foreseen in last section.

The radiation operator and its SVD

The geometry of the problem is depicted in Fig. 1. We consider the two-dimensional case of a strip source with current $\underline{J}(x') = J(x')\hat{i}_y$, radiating in free-space and laying on a radiating panel of size $2a'$ whose field is of interest over a portion Ω of a quiet plane of size $2a$. The only (y) component of the radiated field is denoted by $E(x, z)$ and is expressed, apart from the unessential factor $-\omega\mu_0/4$, as:

$$E(x, z) = \mathcal{A}(J) = \int_{-a'}^{a'} J(x') H_0^{(2)}(\beta R) dx', \quad z = d,\ x \in (-a, a), \tag{1}$$

where ω is the angular frequency, μ_0 is the free-space magnetic permeability of the embedding medium, $H_0^{(2)}$ is the Hankel function of zero-th order and second kind, $R = \sqrt{(x - x')^2 + d^2}$ and $\mathcal{A}$ is the radiation operator mapping J into the field on $(-a, a)$.

We denote by $\{\sigma_l, u_l(x'), v_l(x)\}_{l=0}^{+\infty}$ the singular system of $\mathcal{A}$, where the σ_l's are the singular values, the u_l's are the right singular functions expanding the radiating current J and the v_l's are the left singular functions expanding the radiated field E. From a practical point of view, the spectral representation of the radiation operator can be limited to the only part of the singular system corresponding to the singular values deemed to be significant, namely, $\{\sigma_l, u_l(x'), v_l(x)\}_{l=0}^{L-1}$. In other words,

$$E(x, d) = \mathcal{A}(J) = \sum_{l=0}^{L-1} \sigma_l < J, u_l >_{(-a', a')} v_l(x), \tag{2}$$

where $< \cdot, \cdot >_{(-a', a')}$ is the scalar product in $\mathcal{L}^2(-a', a')$. As a consequence, the field radiated by J belongs essentially to the finite dimensional space spanned by the v_l's, $l = 0, \ldots, L-1$.

Legendre quadrature

In this section, we shortly review Legendre quadrature which will be used in section "Panel Discretization".

To this end, let us firstly state that, following a change of variables, any one-dimensional integral over a domain (b, c) can be set up as an integral over $(-1, 1)$, namely:

$$\int_b^c f(x) dx = \frac{c-b}{2} \int_{-1}^{1} f\left(\frac{c-b}{2}\xi + \frac{b+c}{2}\right) d\xi, \tag{3}$$

where f is a generic complex-valued function of a real variable.

Following the application of an N points Gaussian quadrature, the integral in (3) can be expressed as:

$$\int_b^c f(x) dx = \frac{c-b}{2} \sum_{n=0}^{N-1} w_n f\left(\frac{c-b}{2}\xi_n + \frac{b+c}{2}\right), \tag{4}$$

where the $\{\xi_n\}_{n=0}^{N-1}$ are the quadrature nodes and the $\{w_n\}_{n=0}^{N-1}$ are the quadrature weights. The quadrature nodes are expressed as the zeros of the Nth degree orthogonal polynomials $p_n(\xi)$, while the quadrature weights, which are all real and positive, are also expressible in terms of the same orthogonal polynomials [11]. The polynomials $p_n(\xi)$ are defined by the following recursive relation:

$$p_{n+1}(\xi) = (\xi - \alpha_n) p_n(\xi) - \beta_n p_{n-1}(\xi), \quad n = 0, 1, \ldots, \tag{5}$$

with $p_0(\xi) = 1$ and $p_{-1}(\xi) = 0$ and the α_n's and the β_n's specifically define the relevant polynomials. The Legendre quadrature theory states that, once defined the tridiagonal symmetric Jacobi matrix:

$$\underline{\underline{M}}_N = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & & & \ddots \\ & & \ddots & \ddots & \sqrt{\beta_{N-1}} \\ 0 & & & \sqrt{\beta_{N-1}} & \alpha_{N-1} \end{bmatrix}, \tag{6}$$

the quadrature nodes can be determined as its eigenvalues, while the quadrature weights can be retrieved according to its eigenvectors [12]. In particular, the weights are expressed as:

$$w_n = \beta_0 \gamma_{n1}^2, \quad n = 0, \ldots, N-1, \tag{7}$$

where γ_{n1} is the first component of the corresponding eigenvector $\underline{\gamma}_n$.

For our purposes, Legendre quadrature is employed for which $\alpha_n = 0$, $n = 0, \ldots, N-1$, and $\beta_0 = 2$ and $\beta_n = (4 - n^{-2})^{-1}$, $n = 1, \ldots, N-1$.

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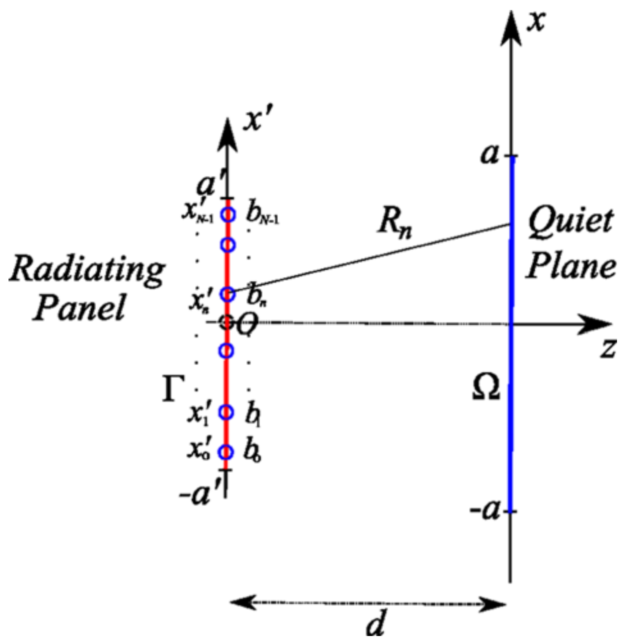

Figure 2. Array geometry.

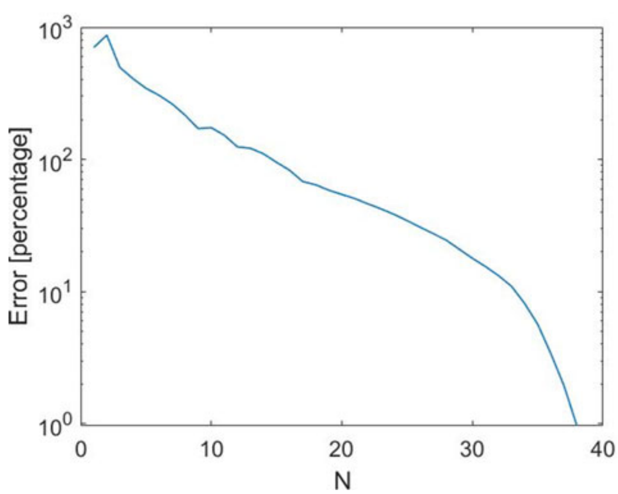

Figure 3. Percentage mean square error.

Panel discretization

In this section, we introduce a discrete version of the panel (array) capable to synthesize any field radiated by J over Ω. The array will generally be non-uniform and since, according to eq. (2), any field radiated over Ω is represented as a sum of functions v_l's, $l = 0, \dots, L-1$, then the array must be capable to radiate any individual v_l over Ω within the prefixed accuracy.

Here, the radiating panel is replaced by a linear, non-uniform array made of N elements located on the x' axis, having unique positions $\underline{x}' = (x'_0, x'_1, \dots, x'_{N-1})$ and complex excitation coefficients $\underline{b}^l = (b^l_0, b^l_1, \dots, b^l_{N-1})$, $l = 0, \dots, L-1$, which depend on the field v_l to be represented (see Fig. 2). The element positions and excitation coefficients are determined according to quadrature rules [8–10], in this paper dealt with as the above recalled Legendre quadrature.

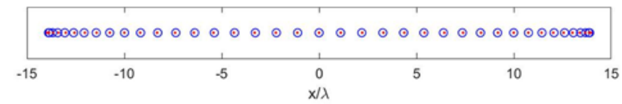

Figure 4. Quadrature nodes.

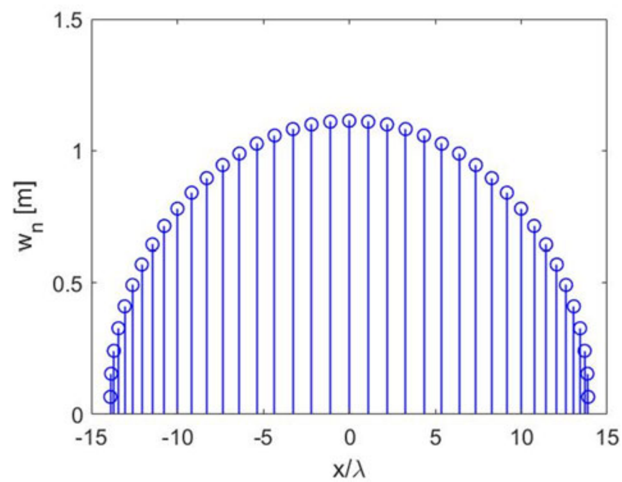

Figure 5. Quadrature weights.

Table 1. Element spacings

n	d_n	n	d_n
0	0.11λ	19	1.1λ
1	0.20λ	20	1.1λ
2	0.28λ	21	1.1λ
3	0.37λ	22	1.1λ
4	0.45λ	23	1.0λ
5	0.53λ	24	1.0λ
6	0.61λ	25	0.97λ
7	0.68λ	26	0.92λ
8	0.75λ	27	0.87λ
9	0.81λ	28	0.82λ
10	0.87λ	29	0.75λ
11	0.92λ	30	0.70λ
12	0.97λ	31	0.61λ
13	1.0λ	32	0.53λ
14	1.0λ	33	0.45λ
15	1.1λ	34	0.37λ
16	1.1λ	35	0.28λ
17	1.1λ	36	0.20λ
18	1.1λ	37	0.11λ

To reach the targeted goal, the field radiated by the array over Ω is represented as:

$$E^l(x, d) = \sum_{n=0}^{N-1} b^l_n H^{(2)}_0(\beta R_n), \quad l = 0, \dots, L-1, \tag{8}$$

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where $R_n = \sqrt{(x - x'_n)^2 + d^2}$. Synthesizing an array discretizing the radiating panel amounts to finding $\underline{x}'$ and $\underline{b}^l$, $l = 1, \ldots, L$, so that:

$$E^l(x, d) = v_l(x), \quad l = 1, \ldots, L, \quad |x| \leq a. \tag{9}$$

Due to the link between left and right singular functions:

$$v_l(x) = \frac{1}{\sigma_l} \int_{-a'}^{a'} H_0^{(2)}(\beta R) u_l(x') dx', \tag{10}$$

the quadrature enables representing approximately the v_l's by summations as:

$$v_l(x) \simeq \tilde{v}_l(x) = \frac{1}{\sigma_l} \sum_{n=0}^{N-1} w_n H_0^{(2)}(\beta R_n) u_l(x'_n), \tag{11}$$

where $\underline{w} = (w_0, w_1, \ldots, w_{N-1})$ is a unique set of quadrature weights. On comparing eqs. (8), (9) and (11), then the array excitation coefficients are:

$$b_n^l = w_n \frac{u_l(x'_n)}{\sigma_l}, \quad l = 0, \ldots, L-1. \tag{12}$$

We remark that the developed approach does not introduce limitations on the class of fields that can be equivalently radiated by the considered array since $E(x, d)$ belongs to the space spanned by $v_l(x), l = 0, \ldots, L-1$, and the addressed quadrature rule, as it will be clearer in the next section, allows to adequately represent all the relevant singular functions $v_l(x)$'s.

Numerical results

In this section, we provide results to validate the quadrature technique for the discretization of the radiating panel.

To this end, we consider the same results of the panel dimensioning problem in [6] and consider a domain Γ with $a' = 14\lambda$, a domain Ω with $a = 5\lambda$ and a reciprocal distance of $d = 10\lambda$. With this setup, $L = 18$.

The first step toward the array definition is the choice of the number of elements N. The order of the quadrature depends on the order of the polynomials one wants to integrate exactly. An N-nodes quadrature integrates exactly Legendre polynomials of degree up to $2N-1$. From this point of view, when increasing N, we expect that the overall performance of the array improve. In order to assess the array performance, the percentage mean square error (PMSE) is adopted:

$$\Phi(\underline{x}', \underline{w}) = 100 \cdot \frac{\sum_{l=0}^{L-1} \|\tilde{v}_l(x) - v_l(x)\|^2}{L}. \tag{13}$$

For the considered test case, the PMSE against N is reported in Fig. 3. Once assigned the maximum tolerable PMSE, the number N of nodes to be employed can be determined. In particular, from Fig. 3, to reach a maximum PMSE of 1%, we should have $N \geq 39$. As a consequence, henceforth a Legendre quadrature rule with $N = 39$ is considered for which $PMSE = 0.96\%$.

Figs. 4 and 5 display the array element positions and weights w_n's corresponding to a Legendre quadrature rule with $N = 39$. On the other side, Table 1 reports the interelement spacings $d_n = x_{n+1} - x_n$. As it can be seen, the spacing between outermost two elements on the left and on the right is very small and thus unpractical to realize. In order to avoid too close interelement spacings, we change the quadrature rule by replacing elements electrically too close each other with a single element having coordinate and weight equal to the average of their respective coordinates and weights. Obviously, such a procedure introduces further errors. Nevertheless, as it will be shortly clear, the performance of the array is not impaired. Indeed, Figs. 6, 7, 8 and 9 enable the comparison between the singular functions v_l's radiated by the panel and the $\tilde{v}_l$'s approximated by the array for $l = 0, 4, 10, 13$. A very good match between the twos can be appreciated. Analogous results have been observed for all the other relevant singular functions.

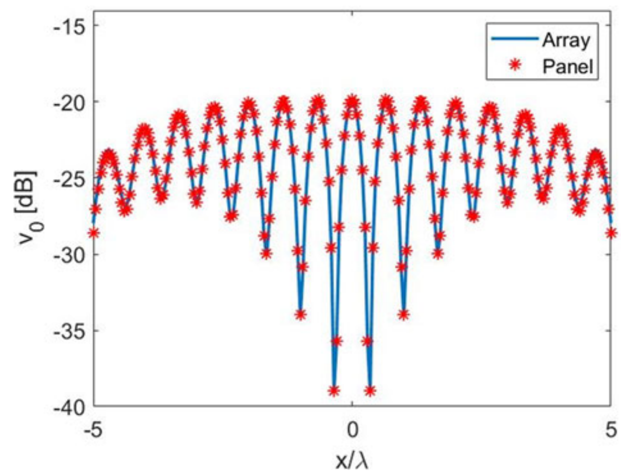

Figure 6. Radiated singular function $v_0(x)$.

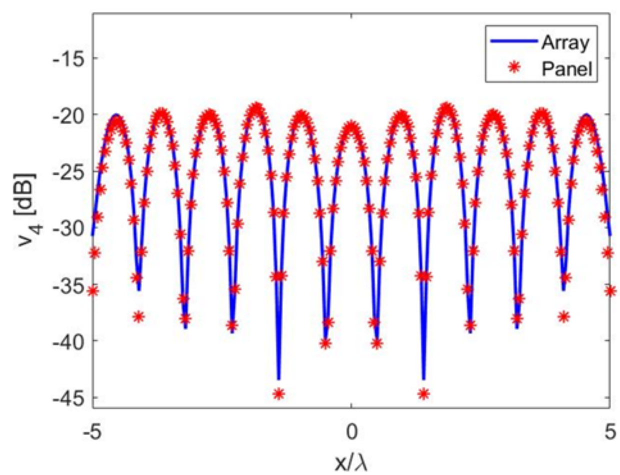

Figure 7. Radiated singular function $v_4(x)$.

Conclusions and future developments

A practical implementation of a continuous equivalent radiating panel synthesized according to [6] has been considered and provided by a non-uniform array. The element positions have been chosen by a quadrature rule so that the array is capable to approximate, with an adequate degree of accuracy, the fields radiated by the equivalent radiator. The quadrature rule has enabled also to choose a set of weights which are essential in the definition of the element excitation coefficients from the knowledge of the source distribution on the equivalent panel. Indeed, the capability of switching from a radiated field to another is based on the reconfigurability of the element excitation coefficients.

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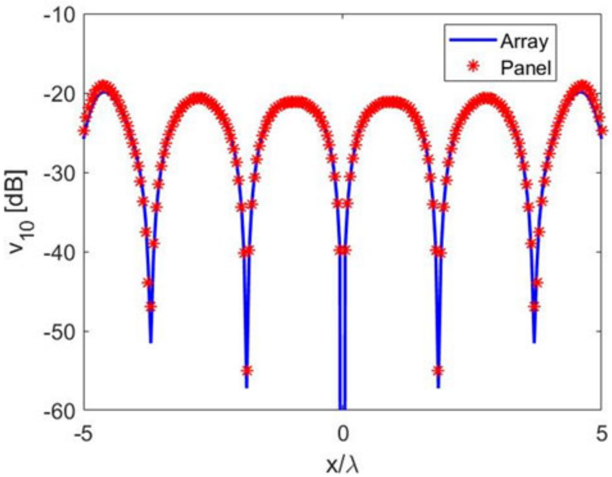

Figure 8. Radiated singular function $v_{10}(x)$.

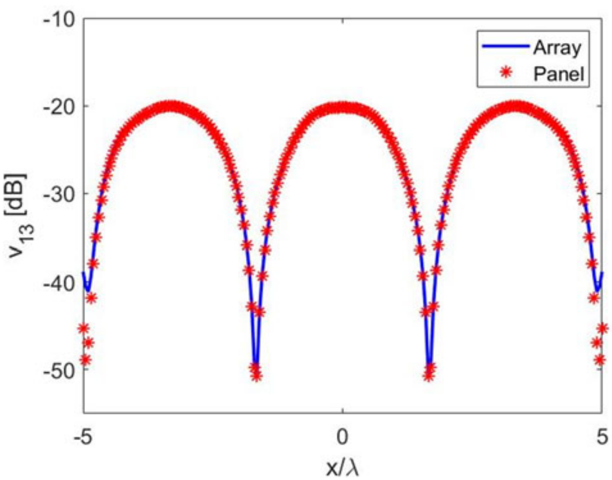

Figure 9. Radiated singular function $v_{13}(x)$.

For the sake of simplicity, a one-dimensional problem with a Legendre quadrature rule has been considered. The approach has been numerically assessed by checking the capability of the array to radiate, with a satisfactory degree of accuracy, the singular functions associated to the region of interest.

Future developments will involve the optimization of the element position and weights [7] by taking also into account constraints concerning the minimum allowed interelement spacing and the maximum allowed size [13, 14] and the extension to the two-dimensional case using two-dimensional quadrature rules [15].

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References

1. **Bucci OM, Franceschetti G and D'Elia G** (1980) Fast analysis of large antennas – a new computational philosophy. *IEEE Transactions on Antennas and Propagation* **AP-28**, 306–310.
2. **Regué J-R, Ribó M, Garrell J-M and Martín A** (2001) A genetic algorithm based method for source identification and far-field radiated emissions prediction from near-field measurements for PCB characterization. *IEEE Transactions on Electromagnetic Compatibility* **43**, 520–530.
3. **Capozzoli A, Curcio C and Liseno A** (2011) Time-harmonic echo generation. *IEEE Transactions on Antennas and Propagation* **59**, 3234–3245.
4. **Yaghjian AD, Hansen TB and Devaney AJ** (1997) Minimum source region for a given far-field pattern. *IEEE Transactions on Antennas and Propagation* **45**, 911–912.
5. **Zaridze R, Bit-Babik G, Tavzarashvili K, Economou DP and Uzunoglu NK** (2002) Wave field singularity aspects in large-size scatterers and inverse problems. *IEEE Transactions on Antennas and Propagation* **50**, 50–58.
6. **Capozzoli A, Curcio C and Liseno A** (2022) Dimensioning flat, rectangular equivalent radiators. In *Microwave Mediterranean Symposium*, Pizzo Calabro, Italy.
7. **Capozzoli A, Curcio C and Liseno A** (2021) On the optimal field sensing in near-field characterization. *Sensors* **21**, 1–18.
8. **Barclay W and Marinos P** (1966) A new approach to antenna array synthesis. In *IRE International Convention Record*, New York, USA.
9. **Stutzman WL** (1972) Shaped-beam synthesis of nonuniformly spaced linear arrays. *IEEE Transactions on Antennas and Propagation* **20**, 499–501.
10. **Marchaud FBT, de Villiers GD and Pike ER** (2003) Element positioning for linear arrays using generalized Gaussian quadrature. In *IEEE Transactions on Antennas and Propagation* **51**, 1357–1363.
11. **Gautschi W** (1994) Algorithm 726: ORTHPOL – a package of routines for generating orthogonal polynomials and Gauss-type quadrature rules. *ACM Transactions on Mathematical Software* **20**, 21–62.
12. **Golub GH and Welsch JH** (1969) Calculation of Gauss quadrature rules. *Mathematics of Computation* **92**, 221–230.
13. **Capozzoli A, Curcio C, D'Elia G, Liseno A and Vinetti P** (2008) FFT & aperiodic arrays with phase-only control and constraints due to super-directivity, mutual coupling and overall size. In *30th ESA Antenna Workshop on Antennas for Earth Observation, Science, Telecommunication and Navigation Space Missions*, Noordwijk, The Netherlands.
14. **Capozzoli A, Curcio C, D'Elia G, Liseno A and Vinetti P** (2008) FFT & equivalently tapered aperiodic arrays, XXIX General Assembly of the International. Union of Radio Sci, Chicago, USA.
15. **Sukri SA, Hoe YS and Khairuddin TKA** (2020) Numerical evaluation for two-dimensional integral using higher order Gaussian quadrature. *Annals of Mathematical Modeling* **2**, 37–42.

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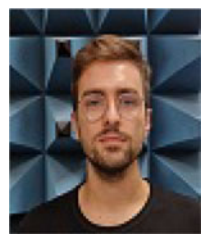

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