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Classification of AF Flows

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Abstract. An AF flow is a one-parameter automorphism group of an AF C^*-algebra A such that there exists an increasing sequence of invariant finite dimensional sub-C^*-algebras whose union is dense in A. In this paper, a classification of C^*-dynamical systems of this form up to equivariant isomorphism is presented. Two pictures of the actions are given, one in terms of a modified Bratteli diagram/path-space construction, and one in terms of a modified K_0 functor.

1 Introduction

In this paper we shall discuss C^*-dynamical systems of the following form: We have a C^*-algebra which is given as an inductive limit of a sequence of nice building blocks, and the group acts on each of these in such a way as to make the maps in the inductive system equivariant and result in an action on the limit algebra.

In the case that the group is compact there is already a long list of classification results for actions of this type. In [8] and [9], Handelman and Rossmann classified actions of a compact group on an AF algebra which left invariant an increasing sequence of finite dimensional subalgebras with dense union for which the action of the group on each of the finite dimensional algebras was by inner automorphisms assumed to arise from a unitary representation of the group. Such actions they referred to as locally representable. In [10] Kishimoto considered actions of finite groups on inductive limit algebras with more complicated building blocks (circles), and in [2] this study was extended to still more complicated inductive systems and to general compact groups. In both of these cases it was assumed that the action still satisfied a local representability condition, namely that the group acted by inner automorphisms on the building blocks, again, arising from a unitary representation. In the case where the group was just $\mathbb{Z}/2\mathbb{Z}$ and the building blocks finite dimensional, this local representability hypothesis was removed in [6].

The story for inductive limit actions of the reals is at a much earlier chapter. In [3] an invariant was introduced which classified product type actions of the reals on UHF algebras. The invariant bore some resemblance to the supernatural numbers introduced by Glimm to classify these algebras. In this paper we shall classify AF flows. We shall present two pictures of them, one based on a generalisation of the Bratteli diagrams of [1], and the other based on an analogue of the dimension groups of [4].

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2 Coloured Bratteli Diagrams

Definition 2.1 We begin with some definitions and conventions. All of our C^*-algebras will be unital, but we do not assume that all *-homomorphisms preserve the units. Let A be a C^*-algebra with an action α of the reals on it. Suppose that there exists an increasing sequence $\{A_n\}$ of finite dimensional sub-C^*-algebras of A such that each A_n is invariant under the action and $\mathfrak{A} = \bigcup_{n=1}^{\infty} A_n$ is norm dense in A. Then the C^*-dynamical system (A, α) will be referred to as an AF *flow*. The dense *-subalgebra $\mathfrak{A}$ along with the action $\alpha|_{\mathfrak{A}}$ will be referred to as a *locally finite* **-dynamical system*, and we shall use the term **-dynamical system* to refer to either a C^*-dynamical system or a locally finite *-dynamical system.

Let (A, α) be a C^*-dynamical system with A a full matrix algebra. Then by Proposition 2.5.5 of [11] there exists a positive operator $h \in A$ such that the generator of α is the derivation $\delta(x) = i[h, x]$. Clearly any two positive operators with this property differ by a scalar multiple of the identity. We shall refer to the unique such positive operator having smallest eigenvalue zero as the *minimal positive Hamiltonian* of the action.

In this section, we shall give a diagrammatic way to classify locally finite *-dynamical systems. Our first step is the following theorem describing the injective, unital, equivariant *-homomorphisms from one finite dimensional C^*-dynamical system to another.

Theorem 2.2 **(Finite Dimensional Embeddings)** *Let (A, α) and (B, β) be two C^*-dynamical systems with A finite dimensional and B a full matrix algebra. Let $\psi\colon A \to B$ be an injective unital equivariant *-homomorphism. Let $A_1, \dots, A_n$ denote the minimal direct summands of A, and let $h_1, \dots, h_n$ denote the minimal positive Hamiltonians of $(A_1, \alpha|_{A_1}), \dots, (A_n, \alpha|_{A_n})$ respectively. Let $F_1 = \{f_1^1, \dots, f_{l_1}^1\}, \dots, F_n = \{f_1^n, \dots, f_{l_n}^n\}$ be the eigenvalue lists, with multiplicities, of $h_1, \dots, h_n$ respectively, and, for each k, let $\{e_{ij}^k\}_{1\le i,j\le l_k}$ be a system of matrix units for A_k consisting of eigenoperators for the action α such that the eigenvalue for e_{ij}^k is $f_i^k - f_j^k$. Suppose that the multiplicities of the embeddings under ψ are $m_1, \dots, m_n$ for $A_1, \dots, A_n$ respectively. Let h_β denote the minimal positive Hamiltonian for (B, β). Then there exist real numbers $c_1, \dots, c_n$, sets of real numbers $S_1 = \{s_1^1, \dots, s_{m_1}^1\}, \dots, S_n = \{s_1^n, \dots, s_{m_n}^n\}$ with $|S_k| = m_k$ for each k, and a system of matrix units $\{f_{st}\}$ for B consisting of eigenoperators for β such that*

$$\psi(e_{ij}^k) = \sum_{r=0}^{m_k-1} f_{((\sum_{u=1}^{k-1} l_u m_u)+rl_k+i)((\sum_{u=1}^{k-1} l_u m_u)+rl_k+j)}$$

and the eigenvalue of f_{st} is

$$\Big(c_{w(s)} + s^{w(s)}_{\lfloor \frac{s-w(s)}{l_{w(s)}} \rfloor} + f^{w(s)}_{s-w(s)-l_{w(s)}\lfloor \frac{s-w(s)}{l_{w(s)}} \rfloor}\Big) - \Big(c_w(t) + s^{w(t)}_{\lfloor \frac{t-w(t)}{l_{w(t)}} \rfloor} + f^{w(t)}_{t-w(t)-l_{w(t)}\lfloor \frac{t-w(t)}{l_{w(t)}} \rfloor}\Big),$$

where $w(s)$ is the smallest integer k such that $\sum_{r=1}^{k} l_r m_r \ge s$.

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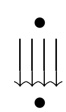
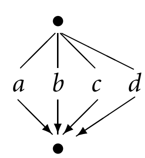
Proof Assume the notation in the statement of the theorem. Let $C_1, \ldots, C_n$ denote $\psi(1_{A_1})B\psi(1_{A_1}), \ldots, \psi(1_{A_n})B\psi(1_{A_n})$ respectively. Then, for each k, $C_k \cong \psi(A_k) \otimes \big(\psi(A_k)' \cap C_k\big)$, and the action β restricts to an action on C_k that respects this tensor product decomposition. Let g_k denote the minimal positive Hamiltonian of the action on $\big(\psi(A_k)' \cap C_k\big) \cong M_{m_k}$, and let $S_k = \{s_1^k, \ldots, s_{m_k}^k\}$ be its eigenvalue list. Let $\{u_{ij}\}_{1 \leq i,j \leq m_k}$ be a system of matrix units for $\psi(A_k)' \cap C_k$ consisting of eigenoperators such that $g_k = s_1^k u_{11} + \cdots + s_{m_k}^k u_{m_k m_k}$. Then the set $\big\{ \big(\psi(e_{ij})\big) \otimes u_{\mathrm{op}} \mid 1 \leq i, j \leq l_k, 1 \leq o, p \leq m_k \big\}$ is a basis for C_k consisting of eigenoperators. Order the projections $\psi(e_{ii}) \otimes u_{oo}$ lexicographically, first by the e numbers, then the u numbers.

Next, we choose an ordering to place the blocks C_k down the diagonal of B so that filling out the systems of matrix units we already have for the C_k to a system of matrix units consisting of eigenoperators for all of B will result in a system satisfying the requirements of the theorem. For each k, let p_k be a minimal projection of C_k that is a sub-projection of the spectral projection of $h_k \otimes 1 + 1 \otimes g_k$ with eigenvalue zero. Since the p_ks are also minimal projections of B, they are all Murray-von Neumann equivalent. Let v_{jk} be a partial isometry such that $v_{jk}^* v_{jk} = p_k$ and $v_{jk} v_{jk}^* = p_j$. Then since the range and support projections of v_{jk} are in the fixed point subalgebra and are orthogonal, v_{jk} is an eigenoperator. Let c_n denote the largest positive eigenvalue of any of the v_{jk}s, say c_n is the eigenvalue of v_{xy}. Let $c_1 = 0$. Place C_x first in the upper left corner of B, C_y last in the lower right corner, and the other C_ks down the diagonal in increasing order as the eigenvalue of v_{xk}. These eigenvalues give us the other c_ks. With this ordering on the projections $\psi(e_{ii}^k) \otimes u_{oo}^k$, we may now use the v_{jk}s to fill out the matrix units already found for the C_ks to a system $\{f_{st}\}$ of matrix units for B that is easily seen to satisfy the requirements of the theorem.

Perhaps the best way to picture this result is in terms of a path space construction using a Bratteli diagram with labelled edges, as follows (*cf.* [7], Chapter 2, for a discussion of the path space construction). First consider the Bratteli diagram for a single full matrix algebra of size $n \times n$:

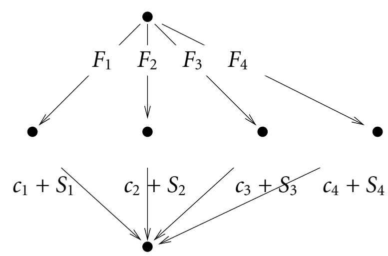


We have n edges, which we may use to index a system of matrix units. Each edge gives a projection in this system. Now suppose the minimal positive Hamiltonian of an action is diagonal with respect to this system of matrix units. Label each edge by the corresponding eigenvalue of the Hamiltonian:



We may think of a matrix unit as a loop, starting at the top, going down to the lower dot, and then back up. These matrix units are all eigenoperators for the action, and the eigenvalue is given by taking the value on the down edge and subtracting the value on the up edge.

Next, consider the two step diagram, consisting of the labelled Bratteli diagrams for each of the minimal direct summands of A and the multiplicity diagram for the embedding of A into B:



We use the label F_k to mean that there are really l_k edges with labels $f^k_1, \dots, f^k_{k_k}$, and the label $c_k + S_k$ to mean that there are really m_k edges which we label with the numbers $c_k + s^k_1, \dots, c_k + s^k_{m_k}$. Now we get a system of matrix units for B indexed by two edge paths from the top dot to the bottom dot, and we again think of our matrix units as loops formed by a pair of such paths. If we suppose that the minimal positive Hamiltonian of B is diagonal with respect to these matrix units, we have that the matrix units are eigenoperators and the eigenvalue of one of them is given by adding up the labels going down and subtracting the labels going back up.

The embedding of A into B is given by the usual relations in the path space construction, *i.e.*, a loop with bottom in the middle row is set equal to the sum of the loops obtained from it by appending the same edge down to the bottom row to both the down and up sides of the loop. As the labels for these extra edges cancel, we see that this indeed gives an equivariant embedding.

With this picture, the theorem above can be thought of as just a careful numbering of the edges and paths. Clearly one may extend this to the case where B has several direct summands, with the partial maps not necessarily all being injective.

Considering now the case of a sequence of embeddings of finite dimensional C^*-dynamical systems giving rise to a locally finite *-dynamical system we are led to the following modification of the definition of a Bratteli diagram.

Definition 2.3 (Coloured Bratteli Diagrams) A *coloured Bratteli diagram* consists of a set V of vertices indexed by a subset of the pairs of positive integers, a set E of edges, a pair of maps $r\colon E \to V$ and $s\colon E \to V$, called *the range and source maps*, and a map $c\colon E \to \mathbb{R}+$, which we shall call the colour of an edge, satisfying the following axioms:

(1) For each positive integer n, the set of vertices with first coordinate n is finite and non-empty, and there is exactly one vertex with first coordinate zero.
(2) If x is an edge and $s(x) = v(m, n)$ and $r(x) = v(u, w)$, then $u = m + 1$.
(3) For each $v(m, n)$ with $m \geq 1$, there exists an edge x such that $r(x) = v(m, n)$ and $c(x) = 0$.

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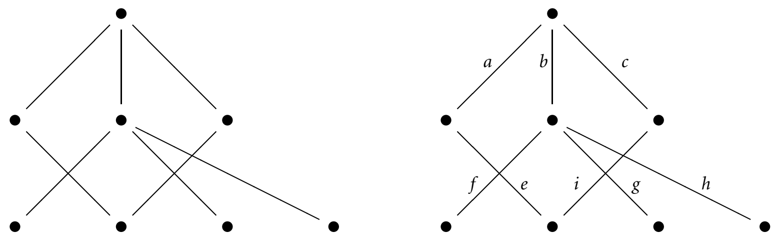
(4) For each vertex v, the set of edges x such that $r(x) = v$ is finite, as is the set of edges y such that $s(y) = v$.



A "black" Bratteli diagram A "coloured" Bratteli diagram

Coloured Bratteli Diagrams

Remark 2.4 (Observations on Coloured Bratteli Diagrams) We note a few facts about these coloured diagrams.

(1) Given an inductive system of finite dimensional C^*-dynamical systems with unital injective maps there is a unique (up to isomorphism) coloured Bratteli diagram determined by the system, the diagram being determined by the multiplicities alone and the colours given by the theorem above.
(2) Given a coloured diagram, the path space construction, with our convention for assigning eigenvalues from the labels on the edges, gives us an inductive system of finite dimensional C^*-dynamical systems having the given coloured diagram.
(3) If two inductive systems of finite dimensional C^*-dynamical systems yield isomorphic coloured diagrams, then the locally finite *-dynamical systems they give rise to are isomorphic.

Let $(\mathbb{C}, \alpha_0) \subseteq (A_1, \alpha_1) \subseteq (A_2, \alpha_2) \subseteq \cdots$ be an inductive system of finite dimensional C^*-dynamical systems giving rise to a locally finite *-dynamical system $(\mathfrak{A}, \alpha)$, and let $\{m_k\}$ be a strictly increasing sequence of positive integers. Then, clearly, the inductive system $(A_{m_1}, \alpha_{m_1}) \subseteq (A_{m_2}, \alpha_{m_2}) \subseteq \cdots$ also defines the *-dynamical system $(\mathfrak{A}, \alpha)$. Considering this process of "passing to subsequences" in an inductive system leads one to a corresponding relation between coloured Bratteli diagrams. To describe it, it will be useful to make a few more definitions. If D is a coloured Bratteli diagram, a *path* in D is a finite sequence of edges such that the range of one edge is the source of the next. We shall say that the source of the first edge is the source of the path, and the range of the last edge is the range of the path. We define the colour of a path to be the sum of the colours of its edges.

Definition 2.5 (Contractions of Coloured Diagrams) Let $D_1 = (E_1, V_1, s_1, r_1, c_1)$ and $D_2 = (E_2, V_2, s_2, r_2, c_2)$ be two coloured Bratteli diagrams. We say that D_2 is a *contraction* of D_1 if there are two maps, $\psi\colon V_2 \to V_1$ and φ mapping a subset of the set of paths in D_1 to E_2 with the following properties:

(1) The unique element of V_2 with first coordinate zero is mapped by ψ to the unique element of V_1 with first coordinate zero.

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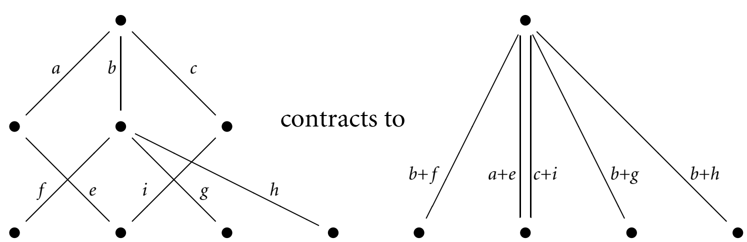
(2) There exists an increasing sequence $0 < n_1 < n_2, \ldots$ of integers such that the subset of V_2 consisting of those elements with first coordinate k is mapped bijectively onto the subset of V_1 consisting of those elements with first coordinate n_k. A path p in D_1 is in the domain of φ if and only if there is an integer k such that the source of p has first coordinate n_k and the range of p has first coordinate n_{k+1}.
(3) For any two vertices v and w in V_2 with the first coordinate of w one greater than the first coordinate of v, the set of paths from $\psi(v)$ to $\psi(w)$ in D_1 is mapped bijectively by φ onto the set of edges from v to w.
(4) For any path p in the domain of φ, $c\big(\varphi(p)\big) = c(p)$.

The following picture illustrates this definition on a finite piece of a coloured Bratteli diagram. The one step diagram on the right is a contraction of the two step one on the left.



Theorem 2.6 *Let $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ be locally finite *-dynamical systems with coloured Bratteli diagrams D_1 and D_2 respectively. Then $(\mathfrak{A}, \alpha) \cong (\mathfrak{B}, \beta)$ if and only if there exist coloured Bratteli diagrams D_1', D_2', and D_3 such that D_1' is a contraction of D_1, D_2' is a contraction of D_2, and D_1' and D_2' are both contractions of D_3.*

Proof If we have two coloured Bratteli diagrams, B_1 and B_2 such that B_2 is a contraction of B_1, then the inductive system described by B_2 is isomorphic to one that arises from the system defined by B_1 by passing to subsequences, so sufficiency is clear. We now establish necessity. If $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are locally finite *-dynamical systems, $\mathfrak{A} = \bigcup_{n=1}^{\infty} A_n$, $\mathfrak{B} = \bigcup_{n=1}^{\infty} B_n$, where $\{A_n\}$ and $\{B_n\}$ are increasing sequences of invariant finite dimensional C^*-algebras each containing the unit of $\mathfrak{A}$, $\mathfrak{B}$ respectively, and $\psi\colon \mathfrak{A} \to \mathfrak{B}$ and $\varphi\colon \mathfrak{B} \to \mathfrak{A}$ is a pair of inverse equivariant *-isomorphisms, then we may, after passing to subsequences, pull ψ and φ back to an intertwining diagram:

$$\begin{array}{ccccccc}
(A_{n_1}, \alpha_{n_1}) & \longrightarrow & (A_{n_2}, \alpha_{n_2}) & \longrightarrow & \cdots & \longrightarrow & (\mathfrak{A}, \alpha) \\
\big\downarrow \psi_1 & \nearrow \varphi_1 & \big\downarrow \psi_2 & & & & \varphi \big\uparrow\big\downarrow \psi \\
(B_{m_1}, \beta_{m_1}) & \longrightarrow & (B_{m_2}, \beta_{m_2} a) & \longrightarrow & \cdots & \longrightarrow & (\mathfrak{B}, \beta)
\end{array}$$

The desired D_1' and D_2' are given by the sequences $A_{n_1} \subseteq A_{n_2} \subseteq \cdots$ and $B_{m_1} \subseteq B_{m_2} \subseteq \cdots$ respectively, and the desired D_3 is given by the sequence of maps $\psi_1, \varphi_1, \psi_2, \varphi_2, \ldots$.

In the next section, we shall develop an analogue of dimension groups for *-dynamical systems. After proving the main result, we shall return briefly to coloured Bratteli diagrams, and see that they actually characterise AF flows, not just locally finite ones.

3 Classification of AF Flows

Definition 3.1 (The Ring of Hamiltonians) Let S denote the set of all finite subsets of the non-negative real numbers, counted with multiplicity. More precisely, $S = \{\varnothing\} \cup \coprod_{n=1}^{\infty} \big((\mathbb{R}^+)^n / S_n \big)$, where the symmetric group S_n acts on $(\mathbb{R}^+)^n$ by permuting the factors. We shall denote the element of S corresponding to an n-tuple $\langle x_1, \dots, x_n \rangle$ of non-negative real numbers by $[x_1, \dots, x_n]$. The element corresponding to the empty set will be denoted $[\varnothing]$. We define two operations, $\oplus$ and $\odot$, on S as follows:

$$[x_1, \dots, x_n] \oplus [y_1, \dots, y_m] = [x_1, \dots, x_n, y_1, \dots, y_m]$$

$$[\varnothing] \oplus X = X \oplus [\varnothing] = X \quad X \text{ any element of } S$$

$$[x_1, \dots, x_n] \odot [y_1, \dots, y_m] = [x_i + y_j, 1 \leq i \leq n, 1 \leq j \leq m]$$

$$[\varnothing] \odot X = X \odot [\varnothing] = [\varnothing] \quad X \text{ any element of } S$$

$(S, \oplus)$ is then an Abelian semi-group with neutral element $[\varnothing]$. Furthermore, this semi-group has cancellation: if $x, y, z \in S$, and $x \oplus z = y \oplus z$, then $x = y$. Let $\mathfrak{R}$ denote the Grothendieck group of $(S, \oplus)$. Since $(S, \oplus)$ has cancellation, the natural map from S to $\mathfrak{R}$ is an injection. We write $\mathfrak{R}^+$ for the image of this map. Since the multiplication operation $\odot$ is associative and distributes over the addition operation $\oplus$ on S, we get a ring structure on $\mathfrak{R}$ by extending $\odot$ in the obvious way. The ring $(\mathfrak{R}, \oplus, \odot)$ is then an ordered ring with identity with positive cone $\mathfrak{R}^+$ and identity element $[0]$. We call this ring the *ring of Hamiltonians*, for reasons which will become obvious below.

There is a natural ring homomorphism, which we shall denote N, from $\mathfrak{R}$ to $\mathbb{Z}$ given by defining

$$N([\varnothing]) = 0$$

$$N([x_1, \dots, x_n]) = n,$$

and extending by additivity (*i.e.* one just counts the number of elements in the sets). It will be important below that this map is positive, $N(\mathfrak{R}^+) \subseteq (\mathbb{Z}^+)$.

Definition 3.2 (The Invariant) Let (A, α) be either an AF flow or a locally finite *-dynamical system and let A^α denote the fixed point subalgebra of A. Let $D(A^\alpha)$ denote the dimension range of A^α, *i.e.* the set of Murray-von Neumann equivalence classes of projections in A^α. We define $K\mathfrak{R}(A, \alpha)$, which we shall refer to as *the*

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coloured K_0 *module of* (A, α), to be the universal right $\mathfrak{R}$ module generated by $D(A^\alpha)$ subject to the following relations:

(1) If p and q are projections in A^α and $p \perp q$, then $[p + q] = [p] + [q]$.
(2) If v is a partial isometry in A which is also an eigenoperator with eigenvalue a, then $[v^*v] = [vv^*][a]$, where $[a] \in \mathfrak{R}$.

We let $\Sigma\mathfrak{R}(A, \alpha)$ denote the image of $D(A^\alpha)$ in $K\mathfrak{R}(A, \alpha)$ under the natural map, and we refer to this set as *the coloured scale of* (A, α). We make $K\mathfrak{R}(A, \alpha)$ into an ordered $\mathfrak{R}$ module by taking as positive cone the semi-group generated by positive multiples of elements of the coloured scale.

If (A, α) and (B, β) are two *-dynamical systems, and $\psi\colon A \to B$ is an equivariant *-homomorphism, then we may define a map $\Sigma\mathfrak{R}(\psi)\colon \Sigma\mathfrak{R}(A, \alpha) \to \Sigma\mathfrak{R}(B, \beta)$ by $\Sigma\mathfrak{R}[p] = [\psi(p)]$ for each projection p in A^α. This map extends to an $\mathfrak{R}$-module homomorphism from $K\mathfrak{R}(A, \alpha)$ to $K\mathfrak{R}(B, \beta)$ which we shall call $K\mathfrak{R}(\psi)$. It is easy to see that these definitions give a functor from the category of *-dynamical systems and equivariant *-homomorphisms to the category of right $\mathfrak{R}$-modules with distinguished positive sub-semi-groups and generating subsets with $\mathfrak{R}$-module maps respecting these additional structures. We shall refer to this functor as *the invariant*, and denote it by Inv.

Remark 3.3 (Finite Dimensional and Locally Finite *-Dynamical Systems) In this remark we collect some useful observations about $K\mathfrak{R}$.

(Matrix Algebras) Let (B, β) be a C^*-dynamical system with B a full matrix algebra. Then $K\mathfrak{R}(B, \beta)$ is generated as an $\mathfrak{R}$-module by a minimal projection subordinate to the spectral projection of the minimal positive Hamiltonian of (B, β) with eigenvalue zero. If p is such a projection, then the correspondence $[p] \mapsto 1$ extends to an $\mathfrak{R}$-module isomorphism of $K\mathfrak{R}(B, \beta)$ with $\mathfrak{R}$.

(Direct Sums) We next observe that $K\mathfrak{R}$ respects direct sums, *i.e.* $K\mathfrak{R}(A \oplus B, \alpha \oplus \beta) \cong K\mathfrak{R}(A, \alpha) \oplus K\mathfrak{R}(B, \beta)$. Let i_A and i_B denote the inclusions of A and B into $A \oplus B$ and let p_A and p_B denote the projections of $A \oplus B$ onto A and B. Then it follows just by functoriality applied to the compositions of i_A, i_B with p_A, p_B that $K\mathfrak{R}(i_A)K\mathfrak{R}(A, \alpha) \cap K\mathfrak{R}(i_B)K\mathfrak{R}(B, \beta) = \{0\}$ and that $K\mathfrak{R}(i_A)$ and $K\mathfrak{R}(i_B)$ are both injective. That these two submodules generate all of $K\mathfrak{R}(A \oplus B, \alpha \oplus \beta)$ follows from the fact that every projection in $A \oplus B$ is the sum of a projection in A and a projection in B.

Combining the above two observations we see that if (A, α) is a finite dimensional C^*-dynamical system then $K\mathfrak{R}(A, \alpha)$ is a free $\mathfrak{R}$-module generated by n positive elements $x_1, \dots, x_n$, where n is the number of direct summands of A, and $K\mathfrak{R}^+(A, \alpha) = x_1\mathfrak{R}^+ \oplus \cdots \oplus x_n\mathfrak{R}^+$.

(Inductive Limits) Let $(\mathfrak{A}, \alpha)$ be a locally finite *-dynamical system and let A_n be an increasing sequence of α-invariant finite dimensional subalgebras and let φ_{nm} (resp. $\varphi_{n\infty}$) denote the inclusion of A_n into A_m (resp. A_n into $\mathfrak{A}$). Clearly, if p is an α-invariant projection in $\mathfrak{A}$, then p belongs to some A_n, so that $K\mathfrak{R}(\mathfrak{A}, \alpha) =$

$\bigcup_{n=1}^{\infty} K\mathfrak{R}(\varphi_n)\big(K\mathfrak{R}(A_n, \alpha|_{A,n})\big)$, and similarly for the coloured scales and positive cones. If $[p] = [q][a]$ is one of the defining relations of $K\mathfrak{R}(\mathfrak{A}, \alpha)$, then it is implemented by a partial isometry and this must belong to some A_n, so we see that all relations of this kind come from finite stages in the inductive system. It is clear that all relations of the form $[p] = [q] + [r]$ also come from the finite stages, so we have that

$$\mathrm{Inv}(\mathfrak{A}, \alpha) \cong \varinjlim\{\mathrm{Inv}(A_n, \alpha|A_n), \mathrm{Inv}(\varphi_{nm})\}.$$

Remark 3.4 (Recovering K_0) Recall the homomorphism $N\colon \mathfrak{R} \to \mathbb{Z}$ from Definition 3.1. We may use this to make $\mathbb{Z}$ into a left $\mathfrak{R}$ module by defining $r \cdot z = N(r)z$ for $r \in \mathfrak{R}$ and $z \in \mathbb{Z}$. We then have a functor from right $\mathfrak{R}$-modules and $\mathfrak{R}$-module homomorphisms to Abelian groups and group homomorphisms given by just tensoring: $M \mapsto M \otimes_{\mathfrak{R}} \mathbb{Z}$; $\psi \mapsto \psi \otimes_{\mathfrak{R}} \mathrm{id}_{\mathbb{Z}}$. We may extend this to a functor from the category of $\mathfrak{R}$-modules with distinguished positive sub-semirings to the category of Abelian groups with distinguished positive sub-semigroups by $M^+ \mapsto M^+ \otimes_{\mathfrak{R}} \mathbb{Z}$. We shall denote this extended functor by $\mathfrak{B}$. From the remarks above we see that if (A, α) is a finite dimensional C^*-dynamical system this functor takes $\big(K\mathfrak{R}(A, \alpha), K\mathfrak{R}^+(A, \alpha)\big)$ to $\big(K_0(A), K_0^+(A)\big)$, where we identify $\big(K_0(A)\big)$ with a direct sum of copies of $\mathbb{Z}$ using as basis the classes of minimal projections, and if (B, β) is another finite dimensional C^*-dynamical system and $\psi\colon A \to B$ is an equivariant *-homomorphism, then $\mathfrak{B} \circ K\mathfrak{R}(\psi) = K_0(\psi)$. Furthermore, we easily see that, in this case, the image of the coloured scale is just the usual dimension range in K_0^+.

Lemma 3.5 **(The Existence Lemma)** *Let (A, α) and (B, β) be two finite dimensional C^*-dynamical systems and let $\psi\colon K\mathfrak{R}(A, \alpha) \to K\mathfrak{R}(B, \beta)$ be a positive $\mathfrak{R}$-module homomorphism mapping $\Sigma\mathfrak{R}(A)$ into $\Sigma\mathfrak{R}(B)$ for which $\mathfrak{B}(\psi)$ maps the class of the unit in $K_0(A)$ to the class of the unit in $K_0(B)$. Then there exists a unital equivariant *-homomorphism $\tilde{\psi}\colon A \to B$ such that $\psi = \mathrm{Inv}(\tilde{\psi})$.*

Proof We shall use the coloured Bratteli diagrams of Section 2 as a convenient method of book-keeping. Let $x_1, \dots, x_n$ and $y_1, \dots, y_m$ be the canonical generators of $K\mathfrak{R}(A, \alpha)$ and $K\mathfrak{R}(B, \beta)$ respectively, *i.e.* the classes of minimal projections subordinate to the zero spectral projections of the minimal positive Hamiltonians for each simple direct summand. If we write $[1_A] = x_1a_1 + \cdots + x_na_n$, then we may use the coefficients $a_1, \dots, a_n$ to construct a coloured Bratteli diagram for $\mathbb{C}1 \hookrightarrow A$. We do this by drawing $N(a_j)$ arrows from the dot representing 1 to the dot representing the j-th simple direct summand of A and labelling them with the elements of a_j. The C^*-dynamical system defined from this coloured Bratteli diagram is then isomorphic to (A, α), and we identify them. Repeat the above procedure with (B, β).

Next, we use the matrix for ψ, and the coloured Bratteli diagram for $\mathrm{Inv}(A, \alpha)$ constructed above, to define a two-step coloured Bratteli diagram as follows. The top two rows of our diagram are a copy of the diagram for $\mathrm{Inv}(A, \alpha)$. The bottom row of vertices will have m vertices, and the number of edges from the j-th dot in the middle row to the i-th dot in the bottom row will be $N(\psi_{ij})$. These are then labelled with the elements of ψ_{ij}, with the corresponding multiplicities. Applying the path space construction to this two-step diagram results in an embedding of (A, α) into a C^*-dynamical system having the same $K\mathfrak{R}$-module as (B, β) by a map inducing ψ.

The rest of the proof may now be displayed in a diagram:

$$(A,\alpha) \cong \left\{\begin{matrix}\text{path space}\\ \text{Inv}(A,\alpha)\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{path space}\\ \text{Inv}(A,\alpha),\psi\end{matrix}\right\} \cong \left\{\begin{matrix}\text{path space}\\ \text{Inv}(B,\beta)\end{matrix}\right\} \cong (B,\beta)$$

where the first and last isomorphisms are those constructed above and the inclusion is that coming from the inclusion of the one-step diagram into the two-step one. The remaining isomorphism in the picture will be given by showing that the contraction of the two-step diagram results in a one-step diagram isomorphic to that of $\text{Inv}(B, \beta)$. To see this, note first that the condition on $\mathfrak{B}(\psi)$ implies that there are the same number of edges for each dot in the lower row of the two diagrams. Consider the image in $\Sigma\mathfrak{R}(B, \beta)$ of the class of 1_A in $\Sigma\mathfrak{R}(A, \alpha)$ under ψ. Since there can not be distinct elements of $\Sigma\mathfrak{R}(B, \beta)$ that both map to $[1_B]$ under $\mathfrak{B}$, we see that $\psi([1_A]) = [1_B]$ in $\Sigma\mathfrak{R}(B, \beta)$. In the contracted diagram the list of labels for the edges ending in the j-th dot in the lower row is given by the coefficient of y_j in $\psi([1_A])$. In the diagram for $\text{Inv}(B, \beta)$ it is given by the coefficient of y_j in $[1_B]$. Since these are the same, the diagrams are isomorphic. It is now easy to see that the composition of the maps in the diagram above yields a map satisfying the requirements of the lemma.

Lemma 3.6 **(The Uniqueness Lemma)** *Let (A, α) and (B, β) be two finite dimensional C^*-dynamical systems and let ψ and φ be two unital equivariant *-homomorphisms from A to B. Suppose that* $\text{Inv}(\psi) = \text{Inv}(\varphi)$. *Then there exists an equivariant inner automorphism, γ, of B such that $\psi = \gamma \circ \varphi$.*

Proof It will suffice to consider the case where B has one simple summand. Let $A = A_1 \oplus \cdots \oplus A_n$ be the decomposition of A into simple direct summands. We shall show that, for each j, there is a partial isometry v_j in the fixed point subalgebra of B such that $v_j^* v_j = \psi(1_{A_j})$, $v_j v_j^* = \varphi(1_{A_j})$, and for all x in A_j, $\varphi(x) = v_j \psi(x) v_j^*$. Our desired unitary will then be $v_1 + \cdots + v_n$.

Let $\{p_{kl}\}_{k,l=1}^r$ be a system of matrix units for A_j consisting of eigenoperators such that p_{11} is a subprojection of the zero spectral projection of the minimal positive Hamiltonian of α. Let e be a minimal projection in B lying in the zero eigenspace of the minimal positive Hamiltonian for β, so that $[e]$ is the canonical generator of $K\mathfrak{R}(B, \beta)$. We then have, for some $b_1, \ldots, b_m$, $[\psi(p_{11})] = [\varphi(p_{11})] = [e][b_1, \ldots, b_m] = [e][b_1] + \cdots + [e][b_m]$. There exist partial isometries $s_1, \ldots, s_m$ in B such that $s_i s_i^* = e$, s_i is an eigenoperator with eigenvalue b_i, and $\sum s_i^* s_i = \psi(p_{11})$. There exists a similar set $t_1, \ldots, t_m$ of partial isometries for $\varphi(p_{11})$. Let $w = t_1^* s_1 + \cdots + t_m^* s_m$. Then w is a partial isometry in the fixed point subalgebra of B such that $w^* w = \psi(p_{11})$ and $ww^* = \varphi(p_{11})$. Let $v_j = \sum_{k=1}^r \varphi(p_{k1}) w \psi(p_{1k})$. It is then easy to check that v_j is a partial isometry meeting our requirements.

Lemma 3.7.13 **(Inclusion of a Locally Finite *-Dynamical System Into Its Completion)** *Let $(\mathfrak{A}, \alpha)$ be a locally finite *-dynamical system. Let A denote the completion of $\mathfrak{A}$, and let $\tilde{\alpha}$ denote the unique* AF *flow determined by the inclusion of $\mathfrak{A}$ into A. Then* $\text{Inv}(\text{inclusion})\colon \text{Inv}(\mathfrak{A}, \alpha) \to \text{Inv}(A, \tilde{\alpha})$ *is an isomorphism.*

Proof Let A_n be an increasing sequence of α-invariant finite dimensional subalgebras of $\mathfrak{A}$ such that $1 \in A_1$ and $\bigcup_{n=1}^{\infty} A_n = \mathfrak{A}$, and let α_n denote $\alpha|_{A_n}$. Our first step is to define a sequence of maps $P_n \colon A \to A_n$ having the following properties:

(i) For all n, $P_n|_{A_n} = \mathrm{id}$
(ii) For all n, P_n is positive and linear, and $\|P_n\| = 1$.
(iii) For all n, P_n is equivariant.

From these properties we see easily that $P_n \to \mathrm{id}$ in the topology of norm convergence at each point.

Let $E_1, \dots, E_m$ be the minimal central projections of A_n, *i.e.* the units of the minimal direct summands of A_n, and consider E_1AE_1. We may express this algebra as $(E_1A_nE_1) \otimes (E_1A_nE_1)' \cap (E_1AE_1)$. E_1 is invariant under the action α, so α restricts to an action on E_1AE_1. Since $E_1A_nE_1$ is invariant under this action, its relative commutant is too, so we get an action of the reals by affine homeomorphisms on the state space of $(E_1A_nE_1)' \cap E_1AE_1$. Since $(E_1A_nE_1)' \cap E_1AE_1$ is a unital C^*-algebra, its state space is a compact convex subset of the locally convex space of all continuous linear functionals on $(E_1A_nE_1)' \cap E_1AE_1$ with the weak-* topology, and so the Markov-Kakutani fixed point theorem provides a fixed point for this action, *i.e.* it provides an α-invariant state on $(E_1A_nE_1)' \cap E_1AE_1$. Let φ be such a state. Define a map $\Pi_1 \colon E_1AE_1 \to E_1A_nE_1$ by $\Pi_1 = \mathrm{id} \otimes \varphi(\cdot)E_1$. Define in a similar way maps $\Pi_2, \dots, \Pi_m$ for each of the other minimal direct summands of A_n, and then define P_n by $P_n(x) = \Pi_1(E_1xE_1) + \cdots + \Pi_m(E_mxE_m)$.

It is easy to see from the definition that $P_n|_{A_n}$ is the identity on A_n, that P_n is linear, and that $P_n(1)$ is equal to 1. That P_n is positive follows from Corollary 3.5 in [12], which states that a positive linear functional on a C^*-algebra is completely positive (this shows that each Π_j is positive). That $\|P_n\|$ is equal to 1 follows from positivity and $P_n(1) = 1$.

To see that P_n is also equivariant requires a little more work. Clearly we only have to check that the maps Π_j are equivariant. Notice that the restriction of α to $(E_jA_nE_j)' \cap (E_jAE_j)$ is also an AF-flow, so that the linear span of the eigenoperators is dense in this algebra. Let $\{e_{jk}\}$ be a system of matrix units for $E_jA_nE_j$ consisting of eigenoperators. It will suffice for us to check that P_n is equivariant on operators of the form $e_{jk} \otimes x$ where x is an eigenoperator of $(E_jA_nE_j)' \cap (E_jAE_j)$, since the rest follows from linearity and continuity. Suppose we have $\alpha_t(x) = e^{iat}x$ for all t. Then $\varphi(x) = \varphi\big(\alpha_t(x)\big) = \varphi(e^{iat}x) = e^{iat}\varphi(x)$ for all t, so either $a = 0$ or $\varphi(x) = 0$. Suppose that $\alpha_t(e_{jk}) = e^{ibt}e_{jk}$ for all t. Then, if $a = 0$, we have $P_n\big(\alpha_t(e_{jk} \otimes x)\big) = P_n(e^{ibt}e_{jk} \otimes x) = e^{ibt}e_{jk} \otimes \varphi(x)E_j = \alpha_t\big(P_n(e_{jk} \otimes x)\big)$, and, if $\varphi(x) = 0$, we have $P_n\big(\alpha_t(e_{jk} \otimes x)\big) = 0 = \alpha_t\big(P_n(e_{jk} \otimes x)\big)$. Thus we have that P_n is equivariant.

We are now ready to prove the lemma. Let p be a projection in $A^{\tilde{\alpha}}$. Then $P_n(p)$ is in the fixed point subalgebra of A_n and $P_n(p) \to p$ in norm as $n \to \infty$, so we may find a projection q in the fixed point subalgebra of some A_n such that $[q] = [p]$ in $D(A^{\tilde{\alpha}})$. Thus the map $K\mathfrak{R}(\mathrm{inclusion}) \colon K\mathfrak{R}(\mathfrak{A}, \alpha) \to K\mathfrak{R}(A, \tilde{\alpha})$ is surjective.

To show that the map is injective it will suffice to show that if p, q, e, f, and r are projections in $\mathfrak{A}$, $a \in \mathbb{R}$, and $[p] = [q][a]$ and $[e] + [f] = [r]$ are among the defining relations in $K\mathfrak{R}(A, \tilde{\alpha})$, then $[p] = [q][a]$ and $[e] + [f] = [r]$ in $K\mathfrak{R}(\mathfrak{A}, \alpha)$.

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Let v be a partial isometry in A such that $vv^* = q$, $v^*v = p$, and v is an eigenoperator with eigenvalue a. Choose n large enough so that $p \in A_n$, $q \in A_n$, and $\|P_n(v) - v\|$ is "small", where small just means small enough for each of the approximations below to work out. Let $w|P_n(v)|$ be the polar decomposition of $P_n(v)$ in A_n and consider the function f defined by $f(t) = 0$ if $t < 1/2$ and $f(t) = 1$ if $t \geq 1/2$. Then, since $|P_n(v)|$ is close to a projection, f is continuous on the spectrum of $|P_n(v)|$. Also, w is an eigenoperator with eigenvalue a. Consider the partial isometry $u = wf\big(|P_n(v)|\big)$. We have that u is a partial isometry in A_n such that u is an eigenoperator with eigenvalue a, u^*u is norm close to p and uu^* is norm close to q. Thus we have, in $K\mathfrak{R}(\mathfrak{A}, \alpha)$, $[p] = [u^*u] = [uu^*][a] = [q][a]$, as required.

That the other relations also hold in $K\mathfrak{R}(\mathfrak{A}, \alpha)$ follows easily from the fact that the dimension range respects inductive limits and the observation that the fixed point subalgebra of A is the inductive limit of the fixed point subalgebras of the A_ns, which is easily proved using the P_n's.

Theorem 3.8 **(The Main Result)** *Let (A, α) and (B, β) be two* AF *flows, with A and B unital algebras, and let $\psi\colon \operatorname{Inv}(A, \alpha) \to \operatorname{Inv}(B, \beta)$ be an isomorphism of their invariants. Then there exists an equivariant isomorphism $\tilde{\psi}\colon A \to B$ such that $\operatorname{Inv}(\tilde{\psi}) = \psi$.*

Proof Let $\{A_n\}$ be an increasing sequence of α-invariant finite dimensional sub-C^*-algebras of A each containing the unit of A and whose union, $\mathfrak{A}$, is dense in A, and let $\{B_n\}$ be a similar sequence for B with union $\mathfrak{B}$. Let i_{mn} (resp. j_{mn}) denote the inclusion of A_m (resp. B_m) into A_n (resp. B_n), and let $i_{m\infty}$ (resp. $j_{m\infty}$) denote the inclusion of A_m (resp. B_m) into $\mathfrak{A}$ (resp. $\mathfrak{B}$). Let α_n (resp. β_n) denote the restriction of α (resp. β) to A_n (resp. B_n). Let $\varphi\colon \operatorname{Inv}(B, \beta) \to \operatorname{Inv}(A, \alpha)$ denote the inverse of ψ.

From Lemma 3.7 we have that the inclusions $\mathfrak{A} \subset A$ and $\mathfrak{B} \subset B$ induce isomorphisms of the invariants, which we can use to pull back the isomorphisms ψ and φ, to isomorphisms which we shall call by the same names. Thus we have a commuting diagram:

$$(\dagger)\qquad \begin{array}{ccc} \operatorname{Inv}(\mathfrak{A}, \alpha|_{\mathfrak{A}}) & \xrightarrow{\cong} & \operatorname{Inv}(A, \alpha) \\ {\scriptstyle\varphi}\uparrow\downarrow{\scriptstyle\psi} & & {\scriptstyle\varphi}\uparrow\downarrow{\scriptstyle\psi} \\ \operatorname{Inv}(\mathfrak{B}, \beta|_{\mathfrak{B}}) & \xrightarrow{\cong} & \operatorname{Inv}(B, \beta). \end{array}$$

Using Remark 3.3 we see that $\operatorname{Inv}(\mathfrak{A}, \alpha|_{\mathfrak{A}}) \cong \lim_{\to}\{(A_n, \alpha|_{A_n}), i_{mn}\}$, and similarly for $\operatorname{Inv}(\mathfrak{B}, \alpha|_{\mathfrak{B}})$.

Let $x_1, \dots, x_m$ be the finitely many minimal positive generators of $K\mathfrak{R}(A_1, \alpha_1)$ (*cf.* Remark 3.3). Consider there images in $K\mathfrak{R}(\mathfrak{B}, \alpha|_{\mathfrak{B}})$. Since $K\mathfrak{R}^+(\mathfrak{B}, \alpha|_{\mathfrak{B}}) = \bigcup_{n=1}^{\infty} K\mathfrak{R}(i_{n\infty})K\mathfrak{R}^+(B_n, \beta_n)$, there exist a k and elements $y_1, \dots, y_m \in K\mathfrak{R}^+(B_k, \beta_k)$ such that y_i is mapped to the image of x_i by $K\mathfrak{R}(j_{k\infty})$ for $i = 1, \dots, m$. Since $K\mathfrak{R}(A_1, \alpha_1)$ is freely generated by $x_1, \dots, x_m$, the correspondence $x_1 \mapsto y_1, \dots, x_m \mapsto y_m$ extends to a positive $\mathfrak{R}$-module homomorphism $\psi_1\colon K\mathfrak{R}(A_1, \alpha_1) \to K\mathfrak{R}(B_k, \beta_k)$ which sends the positive cone of $K\mathfrak{R}(A_1, \alpha_1)$ into the positive cone of $K\mathfrak{R}(B_k, \beta_k)$. At this point, ψ_1 need not, a *priori*, send the coloured scale $\Sigma\mathfrak{R}(A_1, \alpha_1)$ into $\Sigma\mathfrak{R}(B_k, \beta_k)$. However, since $\Sigma\mathfrak{R}(\mathfrak{B}, \alpha|_{\mathfrak{B}})$ is the increasing union of the images of the $\Sigma\mathfrak{R}(B_n, \beta_n)$'s

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and $\Sigma\mathfrak{R}(A_1, \alpha_1)$ is a finite set, we may push forward in the lower inductive system to achieve this too. We now have a commuting diagram:

$$\begin{array}{ccc}
\operatorname{Inv}(A_1, \alpha_1) & \xrightarrow{\operatorname{Inv}(i_{1\infty})} & \operatorname{Inv}(\mathfrak{A}, \alpha|_{\mathfrak{A}}) \\
\downarrow \psi_1 & & \varphi \uparrow\downarrow \psi \\
\operatorname{Inv}(B'_k, \beta'_k) & \xrightarrow{\operatorname{Inv}(j_{k'\infty})} & \operatorname{Inv}(\mathfrak{B}, \alpha|_{\mathfrak{B}})
\end{array}$$

Referring to Remark 3.4, we push forward in the inductive system one last time, to $B_{k''}$ say, to ensure that the K_0 map, $\mathfrak{B}(j_{k',k''} \circ \psi_1)$ sends the class of the unit in $K_0(A_1)$ to the class of the unit in $K_0(B_{k''})$.

We now repeat this process with the map from $\operatorname{Inv}(B_{k''}, \beta_{k''})$ to $\operatorname{Inv}(\mathfrak{A}, \alpha|_{\mathfrak{A}})$, with one more step included: we push forward one more time until the triangle

$$\begin{array}{ccc}
\operatorname{Inv}(A_1, \alpha_1) & \longrightarrow & \operatorname{Inv}(A_r, \alpha_r) \\
\downarrow & \nearrow & \\
\operatorname{Inv}(B_{k''}, \beta_{k''}) & &
\end{array}$$

commutes. That this may be done follows from the observations in Remark 3.4 that $K\mathfrak{R}(A_1, \alpha_1)$ is finitely generated as an $\mathfrak{R}$-module, so that only finitely many relations have to be killed, and that $\operatorname{Inv}(\mathfrak{A}, \alpha|_{\mathfrak{A}})$ is the inductive limit of the $\operatorname{Inv}(A_n, \alpha_n)$s.

We continue this process to get a commuting diagram:

$$\begin{array}{ccccccc}
\operatorname{Inv}(A_1, \alpha_1) & \longrightarrow & \operatorname{Inv}(A_2, \alpha_2) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Inv}(\mathfrak{A}, \alpha|_{\mathfrak{A}}) \\
\downarrow & \nearrow & \downarrow & \nearrow & & & \varphi \uparrow\downarrow \psi \\
\operatorname{Inv}(B_1, \beta_1) & \longrightarrow & \operatorname{Inv}(B_2, \beta_2) & \longrightarrow & \cdots & \longrightarrow & \operatorname{Inv}(\mathfrak{B}, \alpha|_{\mathfrak{B}})
\end{array}$$

(after passing to subsequences and relabelling). Next, we use Lemma 3.5 to lift the φ_ns and ψ_ns to equivariant *-homomorphisms, to arrive at a diagram

$$\begin{array}{ccccccc}
(A_1, \alpha_1) & \longrightarrow & (A_2, \alpha_2) & \longrightarrow & \cdots & \longrightarrow & (\mathfrak{A}, \alpha|_{\mathfrak{A}}) \\
\downarrow & \nearrow & \downarrow & \nearrow & & & \\
(B_1, \beta_1) & \longrightarrow & (B_2, \beta_2) & \longrightarrow & \cdots & \longrightarrow & (\mathfrak{B}, \alpha|_{\mathfrak{B}})
\end{array}$$

for which the diagram above (excluding the limit maps) is the image under the invariant. At this point, this diagram need not commute, but we may apply Lemma 3.6 to each triangle, moving left to right through the diagram, to alter the vertical maps by equivariant inner automorphisms of their target algebras to get a diagram that does commute and still has the same image under the invariant.

It is well known (*cf.* [4]) that such a diagram gives rise to a pair of inverse isomorphisms between the locally finite algebras $\mathfrak{A}$ and $\mathfrak{B}$ that make the whole diagram commute. Since all of our maps are equivariant, it follows that these isomorphisms are also equivariant. Commutativity of the diagram of invariants implies that these isomorphisms induce the maps φ and ψ on the invariants. Finally, these equivariant *-isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$ extend uniquely to a pair of inverse equivariant *-isomorphisms between A and B, and it follows from the commutativity of the diagram (†) that these extensions induce the original maps φ and ψ.

Remark 3.9 (Coloured Bratteli Diagrams Again) It follows from the above theorem that two locally finite *-dynamical systems are isomorphic if and only if their completions are, so we see that the coloured Bratteli diagrams actually characterise AF flows, not just locally finite *-dynamical systems.

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