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A GENERALIZATION OF SUPERSOLVABILITY

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Abstract. In this paper we consider a generalization of supersolvability called groups of polycyclic breadth n for $n \geq 1$, we see that a number of well known results for supersolvable groups generalize to groups of polycyclic breadth n. This generalization of supersolvability is especially strong for the groups of polycyclic breadth 2.

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1. Introduction.

DEFINITION 1.1. A group G is of polycyclic breadth n (PBn-group) if it has a normal series whose factors are all abelian groups with no more than n generators [**7**, page 57].

Note that PBn-groups are polycyclic groups and that every polycyclic group is a PBn-group for some n, in particular PB1-groups are the same as supersolvable groups. Finite groups of polycyclic breadth n, are also known in the literature as solvable groups of rank n [**2**]. We will show that some well known results about supersolvable such as [**5**],

THEOREM 1.2. *The elements of odd order form a finite characteristic subgroup in a supersolvable group,*

and

THEOREM 1.3. *The derived group of a supersolvable group is nilpotent,*

have nice natural generalizations for polycyclic groups of breadth n. We see that if G is a PBn-group, then $G^{(n+3)}$ is nilpotent, so the derived length of a polycyclic group quotient by its Fitting subgroup is bounded by a function of of its breath. We prove that the torsion elements of $6'$ order in a PB2-group form a finite characteristic subgroup.

2. Breadth of a polycyclic group.

In this section we will look at general results about PBn-groups.

DEFINITION 2.1. Given a polycyclic group G let $B(G) = n$ if G is PBn-group, but not a PBk-group for $k < n$.

DEFINITION 2.2. Let $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where the p_i's are distinct primes, then set

$$\alpha(n) = \max_{1 \leq i \leq k} \{\alpha_i\}.$$

The first three lemmas we introduce have very standard proofs and are thus omitted.

LEMMA 2.3. *Let G be a finite solvable group of order n, then it is a* $PB\alpha(n)$*-group.*

LEMMA 2.4. *Let G be a PBn-group. Then a principal factor of G is an elementary abelian p-group of* $B(G) \leq n$ *generators (note that equality is achieved at least once).*

LEMMA 2.5. *The class of PBn-groups is closed with respect to forming subgroups, images, and finite direct products.*

LEMMA 2.6. *If G is a solvable subgroup of* $GL_n(\Delta)$ *where* Δ *is an arbitrary field or the integers, then G has derived length* $\rho(n) \leq n + 3$.

Proof. See [**1**, **4**]. □

REMARK 2.7. The Fitting subgroup of a polycyclic group is nilpotent [**7**].

REMARK 2.8. We will denote the Fitting subgroup by $\mathcal{F}$.

THEOREM 2.9. *If G is a PBn-group, then* $G/\mathcal{F}$ *has derived length less than or equal to* $\rho(n)$. *In particular* $G^{(\rho(n))}$ *is nilpotent.*

Proof. Let

$$1 = G_0 \leq G_1 \cdots \leq G_k = G$$

be a normal series of G whose factors are abelian groups with less than or equal to n generators. Set $F_i = G_i/G_{i-1}$ and

$$C = \bigcap_{1 \leq i \leq k} C_G(F_i).$$

Now since every solvable subgroup of $Aut(F_i)$ is has derived length $\leq \rho(n)$. Thus so is G/C. Moreover C is nilpotent; for it has the central series $(C \cap G_i)_{i=0,\ldots,k}$. □

COROLLARY 2.10. *If G is a supersolvable group, then* $G/\mathcal{F}$ *is abelian. In particular* G' *is nilpotent.*

Proof. Since, $\rho(1) = 1$ by Theorem 2.9 $G/\mathcal{F}$ has derived length 1. □

COROLLARY 2.11. *If G is a PB*2*-group, then* $G/\mathcal{F}$ *has derived length less than or equal to* 4.

Proof. Since, $\rho(2) = 4$ by Theorem 2.9 $G/\mathcal{F}$ has derived length less than or equal to 4. □

COROLLARY 2.12. *If G is a polycyclic group, then* $G^{\rho(B(G))}$ *is nilpotent.*

REMARK 2.13. From Corollary 2.12 we get that derived length of a polycyclic group quotient by its Fitting subgroup is bounded by a function of $B(G)$.

DEFINITION 2.14. Given a group G, let

$$P(G) = \{\text{the set of all primes } p \text{ such that } p = |g| \text{ for some } g \in G\}.$$

REMARK 2.15. If $P(G) = \emptyset$, then G is trivial or torsion free.

REMARK 2.16. If G is a polycyclic group, then $P(G)$ is a finite set.

DEFINITION 2.17. An element of a group G is called an n'-element if it is a torsion element and its order is relatively prime to n. A group G is called an n'-group if every $p \in P(G)$ is relatively prime to n (i.e. if every torsion element in G is an n'-element).

DEFINITION 2.18. For any integer $n > 0$:

$$g(n) = \prod_{\{\text{prime } p \leq n+1\}} p.$$

LEMMA 2.19. *If $A \in GL_n(\mathbb{Z})$ is a nontrivial torsion element, then the greatest common divisor of $|A|$ and $g(n)$ is greater than* 1.

Proof. See [**3**]. □

THEOREM 2.20. *If G is a PBn-group, then the $g(n)'$-elements generate a finite subgroup of G.*

Proof. Let $H = \langle X \rangle$, where X is the set of $g(n)'$-elements in G. By [**6**, 5.4.15] there exists a torsion-free normal subgroup L of finite index in H. We first claim that $L \leq Z(H)$. To establish our claim, we will use induction on the derived length of L. If $L^{(k)} \neq 1$ is abelian, then given $a \in L^{(k)}$ let $K = \langle a \rangle^H$, since $r(K) \leq n$, by Lemma 2.19 any $x \in X$ acts trivially on K, so $L^{(k)} \leq Z(H)$. Assume that L' is abelian but L is not. By induction $[x, a] \in L' \leq Z(H)$. So by [**6**, Exercise 5.1.4] $[x, a]^{|x|} = [x^{|x|}, a] = 1$, and since L' is torsion-free we get that $[x, a] = 1$. Thus $L \leq Z(H)$ and H is central-by-finite. By [**5**, Theorem 4.12], $|H'| < \infty$, thus H is finite. □

COROLLARY 2.21. *The torsion elements in a $g(n)'$-PBn-group form a finite characteristic subgroup.*

COROLLARY 2.22. *The elements of odd order in a $2'$-supersolvable group form a finite characteristic subgroup.*

3. Polycyclic breadth 2.

In this section we will look at some results about PB2-groups, this results are very similar to the supersolvable case.

THEOREM 3.1. *If G is a finite PB2-group of odd order, then there is a normal series*

$$1 = G_0 \leq G_1 \cdots \leq G_k = G$$

with the factors of descending prime exponent.

Proof. [**2**, Satz VI.9.1.d] and induction. □

THEOREM 3.2. *If G is a PB2-group, then there is a normal series*

$$1 = G_0 \leq G_1 \cdots \leq G_k = G$$

with finite elementary abelian $6'$ factors of descending prime exponent, followed by free abelian factors, followed by factors of exponents 2 *and* 3.

Proof. Since G is a PB2-group, we obtain a normal series

$$1 = H_0 \leq H_1 \cdots \leq H_m = G$$

whose factors are elementary abelian p-groups or free abelian groups of rank at most 2.

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Suppose that H_{i+1}/H_i is an elementary abelian p-group, where $p > 3$ and H_i/H_{i-1} is free abelian. Then $Aut(H_i/H_{i-1})$ is isomorphic to a subgroup of $GL_n(\mathbb{Z})$. If H_{i+1}/H_{i-1} is free abelian, delete H_i. Otherwise there is an elementary abelian p-subgroup $\overline{H}_i/H_{i-1}$ of H_{i+1}/H_{i-1}. By Lemma 2.19 this subgroup acts trivially on H_i/H_{i-1}. So H_{i+1}/H_{i-1} is abelian and $H_{i+1}/\overline{H}_i$ is free abelian. So, we can replace H_i by $\overline{H}_i$.

Suppose that H_{i+1}/H_i is an elementary abelian p-group, where $p > 3$, and H_i/H_{i-1} is an elementary abelian q-group where $q = 2$ or 3. Then $|Aut(H_i/H_{i-1})|$ divides 48, which is not divisible by p. Let $\overline{H}_i/H_{i-1}$ be an elementary abelian p-subgroup of H_{i+1}/H_{i-1}. This subgroup acts trivially on H_i/H_{i-1}. Hence, we see that H_{i+1}/H_{i-1} is abelian, and $\overline{H}_i \lhd G$; also $H_{i+1}/\overline{H}_i$ is an elementary abelian q-group. So, we can replace H_i by $\overline{H}_i$.

Suppose that H_{i+1}/H_i is free abelian and H_i/H_{i-1} is an elementary abelian p-group where $p = 2$ or 3. Since $|\ Aut(H_i/H_{i-1})\ |$ divides 48, we may replace H_{i+1} by $\overline{H}_{i+1}$ where $\overline{H}_{i+1}/H_i = (H_{i+1}/H_i)^{48}$, if needed, and assume that H_{i+1}/H_i acts trivially on H_i/H_{i-1} forcing H_{i+1}/H_{i-1} to be abelian. Let $\overline{H}_i \leq G$ be such that $\overline{H}_i/H_{i-1} = (H_{i+1}/H_{i-1})^p$. It follows that $\overline{H}_i/H_{i-1}$ is an infinite cyclic group. Also $\overline{H}_i \lhd G$ and $[H_{i+1} : \overline{H}_i] = k$ which divides p^4. So there exists $\widehat{H}_i \lhd G$ such that

$$[H_{i+1} : \widehat{H}_i] \mid p^2 \text{ and } [\widehat{H}_i : \overline{H}_i] \mid p^2$$

Delete H_i and insert $\widehat{H}_i$ and $\overline{H}_i$. Thus we move the infinite factors to the left.

Applying Theorem 3.1 completes the proof. □

REMARK 3.3. A_4 and S_3 are PB2-groups, A_4 has a normal series of a factor of exponent 2 followed by one of exponent 3, while S_3 has a normal series of a factor of exponent 3 followed by one of exponent 2.

COROLLARY 3.4. *The elements of* $6'$ *order in a PB2-group form a finite characteristic subgroup.*

REMARK 3.5. In the supersolvable case the elements of odd order form a finite characteristic subgroup [**5**].

COROLLARY 3.6. *If G is a PB2-group and*

$$p_{\max} = \max\{\text{prime } p \in P(G)\} > 3,$$

then G has a charecteristic subgroup H of exponent $p_{\max}$.

Proof. By Theorem 3.2 G has a normal subgroup G_1 of exponent $p_{\max}$, let

$$H = \Pi_{f \in \mathrm{aut}(G)} f(G_1).$$

□

REMARK 3.7. If G is a supersolvable group and

$$p_{\max} > 2,$$

then G has a charecteristic subgroup H of exponent $p_{\max}$.

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