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Chunikhin's existence theorem for subgroups of a finite group

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We give a simplified proof of a general theorem of Chunikhin on existence of subgroups of a finite group. The proof avoids the technical device of "indexials" which Chunikhin set up for this purpose.

1. Introduction

In [2] (and later in [1], pp. 79-100), Chunikhin proves a very general theorem which asserts, for any finite group and any normal series of that group, the existence of a subgroup having a certain relationship with the terms of the normal series. It includes as special cases the existence of a Hall π-subgroup in a π-soluble group and the existence of subgroups of all possible π-orders in a π-supersoluble group. In this paper we give a much more direct proof than the one in [1], avoiding the elaborate machinery of "indexials" which Chunikhin sets up.

Throughout the paper, all groups are assumed to be finite.

THEOREM 1 (Chunikhin [1], pp. 79-100). *Suppose that the group* G *has a series* $1 = G_0 \leq G_1 \leq \ldots \leq G_{2n} = G$ *such that if* $1 \leq i \leq n-1$, $G_{2i} \trianglelefteq G$ *and* $G = G_{2i}N_G(G_{2i-1})$. *If* $0 \leq i \leq n-1$, *let* θ_i *be the set of primes which divide* $|G_{2j+1}/G_{2j}|$ *for some* $j \in \{i, \ldots, n-1\}$. *Then there exists a subgroup* H *of* G *such that if* $H_i = H \cap G_i$ *for* $0 \leq i \leq 2n$,

(1) $G_{2i+1} = H_{2i+1}G_{2i}$ *for* $0 \leq i \leq n-1$,

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(2) $H_i \trianglelefteq H$ *for* $0 \le i \le 2n$,

(3) H_{2i}/H_{2i-1} *is a nilpotent* θ_i*-group for* $1 \le i \le n$,

(4) $H_{2i+1}/H_{2i} \cong G_{2i+1}/G_{2i}$ *for* $0 \le i \le n-1$,

(5) $|H_{2i}/H_{2i-1}|$ *divides* $|G_{2i} : G_{2i-1}|$ *for* $1 \le i \le n$,

(6) H *is a* θ_0*-group.*

2. Some definitions

If π is any set of primes, π' denotes the complement of π in the set of all primes. A π-*number* is an integer whose only prime divisors are elements of π , and a π-*group* is a group whose order is a π-number. A *Hall* π-*subgroup*, H , of a group G is a π-subgroup of G whose index in G is a π'-number.

A group G is π-*supersoluble* if each chief factor is either a cyclic group of order p for some $p \in \pi$, or a π'-group. G is π-*soluble* if each composition factor (chief factor) is either a p-group for some $p \in \pi$ or a π'-group. G is π-*separable* if each composition factor (chief factor) is a π-group or a π'-group. G is π-*decomposable* if the order of each composition factor (chief factor) is divisible by at most one prime from π . G is π-*partible* if the order of each composition factor (chief factor) is a π-number, or is divisible by at most one prime from π .

We are here following Gorenstein [3] in the use of the term "π-separable". Chunikhin uses "π-separable" to refer to what we call π-decomposable. The fact that the above definitions, with the exception of π-supersolubility, can be stated in terms of composition factors or chief factors follows from the fact that every chief factor is a direct product of isomorphic copies of some composition factor.

A π-supersoluble group is clearly π-soluble. A group is π-soluble if and only if it is both π-separable and π-decomposable. π-separability and π-decomposability each imply π-partibility. Subgroups and factor groups of π-supersoluble, π-soluble, π-separable, π-decomposable and π-partible groups are respectively π-supersoluble, π-soluble, π-separable, π-decomposable and π-partible. π-supersoluble and

π-soluble groups are respectively, π_1-supersoluble and π_1-soluble for all $\pi_1 \subseteq \pi$. A π-separable group is π'-separable.

3. Preliminary lemmas

LEMMA 1. *Suppose* $B \le A \le G$ *and* $C \le G$ *such that* B *permutes with* C . *Then*

(a) $A \cap BC = B(A \cap C)$,

(b) $|A \cap C|/|B \cap C|$ *divides* $|A|/|B|$,

(c) if $A \le BC$ *and* $B \trianglelefteq A$ *then* $A \cap C/B \cap C \cong A/B$.

Proof. *(a)* is the modularity law ([4], p. 124).

(b). $|B(A \cap C)| = |B|.|A \cap C|/|B \cap C|$ whence $|A \cap C|/|B \cap C| = |B(A \cap C)|/|B|$ which divides $|A|/|B|$.

(c). $A = A \cap BC = B(A \cap C)$ by *(a)* whence $A/B \cong A \cap C/(A \cap C) \cap B = A \cap C/B \cap C$.

LEMMA 2. (Schur-Zassenhaus [4], p. 224). *If* H *is a normal Hall* π*-subgroup of* G , G *contains a Hall* π'*-subgroup.*

LEMMA 3. *Conclusions* (4) *to* (6) *of Theorem* 1 *are consequences of* (1) *to* (3).

Proof. (4) follows from (1) by the second isomorphism theorem.

(5). Suppose $1 \le i \le n$. Then

$$G_{2i-1}H = H_{2i-1}G_{2i-2}H \text{ by (1),}$$
$$= HG_{2i-2} ,$$

since $G_{2i-2} \trianglelefteq G$. Similarly $HG_{2i-1} = HG_{2i-2}$ and so G_{2i-1} is permutable with H . (5) now follows from Lemma 1 *(b)* on putting $A = G_{2i}$, $B = G_{2i-1}$, $C = H$.

(6). If $0 \le i \le n-1$, H_{2i+1}/H_{2i} is a θ_i-group by (4), and if $1 \le i \le n$, H_{2i}/H_{2i-1} is a θ_i-group by (3).

Since $\theta_0 \supseteq \theta_1 \supseteq \dots \supseteq \theta_n$, H is a θ_0-group.

LEMMA 4. *If* $1 = G_0 \le G_1 \le \dots \le G_{2n} = G$ *is a chain of subgroups of*

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G such that if $1 \le i \le n$, $G_{2i} \trianglelefteq G$ and $G = G_{2i}N_G(G_{2i-1})$ and if S is a subgroup of G such that $G_1 \le S \le N_G(G_1)$ and $G = G_2S$, then putting $S_i = S \cap G_i$,

(a) $G_i = G_2S_i$ for $2 \le i \le 2n$, and

(b) $S = S_{2i}N_S(S_{2i-1})$ and $S_{2i} \trianglelefteq S$ for $1 \le i \le n$.

Proof. *(a).* Suppose $i \ge 2$. Then

$$\begin{aligned} G_i = G_i \cap G_2S &= G_2(G_i \cap S) \quad \text{by Lemma 1 } (a), \\ &= G_2S_i . \end{aligned}$$

(b). If $i \ge 2$,

$$\begin{aligned} S = S \cap G = S \cap G_{2i}N_G(G_{2i-1}) &, \\ &= S \cap G_2S_{2i}N_G(G_{2i-1}) \quad \text{by } (a), \\ &= S \cap S_{2i}N_G(G_{2i-1}) \quad \text{since } G_2 \le G_{2i-1} \le N_G(G_{2i-1}) , \\ &= S_{2i}\left(S \cap N_G(G_{2i-1})\right) \quad \text{by Lemma 1 } (a), \\ &= S_{2i}N_S(G_{2i-1}) \le S_{2i}N_S(S_{2i-1}) . \end{aligned}$$

But $S_{2i}N_S(S_{2i-1}) \le S$ and so $S = S_{2i}N_S(S_{2i-1})$.

If $i = 1$, $S_{2i-1} = S_1 = G_1 \trianglelefteq S$ whence $N_S(S_{2i-1}) = S$.

4. Proof of Theorem 1

We suppose that the theorem is false and throughout this section G is assumed to be a minimal counter-example. We suppose further that the theorem fails for G in respect of the chain $1 = G_0 \le G_1 \le \ldots \le G_{2n} = G$, $(n \ge 1)$ but holds for every shorter chain. For $0 \le i \le n-1$, θ_i denotes the set of primes which divide some $|G_{2j+1}/G_{2j}|$ for $j \ge i$.

LEMMA 5. *If $G_1 \le S \le N_G(G_1)$ and $G = G_2S$ then $S = G$.*

Proof. Suppose $S < G$. It follows from Lemma 4 *(b)* and the fact that G is a minimal counter-example that there exist $H \le S$ such that if

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for $0 \le i \le n-1$, ϕ_i denotes the set of primes which divide some $|S_{2j+1}/S_{2j}|$ for $j \ge i$, then

(i) $S_{2i+1} = H_{2i+1}S_{2i}$ for $0 \le i \le n-1$,

(ii) $H_i \trianglelefteq H$ for $0 \le i \le 2n$,

(iii) H_{2i}/H_{2i-1} is a nilpotent ϕ_i-group for $1 \le i \le n$,

where H_i is defined to be $H \cap S_i$ for $0 \le i \le 2n$.

Now if $i \ge 1$,

$$\begin{aligned} G_{2i+1} &= G_2S_{2i+1} \quad \text{by Lemma 4 } (a) \\ &= G_2H_{2i+1}S_{2i} \quad \text{by (i)} \\ &= G_2S_{2i}H_{2i+1} \\ &= G_{2i}H_{2i+1} \quad \text{by Lemma 4 } (a). \end{aligned}$$

If $i = 0$,

$$\begin{aligned} G_{2i+1} = G_1 = S_1 &= H_1S_0 \quad \text{by (i)}, \\ &= H_1 . \end{aligned}$$

Thus (1) holds for G.

If $0 \le j \le n-1$ it follows from Lemma 1 *(b)* that $|S_{2j+1}/S_{2j}|$ divides $|G_{2j+1}/G_{2j}|$ and so $\phi_i \subseteq \theta_i$ for $0 \le i \le n-1$. Finally, $H \cap S_i = H \cap G_i$ for $0 \le i \le 2n$ and so from (ii), (iii) it follows that (2) and (3) hold for G. Hence by Lemma 3, the theorem holds for G, a contradiction. Hence $S = G$.

LEMMA 6. *If* $G_1 \trianglelefteq G$ *then* $G_1 = 1$.

Proof. Suppose that $G_1 \ne 1$. Using the symbol "$\overline{\ \ }$" to denote images of subgroups of G in G/G_1, we have by the assumptions on the G_i, $\overline{G}_{2i} \trianglelefteq \overline{G}$ and $\overline{G} = \overline{G}_{2i}\overline{N_G(G_{2i-1})} = \overline{G}_{2i}N_{\overline{G}}(\overline{G}_{2i-1})$. Hence by the minimality of G, there is a subgroup H of G such that $G_1 \le H$ and such that if for $0 \le i \le n-1$, α_i denotes the set of primes which divide

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some $|\overline{G}_{2j+1}/\overline{G}_{2j}|$ for $j \geq i$,

(i) $\overline{G}_{2i+1} = \overline{H}_{2i+1}\overline{G}_{2i}$ for $0 \leq i \leq n-1$,

(ii) $\overline{H}_i \trianglelefteq \overline{H}$ for $0 \leq i \leq 2n$,

(iii) $\overline{H}_{2i}/\overline{H}_{2i-1}$ is a nilpotent α_i-group for $1 \leq i \leq n$,

where H_i is defined to be $H \cap G_i$ for $0 \leq i \leq 2n$.

If $i \geq 1$, it follows from (i) that $G_{2i+1} = H_{2i+1}G_{2i}$. Moreover $G_1 \leq H$ and so $G_1 = H_1 = H_1G_0$. Thus (1) holds for G . From (ii), $H_i \trianglelefteq H$ for $i \geq 1$, and clearly $H_0 \leq H$. Hence (2) holds for G . If $j \geq 1$, $\overline{G}_{2j+1}/\overline{G}_{2j} \cong G_{2j+1}/G_{2j}$. Moreover $\overline{G}_1/\overline{G}_0$ is trivial. Hence if $0 \leq i \leq n-1$, $\alpha_i \subseteq \theta_i$. If $i \geq 1$, $\overline{H}_{2i}/\overline{H}_{2i-1} \cong H_{2i}/H_{2i-1}$ and so by (iii), H_{2i}/H_{2i-1} is a nilpotent θ_i-group and so (3) holds for G . Thus, by Lemma 3, the theorem holds for G , a contradiction. Hence $G_1 = 1$.

LEMMA 7. *If* $G_1 = 1$ *and* G_2 *is nilpotent then* $G_2 = 1$.

Proof. Suppose that $G_2 \neq 1$. Using the symbol "$\overline{}$" to denote images of subgroups of G in G/G_2 , then by the assumptions on the G_i , $\overline{G}_{2i} \trianglelefteq \overline{G}$ and $\overline{G} = \overline{G_{2i}N_G(G_{2i-1})} = \overline{G}_{2i}N_{\overline{G}}(\overline{G}_{2i-1})$. Hence by the minimality of G , there is a subgroup K of G such that $G_2 \leq K$ and such that if for $0 \leq i \leq n-1$, α_i denotes the set of primes which divide some $|\overline{G}_{2j+1}/\overline{G}_{2j}|$ for $j \geq i$,

(i) $\overline{G}_{2i+1} = \overline{K}_{2i+1}\overline{G}_{2i}$ for $0 \leq i \leq n-1$,

(ii) $\overline{K}_i \trianglelefteq \overline{K}$ for $0 \leq i \leq 2n$,

(iii) $\overline{K}_{2i}/\overline{K}_{2i-1}$ is a nilpotent α_i-group for $1 \leq i \leq n$,

(iv) $\overline{K}$ is an α_0-group,

where K_i is defined to be $K \cap G_i$ for $0 \leq i \leq 2n$.

If $j \geq 1$, $\overline{G}_{2j+1}/\overline{G}_{2j} \cong G_{2j+1}/G_{2j}$, and so $\alpha_j = \theta_j$. Since $\overline{G}_1/\overline{G}_0$ is trivial, $\alpha_0 = \alpha_1 = \theta_1$. Thus by (iv), K/G_2 is a θ_1-group.

Since G_2 is nilpotent it contains a unique Hall θ_1'-subgroup, M . M is characteristic in G_2 and hence normal in G . G_2/M is a θ_1-group and so K/M is a θ_1-group. By Lemma 2, there exists a Hall θ_1-subgroup H of K . Thus $K = MH$ and $M \cap H = 1$.

If $i \geq 2$, then since $M \leq G_2$, we have by Lemma 1 *(a)* that $K_i = K \cap G_i = MH \cap G_i = M(H \cap G_i) = MH_i$ where H_i is defined to be $H \cap G_i$ for $0 \leq i \leq 2n$. If $i \geq 1$ we have from (i) that $G_{2i+1} = K_{2i+1}G_{2i} = MH_{2i+1}G_{2i} = H_{2i+1}G_{2i}$. Moreover G_1 and H_1 are trivial, so $G_1 = H_1G_0$. Thus (1) holds for G .

If $i \geq 2$, it follows from (ii) that $K_i \trianglelefteq K$ and so $MH_i \trianglelefteq MH$. Hence

$$H_i^H \leq MH_i \cap H = H_i(M \cap H) \text{ by Lemma 1 } (a),$$
$$= H_i .$$

Thus $H_i \trianglelefteq H$. Moreover $H_2 = G_2 \trianglelefteq H$ and $H_1 = G_1 = 1 \trianglelefteq H$. Thus (2) holds for G .

If $i \geq 2$, $\overline{K}_{2i}/\overline{K}_{2i-1} \cong K_{2i}/K_{2i-1} = MH_{2i}/MH_{2i-1} \cong H_{2i}/H_{2i-1}$ and so by (iii), H_{2i}/H_{2i-1} is a nilpotent α_i-group and so a nilpotent θ_i-group. Since $H_1 = 1$, $H_2/H_1 \cong H_2$ and is nilpotent since G_2 is nilpotent. Since K/M is a θ_1-group, so is H_2 . Hence (3) holds for G and so by Lemma 3, the theorem holds for G , a contradiction. Hence $G_2 = 1$.

Proof of Theorem 1. We obtain our ultimate contradiction through an interplay of Lemmas 5, 6 and 7. Taking $S = N_G(G_1)$ in Lemma 5 we conclude that $G_1 \trianglelefteq G$. Hence by Lemma 6, $G_1 = 1$. Thus if S is any subgroup of G such that $G = G_2S$, then by Lemma 5, $S = G$. Hence G_2

is contained in the Frattini subgroup of G, whence it is nilpotent. By Lemma 7, $G_2 = 1$. Since the theorem holds for G in respect of the shorter chain $1 = G_2 \leq G_3 \leq \dots \leq G_{2n} = G$, it must hold in respect of the original chain, a contradiction.

5. Consequences of Theorem 1

THEOREM 2. *Suppose that*

$$1 = G_0 < G_2 < G_4 < \dots < G_{2n} = G$$

is a normal series for G. *If for* $1 \leq i \leq n-1$, *the factor* G_{2i}/G_{2i-2} *contains a single conjugacy class of Hall* π-*subgroups and if* G/G_{2n-2} *contains a Hall* π-*subgroup then* G *contains a Hall* π-*subgroup. If these Hall* π-*subgroups are soluble,* G *contains a soluble Hall* π-*subgroup.*

Proof. For $1 \leq i \leq n$, choose G_{2i-1} so that G_{2i-1}/G_{2i-2} is a Hall π-subgroup of G_{2i}/G_{2i-2}. If $1 \leq i \leq n-1$, all Hall π-subgroups of G_{2i}/G_{2i-2} are conjugate whence $G/G_{2i-2} = G_{2i}/G_{2i-2}$, $N_{G/G_{2i-2}}\left(G_{2i-1}/G_{2i-2}\right)$ and so $G = G_{2i}N_G\left(G_{2i-1}\right)$. By Theorem 1 there exists a subgroup H of G having properties (1) to (6).

$$\begin{aligned} |G : H| &= \prod_{i=1}^{n} \frac{|G_{2i-1}:G_{2i-2}|}{|H_{2i-1}:H_{2i-2}|} \times \prod_{i=1}^{n} \frac{|G_{2i}:G_{2i-1}|}{|H_{2i}:H_{2i-1}|} \\ &= \prod_{i=1}^{n} \frac{|G_{2i}:G_{2i-1}|}{|H_{2i}:H_{2i-1}|} \quad \text{by (4),} \end{aligned}$$

which divides $\prod_{i=1}^{n} |G_{2i}:G_{2i-1}|$ and so is a π'-number. Since for $0 \leq i \leq n-1$, G_{2i+1}/G_{2i} is a π-group, $\theta_i \subseteq \pi$ for all i. In particular $\theta_0 \subseteq \pi$. H is a θ_0-group by (6) and so a π-group. Hence it is a Hall π-subgroup of G.

If the Hall π-subgroups of the factors of G are soluble, G_{2i-1}/G_{2i-2}, and hence by (4) H_{2i-1}/H_{2i-2} is soluble for $1 \leq i \leq n$.

Since, by (3), H_{2i}/H_{2i-1} is nilpotent for $1 \le i \le n$, H is soluble.*

COROLLARY. *The theorem holds if* $1 = G_0 < G_2 < \ldots < G_{2n} = G$ *is a composition series.*

Proof. If A is a direct product of isomorphic copies of B , then A has a single conjugacy class of Hall π-subgroups if and only if B has. Since each chief factor of G is a direct product of isomorphic copies of some composition factor, the assumptions on the composition factors carry over to the chief factors.

THEOREM 3 ([*1*], Theorem 3.9.1). *If* G *is* π-*partible then it contains a Hall* π-*subgroup. If for some* $\pi_1 \subseteq \pi$, G *is* π_1-*decomposable and* $(\pi-\pi_1)$-*separable then it contains a* π_1-*soluble Hall* π-*subgroup.*

Proof. Let $1 = G_0 < G_2 < G_4 < \ldots < G_{2n} = G$ be a chief series for G . If ρ is the set of prime divisors for some chief factor then by the π-partibility of G ,

(i) $\rho \subseteq \pi$, or

(ii) $\rho \subseteq \pi'$, or

(iii) $\rho \cap \pi = \{p\}$ for some prime p .

In case (i) the factor is a π-group and so has a unique Hall π-subgroup (namely itself). In case (ii) the factor is a π'-group and so has a unique Hall π-subgroup (namely the trivial subgroup). In case (iii) the factor has a single conjugacy class of Hall π-subgroups (namely the Sylow p-groups). Hence by Theorem 2, there exists a Hall π-subgroup H of G satisfying (1) to (6) of Theorem 1.

Suppose that G is π_1-decomposable and $(\pi-\pi_1)$-separable. If ρ is the set of prime divisors of $|G_{2i}/G_{2i-2}|$, $\rho \subseteq \pi_1'$ or $\rho \cap \pi_1 = \{p\}$ by π_1-decomposability. If $\rho \cap \pi_1 = \{p\}$ then by $(\pi-\pi_1)$-separability, $\rho \cap (\pi-\pi_1) = \emptyset$ that is $\rho \cap \pi = \rho \cap \pi_1$. Let τ be the set of prime divisors of $|H_{2i-1}/H_{2i-2}|$. Then by (4), $\tau \subseteq \rho$ and by (6), $\tau \subseteq \theta_0 = \pi$. Hence $\tau \subseteq \rho \cap \pi$. Thus either $\tau \subseteq \rho \subseteq \pi'$ or

* In fact by (5) and (6), each H_{2i}/H_{2i-1} is trivial.

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$\tau \subseteq \rho \cap \pi = \rho \cap \pi_1 = \{p\}$. Finally for $1 \le i \le n$, H_{2i}/H_{2i-1} is, by (5), a π'-group.* Hence H is π_1-soluble.

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* In fact, since H is a π-group, H_{2i}/H_{2i-1} is trivial.

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