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ELEMENTARY ABELIAN CARTESIAN GROUPS CARTESIAN GROUPS

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Throughout the paper, G will denote an additively written, but not always abelian, group of finite order n; and $X = (x_{ij})$ will denote a square matrix of order n with entries from G and whose rows and columns are numbered $0, 1, \ldots, n-1$. We call X a *cartesian array (afforded by G)* if

(1.1) The sequence $\{-x_{mi} + x_{ki}, i = 0, \ldots, n-1\}$ contains all elements of G whenever $k \neq m$.

By a theorem of Jungnickel (see Theorem 2.2 in [**5**]), the transpose of a cartesian array is also a cartesian array. We call G a *cartesian group* if there is a cartesian array X afforded by G. In this case, we also call (G, X) a *cartesian pair*.

The existence of a cartesian group of order n is equivalent to the existence of a projective plane of order n with a (P, L)-transitivity. An r-rowed matrix X satisfying (1.1) is equivalent to the existence of certain nets (partial geometries). These equivalences are well known and appear in one form or another in the papers of Bose, Bruck [**1**], and most directly in Drake's paper [**2**] in the context of generalized Hadamard matrices.

It is the purpose of this paper to examine the question of which groups G afford a cartesian pair (G, X). The following theorem is the main result of this study.

THEOREM 3.1 *Assume (G, X) is a cartesian pair in which G is an abelian group of order n and exponent s. Assume further that X satisfies a homogeneous condition (H) (see* (2, 7)*). Then s is a prime number.*

Theorem (3.1) asserts that finite abelian cartesian groups satisfying condition (H) must be elementary abelian p-groups (vector spaces over F_p). Condition (H) requires that the associated ternary ring R satisfy $r(ab) = (ra)b$ for all elements a, b of R and for each integer r with $(r, s) = 1$. In Section 4 of the paper, abelian cartesian groups are constructed which satisfy Condition (H) of Theorem (3.1) but have only a single (P, L)-transtivity. Condition (H) is shown (see (4.1)) to be equivalent to the geometrical assertion that the associated plane admits $\varphi(s)$ homologies (φ denotes Euler's function) with axis L and center $(0, 0)$. No cartesian groups are known to the author which do not satisfy Condition (H).

The techniques of the paper employ the action of the galois group of a splitting field K for G on the irreducible characters of G. Assuming condition (H), it is

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shown Gal(K/Q) must permute the nonidentity characters of G in a fixed point free manner and it follows that G is elementary abelian.

Although infinite nonabelian cartesian groups have been constructed, it is open as to whether finite cartesian groups must be abelian. In all known examples finite cartesian groups are elementary abelian.

2. Cartesian groups. The interest in cartesian groups stems from the following well known lemma (see, e.g., Theorem 6.2, [**4**]). We sketch a proof below because our definition of cartesian group is not the standard definition and because we wish to use the construction in the sequel.

(2.1) *The existence of a cartesian pair* (1.1) *is equivalent to the existence of a projective plane of order n with a* (P, L)*-transitivity.*

Proof. Let (G, X) be a cartesian pair with $G = \{g_0 = 0, g_1, \ldots, g_{n-1}\}$. Construct an affine plane on the point set $G \times G$ by defining lines,

$$(2.2) \qquad L_{mb} = \{(g_i, x_{mi} + g_b | i = 0, \ldots, n-1\}, \text{for } m, b = 0 \ldots, n-1.$$

$$(2.3) \qquad L_c = \{(g_c, g_i)| i = 0, \ldots, n-1\}, \text{ for } c = 0, \ldots, n-1.$$

Each line (2.2), (2.3) is constructed with exactly n points. A point (g_i, g_j) belongs to $L_{mb} \cap L_{kc}$ if and only if $-x_{mi} + x_{ki} = g_b - g_c$. By (1.1) there is a unique solution whenever $m \neq k$. It is easy to check that the family of lines (2.2) with the same first subscript m form a parallel class of n lines which we naturally call the class of slope (m). The lines (2.3) are a parallel class labeled $x = c$. One completes this affine plane to a projective plane of order n by the addition of an infinite line L_∞ whose $n + 1$ points are labeled (m) for $m = 0, \ldots, n-1$ with infinite point (∞). The point (m) is assigned to each L_{mb} and each L_c is augmented to include (∞). Define α_c to fix each point of L_∞ and to act on the affine points by the rule:

$$\alpha_c(g_i, g_j) = (g_i, g_j + g_c).$$

One checks to see $\alpha(L_{mb}) = L_{md}$ where $g_b + g_c = g_d$. The α_c's generate a group E of elations isomorphic to G with each having axis L_∞ and center (∞). For each c the line $x = c$ is fixed setwise with E acting transitively on the n points different from (∞). The converse (2.1) is obtained by retracing the construction just given.

(2.4) *Definition.* A cartesian array (x_{ij}) is said to be *normalized* if $x_{oi} = x_{io}$ for all i.

It is easy to prove that every abelian cartesian group affords a normalized cartesian array. In [**3**] Hayden proves for any cartesian group a coordinatization of the plane (2.1) may be chosen so that the array X is normalized.

(2.5) *Definition.* If $X = (x_{ij})$ is a normalized cartesian array of order n, define maps $\alpha_m, i = 1, \ldots, n-1$ by the rule,

$$\alpha_m(x_{1i}) = x_{mi}, \quad i = 1, \ldots, n-1. \tag{2.6}$$

Call the α_m's the *operators (affording the array X).*

Clearly, the α_m's are permutations, α_1 is the identity map, and $\alpha_m(0) = 0$ for every m. In most known arrays these operators are automorphisms of the vector space G and the resulting geometry is a translation plane. In Section 4 we construct cartesian pairs (G, X) in which G is a vector space but the operators are not automorphisms of G. In particular the plane associated with (G, X) has only one (P, L)-transitivity and the operators satisfy the following definition.

(2.7) *Homogeneity Condition* (H). Let X be a cartesian array afforded by a cartesian group G of exponent s. We say that the array X and the pair (G, X) are *homogeneous* if X is normalized and if every operator α_m of X satisfies

$$\alpha_m(rg) = r(\alpha_m(g)) \tag{2.8}$$

for every g in G and every integer r with $(r, s) = 1$. A cartesian group is said to be *homogeneous* if it affords a homogeneous cartesian array.

3. Homogeneous Cartesian groups. Throughout this section (G, X) is a cartesian pair satisfying the homogeneity condition (2.7). We examine the case in which G is abelian and are able to prove the following main result.

THEOREM 3.1 *Every abelian homogeneous cartesian group is elementary abelian.*

We now begin the proof of (3.1) with G an abelian group of order n and exponent s. Choose ϵ to be a primitive s^{th} complex root of unity and let $K = Q(\epsilon)$ be the filed obtained by adjoining ϵ to the rational field Q. Take $\{\Phi_0 = 1, \Phi_1, \ldots, \Phi_{n-1}\}$ to be the n distinct homomorphisms of G into the multiplicative group $\langle \epsilon \rangle$. These maps are called the irreducible linear charaters of G and satisfy the orthogonality relation

$$(\Phi_i, \Phi_j) = n^{-1} \sum_{g \epsilon G} \Phi_i(g) \Phi_j(-g) = 0 \quad \text{for } i \neq j. \tag{3.2}$$

For each element θ in the galois group $\mathrm{Gal}(K/Q)$, $\theta(\epsilon) = \epsilon^r$ for some positive integer r relatively prime to s. Clearly θ induces an automorphism of G by the rule $\theta : g \longrightarrow rg$ and permutes the nonidentity characters by the rule $\theta : \Phi \longrightarrow \Phi^r$ where

$$\Phi^r(g) = \Phi(rg) = (\Phi(g))^r = \Phi(\theta(g)).$$

The first observation to be made is that a nonidentity character may be applied to each entry of a normalized cartesian array with the resulting matrix a complex orthogonal matrix.

(3.3) *Let* $X = (x_{ij})$ *be a normalized cartesian array for* G *and let* Φ *be any nonidentity character. Then* $\Phi(X) = (\Phi(x_{ij}))$ *is an orthogonal matrix of order* n *whose first row and column is the vector with each coordinate one.*

Proof. For $m \neq k$, the (m,k) entry of $\Phi(X)\Phi(X)^*$ is

$$\sum_{i=0}^{y-1} \Phi(x_{mi})\Phi(-x_{ki}) = \sum_{i=0}^{n-1} \Phi(x_{mi} - x_{ki}) = \sum_{g \in G} \Phi(g) = (\Phi_0, \Phi) = 0.$$

COROLLARY (3.4) *Assume* G, Φ *and the normalized array* X *are given as in* (3.3). *For* $m = 0, \dots, n-1$ *define* v_m *and* $\Phi(v_m)$,

$$\text{(3.5)} \qquad \begin{aligned} v_m &= (x_{m0}, x_{m1}, \dots, x_{mn-1}), \\ \Phi(v_m) &= (\Phi(x_{m0}), \Phi(x_{m1}), \dots, \Phi(x_{mn-1})). \end{aligned}$$

Then the vectors $\{\Phi(v_m) | m = 0, \dots, n-1\}$ *form an orthogonal basis for the vector space* $V_n(K)$.

The main lemma of the paper can now be stated.

(3.6) *Assume* G *is an abelian group of order* n *and exponent* $s, K = Q(\epsilon), \epsilon$ *a primitive* s^{th} *root of unity. If* G *affords a normalized cartesian array* X *which satisfies the homogeneous property* (H), *then* $\mathrm{Gal}(K/Q)$ *permutes the nonidentity characters of* G *in a semiregular manner.*

Proof. Let Φ be a nonidentity character of G and choose an element θ of $\mathrm{Gal}(K/Q)$. Let $\theta(\epsilon) = \epsilon^r$ and regard θ as a permutation of G under the rule $\theta : g \longrightarrow rg$. Assume X is a normalized homogeneous array afforded by G which lists the $n-1$ nonidentity elements of G as $\{x_{1i} | i = 1, \dots, n-1\}$. Define P_θ to be the permutation matrix of order n with (u, v) entry one provided

$$\theta(x_{1v}) = rx_{1v} = x_{1u}.$$

Use (3.5) and regard $\Phi(v_1)$ as a row matrix over K. We may apply θ to each entry of $\Phi(v_1)$ and denote the resulting vector by $(\Phi(v_1))^\theta$. The i^{th} coordinate of this vector is

$$(\Phi(x_{1i}))^\theta = (\Phi(x_{1i}))^r = \Phi(rx_{1i}) = \Phi(\theta(x_{1i})).$$

This may be expressed in the matrix form,

$$\text{(3.7)} \qquad (\Phi(v_1))^\theta = \Phi(v_1)P_\theta.$$

Define P_m to be the permutation matrix for the operator α_m (see (2.5)) so that P_m has (u, v) entry one provided $x_{mv} = \alpha_m(x_{1v}) = x_{1u}$. The v^{th} coordinate of $\Phi(v_m)$ and $\Phi(v_1)P_m$ is $\Phi(x_{mv}) = \Phi(x_{1u})$ where $x_{mv} = x_{1u}$. This can be expressed as the matrix equation,

$$(3.8) \qquad \Phi(v_m) = \Phi(v_1)P_m.$$

Now use (3.7) and (3.8) to conclude,

$$(3.9) \qquad (\Phi(v_m))^\theta = (\Phi(v_1)P_m)^\theta = (\Phi(v_1))^\theta P_m = \Phi(v_1)P_\theta P_m.$$

Hypotheses (H) asserts $\alpha_m\theta(g) = \theta\alpha_m(g)$ for each g. It follows $P_mP_\theta = P_\theta P_m$. Use (3.8) to rewrite (3.9),

$$(3.10) \qquad (\Phi(v_m))^\theta = \Phi(v_1)P_mP_\theta = \Phi(v_m)P_\theta.$$

If Φ is fixed by $\theta, (\Phi(g))^\theta = \Phi(g)$ for each g and (3.10) implies

$$\Phi(v_m) = \Phi(v_m)P_\theta \quad \text{for } m = 0, \ldots, n-1.$$

This means each $\Phi(v_m)$ is in the kernel of $P_\theta - I$. By (3.4), $\{\Phi(v_m)\}$ is an orthogonal basis of $V_n(K)$ and we conclude $P_\theta = I$ or equivalently $\theta = 1$. This proves that A permutes the nonidentity characters of G into orbits of length $|A| = \varphi(s)$.

The next lemma holds for any finite abelian group.

(3.11) *Assume G is an abelian group of order n and exponent s. Let K be a splitting field for G obtained by adjoining a primitive s^{th} root of unity to the rational field Q. Then G is an elementary abelian p-group for some prime p if and only if* Gal(K/Q) *permutes the nonidentity characters of G in a semiregular manner.*

Proof. Assume first that G is an elementary abelian p-group for some prime $p > 2$. Then $[K : Q] = p - 1$ and Gal(K/Q) is cyclic of order $p - 1$. For each nontrivial character Φ choose an element g of G such that $\Phi(g)$ is a primitive p^{th} root of unity. Clearly $\Phi(g)$ is fixed only by the identity of Gal(K/Q) and the assertion follows. The case $p = 2$ is trivial as $K = Q$.

Assume the condition on $A =$ Gal(K/Q) and let $p_1, \ldots, p_t$ be the distinct prime divisors of s. If some prime divisor p of s occurs with exponent $f > 1, \varphi(p^f) = p^{f-1}(p - 1)$ and, because A is isomorphic to the group of units modulo s, it follows that A has an element θ of order p. Since θ permutes the $n - 1$ nonidentity characters into orbits of length p, p is a divisor of $n - 1$. On the other hand, p is a divisor of s so must also divide n. This is impossible and we conclude

$$|A| = (p_1 - 1) \cdots (p_t - 1).$$

If $t > 1$, let β be a primitive p_1^{th} root so that $Q(\beta)$ is a proper subfield of index $p_1 - 1$ over Q. The galois theory shows $Q(\beta)$ is the fixed field for a proper subgroup B of A. This means that any character Φ of G whose range is in $K(\beta)$ is fixed by B. To construct such a character write $G = P \times U$ with $P = \langle x \rangle, |x| = p_1$. Define Φ by $\Phi(x) = \beta; \Phi(u) = 1$ for all elements u of U. Extend Φ in the natural way to a homomorphism of G into $\langle \beta \rangle$. It follows that $Q(\beta) = K$ and s is a prime number.

(3.12) *Proof of Theorem* (3.1).

Proof. As a consequence of (3.6), Gal(K/Q) acts in a semiregular manner on the nonidentity characters. By (3.11) $s = p$ is prime and G is elementary abelian.

4. Examples. In this section we prove that the homogeneity of an abelian cartesian pair is equivalent to the existence of certain homologies in the associated projective plane. We also present homogeneous cartesian pairs for which the operators α_m are not automorphisms and whose associated planes are not translation planes.

(4.1) *Let π be the projective plane associated with a cartesian pair* (G, X). *Assume G is abelian of exponent s. Then X is homogeneous if and only if π has a group of $\varphi(s)$ homologies with axis L_∞ and center* (0.0). *The homologies are the maps γ_r with $(r, s) = 1$ defined by the rule $\gamma_r(a, b) = (ra, rb)$.*

Proof. Assume X is homogeneous and let π be the associated plane constructed by the method (2.1). Consider the map

$$\gamma_r : (a, b) \longrightarrow (ra, rb).$$

If $(r, s) = 1, \gamma_r$ is a permutation of the affine points $G \times G$ of π. A typical point of L_{mb} is

$$(g_i, x_{mi} + g_b) = (g_i, \alpha_m(g_i) + g_b).$$

Condition (H) asserts that $r\alpha_m(g_i) = \alpha_m(rg_i)$ and it follows that

$$\gamma_r(g_i, x_{mi} + g_b) = (rg_i, \alpha_m(rg_i) + rg_b) \in L_{mc} \quad \text{with } g_c = rg_b.$$

This proves that $\gamma_r : L_{mb} \longrightarrow L_{mc}$ with c determined by $g_c = rg_b$. Extending γ_r to a collineation of π fixing each infinite point, we see γ_r is a homology of π with axis L_∞ and center (0.0). The group $< \gamma_r | (r, s) = 1 >$ is isomorphic to the multiplicative group of units of the integers modulo s.

Conversely if each γ_r is a homology, it follows easily that X satisfies (H).

One may construct a plane π of order q^2 with a single (P, L)-transitivity by using techniques described in **[6]**. Essentially one takes a Hughes plane of order

q^2 whose coordinates are based on a nearfield T which is a left vector space over a subfield F of order q. A Hall ternary ring for π is denoted by R with $R = T$, as sets, and addition in R is taken as the same as in T. "New multiplication" in R by elements of F is the same as in T and one may examine Theorem 11, [**7**] to see,

$$(4.2) \qquad r(ab) = (ra)b \quad \text{for each } a, b \varepsilon R, r \varepsilon F.$$

Now assume $G = \{g_0 = 0, \dots, g_{n-1}\}$ is any cartesian group of order n and exponent s. We may take $R = G$ as a ternary ring which coordinatizes the associated plane π with $(R, +) = (G, +)$. Multiplication in R is defined using the G-permutations α_m and the rule,

$$(4.3) \qquad g_i \cdot g_m = \alpha_m(g_i), \quad m = 0, \dots, n-1.$$

Compare (4.3) and (2.7) to see that condition (H) is the statement,

$$(4.4) \qquad r(g_i \cdot g_m) = (rg_i) \cdot g_m, \quad (r, s) = 1.$$

The derived Hughes planes with multiplication (4.2) satisfies (4.4) for $r = 1, \dots, p-1$ where p is the characteristic of F. The case $n = 5^2$ is the smallest value of n in which this occurs. These planes provide examples in which the homogeneous condition (H) is satisfied yet the planes are quite removed from the class of translation planes. In fact there is exactly one point-line pair affording a coordinatization with a ternary ring R in which $(R, +)$ is a cartesian group.

Several questions remain to be studied. For example, do finite nonabelian cartesian groups exist? Does condition (H) always occur?

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