NON-EXISTENCE OF SOME UNSYMMETRICAL PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

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1. Summary and introduction. A partially balanced incomplete block (PBIB) design with m-associate classes is defined by Bose and Shimamoto **(4)** as follows:

(i) The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.

(ii) There are v treatments each of which occurs in r blocks.

(iii) There can be established a relation of association between any two treatments satisfying the following requirements:

(a) Two treatments are either 1st, 2nd, . . . , or mth associates.

(b) Each treatment has exactly n_i ith associates.

(c) Given any two treatments which are ith associates, the number of the jth associates of the first and the kth associates of the second is p^i_{jk} and is independent of the pair of treatments with which we start. Also, $p^i_{jk} = p^i_{kj}$.

(iv) Two treatments which are ith associates occur together in exactly λ_i blocks.

(v) The design is connected, i.e. given any two treatments t_i and t_j there exists a sequence of treatments $t_{i_0} = t_i, t_{i_1}, \ldots, t_{i_n} = t_j$ and a sequence of blocks $b_{i_0}, b_{i_1}, \ldots, b_{i_{n-1}}$ such that b_{i_p} contains the treatments t_{i_p} and $t_{i_{p+1}}$, $p = 0, 1, \ldots, n-1$.

PBIB designs were introduced for the first time in experimental designs by Bose and Nair **(2)**. Nair **(10)** gave a necessary condition for the existence of PBIB designs with $b < v$ based on the vanishing of a certain determinant. In this paper we obtain necessary conditions for the existence of certain PBIB designs with $b < v$ based on the characteristic roots and vectors of NN' where $N = (n_{ij})$ is the $v \times b$ incidence matrix defined by $n_{ij} = 1$ or 0 according as the ith treatment occurs in the jth block or not.

2. Notations and preliminaries. Throughout this paper I_n stands for an identity matrix of order n, $E_{m,n}$ stands for the $m \times n$ matrix with all elements equal to unity, $O_{m,n}$ stands for the $m \times n$ null matrix, and diag$(A_1, A_2, \ldots, A_m)$ stands for the diagonal matrix with the elements or matrices

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$A_1, A_2, \ldots, A_m$ in the diagonal positions. Let A and B be two rational, symmetric, and non-singular matrices of the same order n such that $A = CBC'$, where C is a rational non-singular matrix and C' is its transpose. Then A and B are said to be rationally congruent. The rational congruence relation between A and B is denoted, symbolically, by

$$A \sim B. \tag{2.1}$$

Let D_r be the leading principal minor determinant of order r and suppose that $D_r \neq 0$ for all r. Define $D_0 = 1$. Then the Hasse-Minkowski invariant of A is given by

$$C_p(A) = (-1, -1)_p \prod_{i=0}^{n-1} (D_{i+1}, -D_i)_p, \tag{2.2}$$

for each prime p, where $(a, b)_p$, denotes the extended Hilbert norm residue symbol defined by

$$(a, b)_p = \begin{cases} 1, & \text{if } ax^2 + by^2 = 1 \text{ has a } p\text{-adic solution;} \\ -1, & \text{otherwise.} \end{cases} \tag{2.3}$$

The following theorem is well known (cf. Jones **8**).

THEOREM 2.1. *The necessary and sufficient conditions for two positive definite, rational and symmetric matrices A and B of the same order to be rationally congruent are that the square-free parts of their determinants are the same and that their Hasse-Minkowski invariants are equal for all primes p including p_∞.*

We now state, without proofs, certain lemmas regarding the Hasse-Minkowski invariant. They are taken from Bose and Connor **(1)** and Ogawa **(11)**.

LEMMA 2.1. *If $A_1, A_2, \ldots, A_m$ are rational, non-singular, and symmetric matrices, and if*

$$A = \operatorname{diag}(A_1, A_2, \ldots, A_m), \tag{2.4}$$

then

$$C_p(A) = (-1, -1)_p^{m-1} \left\{ \prod_{i=1}^{m} C_p(A_i) \right\} \left\{ \prod_{i<j=1}^{m} (|A_i|, |A_j|)_p \right\}. \tag{2.5}$$

As a particular case of the above lemma, we have the following lemma.

LEMMA 2.2. *The Hasse-Minkowski invariant of*

$$A = I_m \times B \tag{2.6}$$

is

$$C_p(A) = (-1, -1)_p^{m-1} \{C_p(B)\}^m (|B|, -1)_p^{m(m-1)/2}, \tag{2.7}$$

where B is a rational, non-singular and symmetric matrix, and $\times$ denotes the Kronecker product of matrices.

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LEMMA 2.3. *For an $m \times m$ diagonal matrix Δ_m, with each diagonal element d,*

$$C_p(\Delta_m) = (-1, -1)_p \, (-1, d)_p^{m(m+1)/2}. \tag{2.8}$$

LEMMA 2.4. *If ρ is a non-zero rational and A is a rational, non-singular and symmetric matrix of order m, then*

$$C_p(\rho A) = (-1, \rho)_p^{m(m+1)/2} \, (\rho, |A|)_p^{m-1} \, C_p(A). \tag{2.9}$$

LEMMA 2.5. *If*

$$A = eI_m + fE_{m,m}, \tag{2.10}$$

where e and f are non-zero rationals, then

$$C_p(A) = (-1, -1)_p \, (-1, e)_p^{m(m-1)/2} \, (-1, g)_p \, (m, g)_p \, (m, e)_p \, (g, e)_p^{m-1}, \tag{2.11}$$

where

$$g = e + mf. \tag{2.12}$$

LEMMA 2.6. *If the $m - 1$ rational column vectors $\mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_m$ of dimension m are linearly independent and are orthogonal to $\mathbf{E}_{m,1}$, then the gramian of the set, that is,*

$$U = \begin{pmatrix} \mathbf{a}_2' \\ \mathbf{a}_3' \\ \vdots \\ \mathbf{a}_m' \end{pmatrix} (\mathbf{a}_2 \, \mathbf{a}_3 \ldots \mathbf{a}_m), \tag{2.13}$$

satisfies

$$C_p(U) = (-1, -1)_p. \tag{2.14}$$

LEMMA 2.7. *So long as we restrict ourselves to rational vectors, the p-invariant of the gramian of the set is uniquely determined by the linear subspace generated by the set.*

We quote for completeness the properties of the Hilbert symbol and some of the useful properties of the Legendre symbol (a/p), where p is a prime (cf. Jones **8**, Pall **12**, Ogawa **11**).

LEMMA 2.8. *If m and m' are integers not divisible by the odd prime p, then*

$$(m, m')_p = +1, \tag{2.15}$$

$$(m, p)_p = (m/p). \tag{2.16}$$

Moreover, if $m \equiv m' \not\equiv 0 \pmod p$,

$$(m, p)_p = (m', p)_p. \tag{2.17}$$

LEMMA 2.9. *For arbitrary non-zero integers m, m', n, n', and s and for every prime p,*

$$(m, -m)_p = +1, \tag{2.18}$$

$$(m, n)_p = (n, m)_p, \tag{2.19}$$

$$(m, nn')_p = (m, n)_p \, (m, n')_p, \tag{2.20}$$

$$(mm', m-m')_p = (m, -m')_p, \tag{2.21}$$

$$\prod_{j=1}^{m} (j, j+1)_p = ((m+1)!, -1)_p, \tag{2.22}$$

and

$$(as^2, b)_p = (a, b)_p. \tag{2.23}$$

LEMMA 2.10. *For the Legendre symbol, we have*

$$(a/p) = (b/p), \quad \text{if} \quad a \equiv b \pmod{p}, \tag{2.24}$$

$$(ab/p) = (a/p)(b/p), \tag{2.25}$$

$$(p/q)(q/p) = (-1)^{\frac{1}{2}(p-1)\cdot\frac{1}{2}(q-1)} \tag{2.26}$$

$$(-1/p) = (-1)^{(p-1)/2}, \tag{2.27}$$

$$(2/p) = (-1)^{(p^2-1)/8}, \tag{2.28}$$

where p and q denote odd primes.

If the square-free parts of two rational numbers a and b are equal, we denote this fact by

$$a \sim b. \tag{2.29}$$

3. Main results. Let M be a semi-positive definite rational and symmetric matrix of order v. If the rank of M is $v - \alpha$, then exactly α of the characteristic roots of M are zero. Let the remaining roots be $\rho_0, \rho_1, \ldots, \rho_s$ with multiplicities $\alpha_0, \alpha_1, \ldots, \alpha_s$ respectively. Let $X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\alpha)$ be a $v \times \alpha$ matrix, whose columns represent a set of mutually orthogonal rational vectors corresponding to the zero root of M. Let

$$\mathbf{x}_i'\mathbf{x}_i = C_i. \tag{3.1}$$

Then it is easy to see that $M + XX'$ is also a rational symmetric matrix with

$$|M + XX'| = \left(\prod_{i=0}^{s} \rho_i^{\alpha_i}\right)\left(\prod_{j=1}^{\alpha} C_j\right). \tag{3.2}$$

If further M is irreducible and generalized stochastic, then one of the non-zero roots, say ρ_0, is of multiplicity $\alpha_0 = 1$. Suppose further that the remaining roots $\rho_1, \rho_2, \ldots, \rho_s$ are also rational, then we have the spectral representation

$$M = \sum_{i=0}^{s} \rho_i A_i, \tag{3.3}$$

where the matrices A_i are rational, symmetric, and idempotent and further

$$A_i A_j = O_{v,v}, \qquad i \neq j = 0, 1, \ldots, s. \tag{3.4}$$

In particular $A_0 = E_{v,v}/v$. We also note that each A_i generates the vector space corresponding to ρ_i. Put

$$Y = X \operatorname{diag}(C_1^{-\frac{1}{2}}, C_2^{-\frac{1}{2}}, \ldots, C_\alpha^{-\frac{1}{2}}), \tag{3.5}$$

and

$$L = M + YY'. \tag{3.6}$$

Obviously YY' is again rational, symmetric, and idempotent, and

$$\begin{aligned} LA_i &= \rho_i A_i, \\ LYY' &= YY'. \end{aligned} \tag{3.7}$$

Hence A_i and YY' generate vector spaces corresponding to the root ρ_i and 1 of L, and

$$|L| = \prod_{i=0}^{s} \rho_i^{\alpha_i}. \tag{3.8}$$

Following the method used by Ogawa **(11)**, we have

$$L \sim \operatorname{diag}(\rho_0 v, \rho_1 Q_1, \ldots, \rho_s Q_s, Q), \tag{3.9}$$

where Q_i and Q are the gramians of the complete set of rational vectors corresponding to the roots ρ_i and zero respectively. Hence

$$|L| \sim \rho_0 v |Q| \prod_{i=1}^{s} \{\rho_i^{\alpha_i} |Q_i|\}. \tag{3.10}$$

Equating the two values of $|L|$, we have

$$|Q| \left\{ \prod_{i=1}^{s} |Q_i| \right\} \sim v. \tag{3.11}$$

Further, since

$$\operatorname{diag}(Q_1, Q_2, \ldots, Q_s, Q) \tag{3.12}$$

is the gramian of $v - 1$ independent rational vectors all orthogonal to E_{v1}, making use of Lemmas 2.1 and 2.6, we have

$$\left\{ \prod_{i<j=1}^{s} (|Q_i|, |Q_j|)_p \right\} \left\{ \prod_{i=1}^{s} (|Q_i|, |Q|)_p \right\} \left\{ \prod_{i=1}^{s} C_p(Q_i) \right\} C_p(Q) = (-1, -1)_p^{s+1}. \tag{3.13}$$

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From (3.9), (3.11), (3.13), and the lemmas of Section 2, we obtain

$$(3.14)\quad C_p(L) = \left(\rho_0 v, -v\prod_{i=1}^{s}\rho_i^{\alpha_i}\right)_p \left\{\prod_{i=1}^{s}(\rho_i^{\alpha_i}, |Q|)_p\right\} \times\left\{\prod_{i<j=1}^{s}(\rho_i^{\alpha_i}, \rho_j^{\alpha_j})_p\right\}\left\{\prod_{i<j=1}^{s}(\rho_i^{\alpha_i}, |Q_j|)_p\right\} \times\left\{\prod_{i<j=1}^{s}(\rho_j^{\alpha_j}, |Q_i|)_p\right\}\left\{\prod_{i=1}^{s}(-1, \rho_i)_p^{\alpha_i(\alpha_i+1)/2}\right\} \times\left\{\prod_{i=1}^{s}(\rho_i, |Q_i|)_p^{\alpha_i-1}\right\}(-1,-1)_p.$$

For $s = 1$, we have

$$(3.15)\quad C_p(L) = (\rho_0, -v\rho_1^{\alpha_1})_p\,(v, \rho_1)_p\,(\rho_1, |Q|)_p\,(-1, \rho_1)_p^{\alpha_1(\alpha_1+1)/2}(-1,-1)_p.$$

We can state the above results as the following theorem.

THEOREM 3.1. *Let M be an irreducible and generalized stochastic, semi-positive definite rational symmetric matrix of order v and rank $v - \alpha$, with all its characteristic roots rational. Let L be defined as in* (3.6). *Then the determinant and the Hasse-Minkowski invariant of L are given by the expressions* (3.8) *and* (3.14) *respectively.*

We now apply the above results to PBIB designs. Let N of order $v \times b$ and rank b be the incidence matrix of a PBIB design with $s + 1$ associate classes, where $b = v - \alpha$, and let $M = NN'$. Then $\rho_0 = rk$ is a root of multiplicity 1, and zero is a root of M of multiplicity α. Let the remaining s positive roots $\rho_1, \rho_2, \ldots, \rho_s$ be distinct and rational with respective multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_s$. We shall call $0, \rho_1, \ldots, \rho_s$ roots of the PBIB design and they can be found by the method of Connor and Clatworthy (**6** or **3**).

Let X, Y, and L be as defined earlier. Then

$$\begin{aligned} L &= M + YY' \\ &= (N|X)\operatorname{diag}(I_{v-\alpha}, C_1^{-1}, \ldots, C_\alpha^{-1})(N|X)'. \end{aligned}$$

Hence

$$(3.16)\qquad L \sim \operatorname{diag}(I_{v-\alpha}, C_1, \ldots, C_\alpha) \sim \operatorname{diag}(I_{v-\alpha}, Q),$$

and

$$(3.17)\qquad |L| \sim C_1C_2\ldots C_\alpha \sim |Q|.$$

Equating the value of $|L|$ in (3.8) and (3.17) we get

$$(3.18)\qquad |Q|\left\{\prod_{i=0}^{s}\rho_i^{\alpha_i}\right\} \sim 1,$$

which is a necessary condition for the existence of a PBIB design.

From Lemma 2.1, for any odd prime p, we have

$$C_p(L) = C_p\{\mathrm{diag}(I_{v-\alpha}, Q)\} = C_p(Q), \tag{3.19}$$

or

$$C_p(L)C_p(Q) = 1, \tag{3.20}$$

where $C_p(L)$ is given by (3.14). Thus (3.20) is another necessary condition for the existence of a PBIB design. These two necessary conditions for the existence of PBIB designs can be stated in the following theorem.

THEOREM 3.2. *Let a* PBIB *design with* $s + 1$ *associate classes and* $b = v - \alpha$ *have distinct positive rational roots* $\rho_0 = rk, \rho_1, \ldots, \rho_s$ *with multiplicities* $\alpha_0 = 1$, $\alpha_1, \ldots, \alpha_s$, *and zero as a root with multiplicity* α. *Then necessary conditions for the existence of the designs are*

$$\text{(i)} \qquad \left\{\prod_{i=0}^{s} \rho_i^{\alpha_i}\right\} |Q| \sim 1$$

and further if (i) *is satisfied, then*

$$\text{(ii)} \quad (\rho_0, -v)_p \left(v, \prod_{i=1}^{s} \rho_i^{\alpha_i}\right)_p \left\{\prod_{i=1}^{s} (-1, \rho_i)_p^{\alpha_i(\alpha_i+3)/2}\right\} (-1, -1)_p$$
$$\times \left\{\prod_{i<j=1}^{s} (\rho_i^{\alpha_i}, \rho_j^{\alpha_j})_p\right\} \left\{\prod_{i<j=1}^{s} (\rho_i^{\alpha_i}, |Q_j|)_p\right\} \left\{\prod_{i<j=1}^{s} (\rho_j^{\alpha_j}, |Q_i|)_p\right\}$$
$$\times \left\{\prod_{i=1}^{s} (\rho_i, |Q_i|)_p^{\alpha_i - 1}\right\} C_p(Q) = 1.$$

As a corollary of the above, we have the following corollary.

COROLLARY 3.2.1. *Let a* PBIB *design with two associate classes and* $b = v - \alpha$ *have roots* $\rho_0 = rk$ *necessarily of multiplicity* 1, *and zero as a root of multiplicity* α, *the remaining root being a positive rational number* ρ_1 *of multiplicity* $b - 1$. *Then necessary conditions for the existence of the design are*

$$\text{(i)} \qquad \rho_0 \rho_1^{\alpha_1} |Q| \sim 1,$$

and if (i) *is satisfied, then*

$$\text{(ii)} \qquad (-1, -1)_p (\rho_0, -v\rho_1^{\alpha_1+1})_p (v, \rho_1)_p (-1, \rho_1)_p^{\alpha_1(\alpha_1+3)/2} C_p(Q) = 1.$$

It follows from **(9)** that any PBIB design with two associate classes of the above corollary is a linked block design and hence its dual **(9)** is a balanced incomplete block design.

We apply the results of this section to group divisible, triangular, and L_i designs in the next three sections. In what follows, we omit the subcript p in the Hilbert norm residue symbol and we calculate the C_p invariants for odd primes only.

4. Application of the main results to group divisible designs. A group divisible (GD) design has $v = mn$ treatments which are divided into m groups of n treatments each, such that any two treatments in the same group are first associates and any two treatments in different groups are second associates. It is known **(4)** that $\rho_0 = rk$, $\theta_1 = rk - v\lambda_2$, and $\theta_2 = r - \lambda_1$ are the distinct characteristic roots of NN' with multiplicities $\alpha_0 = 1$, $\beta_1 = m - 1$, and $\beta_2 = m(n-1)$ respectively. If P_1 and P_2 are the gramians corresponding to the rational characteristic roots θ_1 and θ_2 respectively, we can show by a method similar to Corsten **(7)** that

$$\operatorname{diag}(v, P_1) \sim nI_m. \tag{4.1}$$

From (3.11), (3.13), and (4.1), we have

$$|P_1| \sim vn^m, \tag{4.2}$$

$$|P_2| \sim n^m, \tag{4.3}$$

$$C_p(P_1) = (v, n^m)(-1, n)^{m(m+1)/2}, \tag{4.4}$$

$$C_p(P_2) = (n, -1)^{m(m+3)/2}. \tag{4.5}$$

Now, we consider the case $\theta_1 = 0$ and $\theta_2 \neq 0$. These designs are known as semi-regular GD designs **(1)**. To meet the requirements of Section, 3, we consider the case $b = v - m + 1$ only. From Corollary 3.2.1, necessary conditions for the existence of this class of semi-regular GD designs are that

$$\lambda_2 n^m (r - \lambda_1)^{m(n-1)} \tag{4.6}$$

should be a perfect square and, further, if (4.6) is satisfied, then

$$(\rho_0, -v\theta_2^{\beta_2+1})(v, \theta_2)(-1, \theta_2)^{\beta_2(\beta_2+3)/2}\,(v, n^m)(-1, n)^{m(m+1)/2} = 1. \tag{4.7}$$

We now distinguish three cases: (i) m even, (ii) m odd and n even; and (iii) m and n both odd.

In case (i), (4.6) and (4.7) imply that λ_2 must be a perfect square and if this condition is satisfied, then a further necessary condition is that

$$(-1, \theta_2)^{\beta_2(\beta_2+1)/2}(-1, n)^{m(m+1)/2} = 1.$$

Similarly, a necessary condition in case (ii) is that $\lambda_2 n\theta_2$ be a perfect square and further

$$(-1, \theta_2)^{\beta_2(\beta_2+1)/2}(-1, n)(-1, n)^{m(m+1)/2} = 1.$$

Finally, in case (iii) a necessary condition is that $\lambda_2 n$ be a perfect square and further

$$(-1, \theta_2)^{\beta_2/2}(n, \theta_2)(-1, n)^{\frac{1}{2}m(m+1)+1} = 1.$$

From these considerations we deduce the following theorem.

THEOREM 4.1. *Necessary conditions for the existence of a semi-regular* GD *design with $b = v - m + 1$ are:*

(i) *if m is even, then λ_2 must be a perfect square and further*

(a) *if $m \equiv 2 \pmod 4$ and n is odd, then the square-free part of n contains only primes $\equiv 1 \pmod 4$,*

(b) *if $m \equiv 2 \pmod 4$ and n is even, then the square-free part of $n\theta_2$ contains only primes $\equiv 1 \pmod 4$;*

(ii) *if m is odd and n is even, then $\lambda_2 n\theta_2$ must be a perfect square and further*

(a) *if $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$, then the square-free part of θ_2 contains only primes $\equiv 1 \pmod 4$,*

(b) *if $m \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$, then the square-free part of n contains only primes $\equiv 1 \pmod 4$,*

(c) *if $m \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 4$, then the square-free part of $n\theta_2$ contains only primes $\equiv 1 \pmod 4$;*

(iii) *if m and n are both odd, then $\lambda_2 n$ must be a perfect square, and further*

(a) *if $m \equiv n \equiv 1 \pmod 4$, then $(n, \theta_2) = 1$,*

(b) *if $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, then $(-n, \theta_2) = 1$,*

(c) *if $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$, then $(n, -\theta_2) = 1$,*

(d) *if $m \equiv n \equiv 3 \pmod 4$, then $(n, \theta_2)(n\theta_2, -1) = 1$.*

We easily see that Theorem 3 of Saraf **(14)** is contained in the above theorem.

It can easily be seen that the dual of the affine resolvable BIB design D **(17)**, with parameters

$$v = nk = n^2\{(n-1)t + 1\}, \quad b = nr = n(n^2t + n + 1), \quad \lambda = nt + 1 \tag{4.8}$$

is a semi-regular GD design, D^*, with parameters

$$\begin{aligned} v &= n(n^2t + n + 1), \quad m = n^2t + n + 1, \quad n, \\ b &= n^2\{(n-1)t + 1\}, \quad r = n\{(n-1)t + 1\}, \\ k &= n^2t + n + 1, \quad \lambda_1 = 0, \quad \lambda_2 = (n-1)t + 1, \end{aligned} \tag{4.9}$$

satisfying the conditions of Theorem 4.1, and conversely the dual of the semi-regular GD design with parameters (4.9) is an affine resolvable BIB design with parameters (4.8) (cf. **9**). It is easy to verify that the above theorem rules out D^* exactly for those values of n and t for which D is ruled out by the results in **(17)**.

We now consider GD designs where $\theta_2 = 0$. Designs of this type are known as singular GD designs **(1)**. To meet the requirements of Section 3, let $b = m$. The parameters of this class of singular GD designs are

$$v = mn, \quad b = m, \quad r, \quad k = nr, \quad \lambda_1 = r, \quad \lambda_2. \tag{4.10}$$

There is a one-to-one correspondence between the above GD designs and the symmetrical BIB design

$$v^* = m = b^*, \quad r^* = r = k^*, \quad \lambda^* = \lambda. \tag{4.11}$$

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It can again be shown that the results of Corollary 3.2.1 imply the non-existence of singular GD designs with the parameters (4.1) precisely in those cases where the impossibility of the corresponding symmetrical BIB design was proved in **(15)**.

Now let us consider necessary conditions for the existence of certain generalized GD designs. For the definition and characterization of GD m-associate designs, we refer to **(13)**. It was proved that the distinct characteristic roots of NN' for GD m-associate designs are

$$(4.12)\quad \rho_0 = rk,\ \theta_i = (r-\lambda_{m-i+1})+(\lambda_1-\lambda_{m-i+1})n_1+\cdots+(\lambda_{m-i}-\lambda_{m-i+1})n_{m-i}$$

with multiplicities 1, $N_1N_2 \ldots N_{i-1}(N_i - 1)$ respectively $(i = 1, 2, \ldots, m)$. If P_i is the gramian corresponding to the root θ_i of NN', it can be proved by Corsten's method that

$$(4.13)\qquad \begin{cases} \operatorname{diag}(v, P_1) \sim N_2N_3 \ldots N_m I_{N_1}, \\ \operatorname{diag}(v, P_1, P_2) \sim N_3N_4 \ldots N_m I_{N_1N_2}, \\ \ldots \\ \operatorname{diag}(v, P_1, P_2, \ldots, P_{m-1}) \sim N_m I_{N_1N_2 \ldots N_{m-1}}. \end{cases}$$

For $\theta_m, \theta_{m-1}, \ldots, \theta_2$-regular GD m-associate designs, application of the first part of Theorem 3.2 gives the following result.

THEOREM 4.2. *A necessary condition for the existence of $\theta_m, \theta_{m-1}, \ldots, \theta_2$-regular* GD *m-associate designs where $b = v - N_1 + 1$ is that*

$$(4.14)\qquad (N_2N_3 \ldots N_m)^{N_1}\lambda_m\theta_2^{N_1(N_2-1)}\theta_3^{N_1N_2(N_3-1)} \ldots \theta_m^{N_1N_2\ldots N_{m-1}(N_m-1)}$$

be a perfect square.

The second part of Theorem 3.2 can also be applied to this class of designs and further necessary conditions can be obtained; but this involves tedious calculations without presenting any mathematical difficulty. We shall not discuss these conditions.

5. Application of the main results to triangular designs. A triangular design has $v = n(n - 1)/2$ treatments and the association scheme is a square array of side n with the following properties:

(a) The positions in the principal diagonal are blank.

(b) The $n(n - 1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \ldots, n(n - 1)/2$ corresponding to the treatments.

(c) The array is symmetrical about the principal diagonal.

(d) For any treatment ϕ, the treatments lying in the same row and column as ϕ are its first associates while all the other treatments are its second associates.

It is known **(6)** that for triangular designs the distinct characteristic roots of NN' are $\rho_0 = rk$, $\theta_1 = r + (n - 4)\lambda_1 - (n - 3)\lambda_2$, and $\theta_2 = r - 2\lambda_1 + \lambda_2$, with multiplicities $\alpha_0 = 1$, $\beta_1 = n - 1$, and $\beta_2 = n(n - 3)/2$ respectively. If

P_1 and P_2 are the gramians corresponding to the roots θ_1 and θ_2 of NN', it is shown by Corsten (7) that

$$\operatorname{diag}(v, P_1) \sim (n-2)I_n + E_{nn}. \tag{5.1}$$

From (3.11), (3.13), and (5.1), we have

$$|P_1| \sim n(n-2)^{n-1}, \tag{5.2}$$

$$|P_2| \sim 2(n-1)(n-2)^{n-1}, \tag{5.3}$$

$$\begin{aligned} C_p(P_1) = (-1, n-2)^{n(n-1)/2}(-1, n-1)^n(2, n(n-2)^{n-1}) \\ \times (n, -1)(n, n-2)(-1, v)(v, n(n-2)^{n-1}), \end{aligned} \tag{5.4}$$

and

$$C_p(P_2) = (n(n-2)^{n-1}, 2(n-1)(n-2)^{n-1})C_p(P_1). \tag{5.5}$$

Now let $\theta_1 = 0$ and $b = (n-1)(n-2)/2$. From Corollary 3.2.1, necessary conditions for the existence of this class of designs are that

$$\rho_0\theta_2^{n(n-3)/2}n(n-2)^{n-1} \tag{5.6}$$

should be a perfect square and, further, if (5.6) is satisfied, then

$$\begin{aligned} (\rho_0, -v\theta_2^{\beta_2+1})(v, \theta_2)(-1, \theta_2)^{\beta_2(\beta_2+3)/2} \\ \times(-1, n-2)^{n(n-1)/2}(-1, n-1)^n(-1, n)(n-2, n) \\ \times(-1, v)(2v, \rho_0\theta_2^{\beta_2}) = 1. \end{aligned} \tag{5.7}$$

We distinguish several cases for n and, after easy calculations, we have the following theorem.

THEOREM 5.1. *Necessary conditions for the existence of triangular designs with $\theta_1 = 0$ and $b = (n-1)(n-2)/2$ are*

(i) *if $n \equiv 0 \pmod 4$, then $\rho_0 n(n-2)$ must be a perfect square and further*

$$(\rho_0, -2n\theta_2)(v, -\theta_2) = 1 \quad \textit{if} \quad n \equiv 0 \pmod 8,$$

and

$$(\theta_2, -1)(\rho_0, -2n\theta_2)(v, -\theta_2) = 1 \quad \textit{if} \quad n \equiv 4 \pmod 8;$$

(ii) *if $n \equiv 1 \pmod 4$, then $\rho_0\theta_2 n$ must be a perfect square and further*

$$(2, n-2) = 1 \quad \textit{if} \quad n \equiv 1 \pmod 8$$

and

$$(\theta_2, -1)(2, n-2) = 1 \quad \textit{if} \quad n \equiv 5 \pmod 8;$$

(iii) *if $n \equiv 2 \pmod 4$, then $\rho_0\theta_2 n(n-2)$ must be a perfect square and further*

$$(\rho_0\theta_2, 2)(n, n-2)(v, -1) = 1 \quad \textit{if} \quad n \equiv 2 \pmod 8$$

and

$$(\theta_2, -1)(\rho_0\theta_2, 2)(n, n-2)(v, -1) = 1 \quad \textit{if} \quad n \equiv 6 \pmod 8;$$

(iv) *if* $n \equiv 3 \pmod 4$, *then* $\rho_0 n$ *must be a perfect square and further*

$$(\rho_0, -2(n-2))(\theta_2, 2(n-1))(n-2, -1) = 1 \quad \text{if} \quad n \equiv 3 \pmod 8$$

and

$$(\theta_2, -1)(\rho_0, -2(n-2))(\theta_2, 2(n-1))(n-2, -1) = 1 \text{ if } n \equiv 7 \pmod 8.$$

Let us consider the symmetrical BIB design D with parameters

$$v^* = b^* = 1 + n(n-1)/2, \quad r^* = k^* = n, \quad \lambda^* = 2. \tag{5.8}$$

By removing a block and all the treatments contained in it from the above design, we get the BIB design D_1 with parameters

$$v' = (n-1)(n-2)/2,\ b' = n(n-1)/2,\ r' = n,\ k' = n-2,\ \lambda' = 2. \tag{5.9}$$

The dual of D_1 is a PBIB design D_1^* with parameters

$$v = n(n-1)/2,\ b = (n-1)(n-2)/2,\ r = n-2,\ k = n, \quad \lambda_1 = 1, \quad \lambda_2 = 2, \tag{5.10}$$

which has a triangular association scheme when $n \neq 8$ (cf. Shrikhande **18**). Further, D_1^* satisfies the requirements of Theorem 5.1. It can easily be verified that D and D_1^* have the same necessary conditions of existence, which tallies with the fact that there is a one-to-one correspondence between D and D_1^* (**9**, **16**, **18**).

We now consider triangular designs with $\theta_2 = 0$ and $b = n$. From Corollary 3.2.1, necessary conditions for the existence of this class of designs are that

$$2\rho_0\theta_1^{\beta_1}(n-1)(n-2)^{n-1} \tag{5.11}$$

should be a perfect square and further

$$(\rho_0, -v\theta_1^{\beta_1+1})(v, \theta_1)(-1, \theta_1)^{\beta_1(\beta_1+3)/2} \times (-1, n-2)^{n(n-1)/2}(-1, n-1)^n(-1, n)(n-2, n) \times (-1, v)(2v, n(n-2)^{n-1})(n(n-2)^{n-1}, \rho_0\theta_1^{\beta_1}) = 1.$$

From this result the following theorem can easily be deduced.

THEOREM 5.2. *Necessary conditions for the existence of triangular designs with* $\theta_2 = 0$ *and* $b = n$ *are*

(i) *if* n *is even, then* $2\rho_0\theta_1(n-1)(n-2)$ *must be a perfect square and further*

$$(\rho_0\theta_1, -n(n-2)) = 1 \quad \text{if} \quad n \equiv 0 \pmod 4$$

and

$$(\rho_0\theta_1, -n(n-2))(-1, \theta_1(n-2)) = 1 \quad \text{if} \quad n \equiv 2 \pmod 4;$$

(ii) *if* n *is odd, then* $2\rho_0(n-1)$ *must be a perfect square and further*

$$(2, n-2)(\rho_0\theta_1, v)(\rho_0, -\theta_1) = 1 \quad \text{if} \quad n \equiv 1 \pmod 4$$

and

$$(2, n-2)(\rho_0\theta_1, v)(\rho_0, -\theta_1)(-1, \theta_1(n-2)) = 1 \quad \text{if} \quad n \equiv 3 \pmod 4.$$

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6. Application of the main results to L_i designs. An L_i design has $v = s^2$ treatments. The treatments are arranged in a square L and $i-2$ mutually orthogonal latin squares are superimposed on L. For each treatment θ in L, treatments occurring in the same row or column of L as θ or treatments corresponding to the same letters of the superimposed orthogonal latin squares are first associates and others are second associates. It is known that the distinct characteristic roots for L_i designs are $\rho_0 = rk$,

$$\theta_1 = r + (s-i)\lambda_1 - (s+1-i)\lambda_2,$$

and $\theta_2 = r - i\lambda_1 + (i-1)\lambda_2$ with respective multiplicities $\alpha_0 = 1$, $\beta_1 = i(s-1)$, and $\beta_2 = (s+1-i)(s-1)$. If P_1 and P_2 are the gramians corresponding to the roots θ_1 and θ_2 of NN', it has been proved by Shrikhande and Jain **(19)** that

$$(6.1)\qquad \begin{bmatrix} s^2 & O_{1,\beta_1} \\ O_{\beta_1,1} & P_1 \end{bmatrix} \sim \begin{bmatrix} s^2 & sE_{1,s-1} & sE_{1,s-1} & \dots & sE_{1,s-1} \\ sE_{s-1,1} & sI_{s-1} & E_{s-1,s-1} & \dots & E_{s-1,s-1} \\ sE_{s-1,1} & E_{s-1,s-1} & sI_{s-1} & \dots & E_{s-1,s-1} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ sE_{s-1,1} & E_{s-1,s-1} & E_{s-1,s-1} & \dots & sI_{s-1} \end{bmatrix}.$$

From (3.11), (3.13), and (6.1), we have

$$(6.2)\qquad |P_1| \sim s^{is},$$

$$(6.3)\qquad |P_2| \sim s^{is},$$

$$(6.4)\qquad C_p(P_1) = (s, -1)^{is(s+i)/2},$$

and

$$(6.5)\qquad C_p(P_2) = (s, -1)^{is(s+i+2)/2}.$$

Now let $\theta_1 = 0$ and $b = s(s-i) + i$. From Corollary 3.2.1, necessary conditions for the existence of this class of designs are that

$$(6.6)\qquad \rho_0 \theta_2^{\beta_2} s^{is}$$

should be a perfect square and further

$$(6.7)\qquad (\rho_0, -\theta_2^{\beta_2+1})(-1, \theta_2)^{\beta_2(\beta_2+3)/2}(s, -1)^{is(s+i)/2} = 1.$$

Thus we have the following theorem.

THEOREM 6.1. *Necessary conditions for the existence of L_i designs with $\theta_1 = 0$ and $b = s(s-i) + i$ are:*

(i) *if s is even and i is odd, then ρ_0 must be a perfect square and further* $(-1, \theta_2)^{\beta_2(\beta_2+3)/2}(-1, s)^{is/2} = 1$;

(ii) *if s and i are both even, then $\rho_0\theta_2$ must be a perfect square and further* $(-1, \theta_2)^{\beta_2(\beta_2+1)/2} = 1$;

(iii) *if s is odd and i is even, then ρ_0 must be a perfect square and further* $(-1, s)^{is/2} = 1$;

(iv) *if s and i are both odd, then $\rho_0 s$ must be a perfect square and further* $(s, \theta_2)(-1, \theta_2)^{\beta_2(\beta_2+3)/2}(-1, s)^{(s+i+2)/2} = 1$.

We now consider L_i designs where $\theta_2 = 0$ and $b = v - \beta_2 = i(s-1) + 1$. From Corollary 3.2.1, necessary conditions for the existence of this class of designs are that

$$\rho_0\theta_1^{\beta_1}s^{is} \tag{6.8}$$

should be a perfect square and further

$$(\rho_0, -\theta_1^{\beta_1+1})(-1, \theta_1)^{\beta_1(\beta_1+3)/2}(-1, s)^{is(s+i+2)/2} = 1. \tag{6.9}$$

Thus we have the following theorem.

THEOREM 6.2. *Necessary conditions for the existence of the L_i designs with $\theta_2 = 0$ and $b = i(s-1) + 1$ are:*

(i) *if i is even, then ρ_0 must be a perfect square and further*

$$(-1, \theta_1)^{\beta_1(\beta_1+3)/2}(-1, s)^{is^2/2} = 1;$$

(ii) *if i is odd and s is even, then $\rho_0\theta_1$ must be a perfect square and further* $(-1, \theta_1)^{\beta_1(\beta_1+1)/2}(-1, s)^{s/2} = 1$;

(iii) *if i and s are both odd, then $\rho_0 s$ must be a perfect square and further* $(s, \theta_1)(-1, \theta_1)^{\beta_1(\beta_1+3)/2}(-1, s)^{(s+i)/2} = 1$.

We conclude this section by giving an alternative proof of Bruck and Ryser's theorem **(5)** on the non-existence of projective planes. We know that if PG$(2, s)$ exists, then the orthogonal array $(s^2, s+1, s, 2)$ exists. Let P be the gramian of the rational vectors orthogonal to $E_{s^2,1}$ corresponding to i constraints. Then

$$|P_1| = s^{is}, \tag{6.11}$$

$$C_p(P_1) = (-1, s)^{is(i+s)/2}. \tag{6.12}$$

Similarly changing i into $s + 1 - i$ (i.e. considering rational vectors orthogonal to $E_{s^2,1}$ corresponding to $s + 1 - i$ other constraints), we get

$$|P_2| = s^{(s+1-i)s}, \tag{6.13}$$

$$C_p(P_2) = (-1, s)^{s(s+1-i)(2s+1-i)/2}. \tag{6.14}$$

It follows from **(19)** that the vectors corresponding to P_1 and P_2 are orthogonal to one another and to $E_{s^2,1}$. Hence we have

$$s^2\,|P_1|\,|P_2| \sim 1, \tag{6.15}$$

$$(|P_1|, |P_2|)C_p(P_1)C_p(P_2) = 1. \tag{6.16}$$

We can easily verify that (6.15) is always satisfied and (6.16) reduces to

$$(6.17) \qquad (-1, s)^{s\{2i+i(s+i)+(s+1-i)(2s+1-i)\}/2} = 1.$$

The index of $(-1, s)$ in equation (6.17) will be odd if and only if $s \equiv 1$ or 2 (mod 4). Hence if $s \equiv 1$ or 2 (mod 4) and the square-free part of s contains a prime congruent to 3 (mod 4), then the left-hand side of (6.17) has the value -1, which is a contradiction.

7. Examples of non-existent designs and concluding remarks.

We give below the parameters of non-existing PBIB designs in the form of three tables.

TABLE I

PARAMETERS OF NON-EXISTENT SEMI-REGULAR GD DESIGNS

Sr. No.	v	m	n	b	r	k	λ_1	λ_2	Remarks
1	462	22	21	441	105	110	21	25	Th. 4.1, i(a)
2	506	23	22	484	154	161	44	49	Th. 4.1, ii
3	1190	35	34	1156	408	420	136	144	Th. 4.1, ii
4	3306	58	57	3249	456	464	57	64	Th. 4.1, i(a)

TABLE II

PARAMETERS OF NON-EXISTENT TRIANGULAR DESIGNS

Sr. No.	v	n	b	r	k	λ_1	λ_2	Remarks
1	91	14	14	6	39	4	2	Th. 5.2, i
2	105	15	91	39	45	15	17	Th. 5.1, iv
3	153	18	136	40	45	10	12	Th. 5.1, iii
4	253	23	231	105	115	45	48	Th. 5.1, iv
5	325	26	26	8	100	5	2	Th. 5.2, i
6	703	38	666	126	133	21	24	Th. 5.1, iii
7	990	45	946	301	315	91	96	Th. 5.1, ii

TABLE III

PARAMETERS OF NON-EXISTENT L_i DESIGNS

Sr. No.	v	s	i	b	r	k	λ_1	λ_2	Remarks
1	36	6	6	6	1	6	0	1	Th. 6.1, ii
2	196	14	14	14	1	14	0	1	Th. 6.1, ii
3	441	21	21	21	1	21	0	1	Th. 6.1, iv

Adjoining some rows or columns to N, to make N a square and non-singular matrix, and studying the combinatorial properties of the designs is not a new technique. However, adjoining N with the characteristic vectors corresponding

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to the zero root of NN' provides a simple method of deriving necessary conditions for the existence of a particular class of designs. The methods developed in this paper, the authors feel, may be utilized in deriving necessary conditions for the existence of various designs. For example, the methods of this paper may conveniently be used in deriving necessary conditions for the existence of a class of PBIB designs with rectangular type of association scheme defined by Vartak **(20)**.

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