6

Quantization using path integral

A quantization of a theory can be done either by considering the quark and gluon fields as operators with canonical commutation relations or by introducing Feynman integrals in functional spaces. The second procedure is very convenient for gauge theories, and especially for the non-perturbative approaches. However, although this second method preserves Lorentz invariance, it is not clear that the S-matrix calculated in this way is unitary. On the contrary, Lorentz invariance is obscure from the canonical commutation relations, while unitarity is obvious.

6.1 Path integral technique for QCD

The expression of the path integral can be obtained following the derivation discussed in a previous chapter. However, the quantization of the gauge fields is more peculiar as the source term $A_\mu^a J_a^\mu$ is not gauge invariant and hence the generating functional itself. A gauge invariant functional can be obtained (detailed derivations are given in many textbooks). This can be achieved by first introducing an invariant measure (Faddeev–Popov ansatz). One considers that the action is invariant under the gauge tranformation:

$$A_\mu \to A_\mu^\theta \,, \tag{6.1}$$

where:

$$\frac{\lambda}{2} A_\mu^\theta = U(\theta)\left[\frac{\lambda}{2} A_\mu + \frac{1}{ig} U^{-1}(\theta)\partial_\mu U(\theta)\right] U^{-1}(\theta) \,, \tag{6.2}$$

and:

$$U(\theta) = \exp\left[-i\frac{\lambda}{2}\theta(x)\right] \,. \tag{6.3}$$

Then, one makes an expansion for small θ, which leads to:

$$\left(A_\mu^\theta\right)^a = A_\mu^a + f^{abc}\theta_b A_{\mu,c} - \frac{1}{g}\partial_\mu\theta^a \,. \tag{6.4}$$

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The proper invariant measure becomes:

$$\mathcal{D}A \to \mathcal{D}A\ \Delta_G[A] \prod_{a,x} \delta\big[G^\mu A^a_\mu(x) - B^a(x)\big] . \tag{6.5}$$

G^μ has been introduced as:

$$G^\mu A^a_\mu = B^a , \tag{6.6}$$

as a generalization of the (Lorentz) gauge fixing condition $\partial_\mu A^\mu = 0$; B^a is an arbitrary space–time function, independent of the gauge field; Δ_G can be obtained from the volume normalization condition:

$$\begin{aligned} 1 &= \Delta_G[A] \int \mathcal{D}\theta^a \prod_{a,x} \delta\big[G^\mu A^{\theta_a}_\mu(x) - B^a(x)\big] , \\ &= \frac{\Delta_G[A]}{\det M_G} , \end{aligned} \tag{6.7}$$

where:[1]

$$[M_G(x,y)]^{ab} = \frac{\delta\big[G^\mu \left(A^\theta_\mu\right)^a(x)\big]}{\delta\theta^b(y)} . \tag{6.8}$$

By integrating over $B^a(x)$ by the suitable choice of weight:

$$\exp\left[-\frac{i}{2\alpha_G}\int d^4x [B(x)]^2\right] , \tag{6.9}$$

where α_G is the gauge-fixing term, the generating functional reads:

$$\begin{aligned} Z[J] &= \int \mathcal{D}A \det M_G \prod_{a,x} \delta\big[G^\mu A^a_\mu(x) - B^a(x)\big] \\ &\times \exp\left\{ i \int d^4x \left[\mathcal{L}_{\rm kin} - \frac{1}{2\alpha_G}\left(G^\mu A^a_\mu\right)^2 + A^a_\mu J^\mu_a\right]\right\} , \end{aligned} \tag{6.10}$$

where:

$$\mathcal{L}_{\rm kin} = -\frac{1}{4} G^{\mu\nu} G_{\mu\nu} , \tag{6.11}$$

is the gluon kinetic term; the last term in the exponent is the external source term. In a *covariant gauge*, the matrix M_G reads:

$$[M_G(x,y)]_{ab} = \left[\delta_{ab}\left(\frac{\partial}{\partial x_\mu}\right)^2 - g f_{abc}\partial^\mu A^c_\mu\right]\delta^4(x-y) , \tag{6.12}$$

which depends on the gauge field A^a_μ such that a simple perturbative expansion of the previous generating functional is not allowed. In this case, one needs to exponentiate $\det M_G$

[1] One should note that in the case of axial ($n.A_0 = 0$, $n \equiv$ a space-like constant vector) and in a temporal gauge ($A_0 = 0$), $\det M_G$ is a constant like in the case of an Abelian theory, where the canonical quantization can be easily done. This is not the case of the covariant gauge as we shall see later on.

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and consider it as a part of the effective Lagrangian. This can be done by introducing the Faddeev–Popov fictious ghost anti-commuting fields φ and $\bar{\varphi}$:

$$\det M_G = \int \mathcal{D}\varphi\mathcal{D}\bar{\varphi} \exp\left[-i\int d^4x d^4y\, \bar{\varphi}^a(x)\,(M_G(x,y))_{ab}\, \varphi^b(x)\right] . \tag{6.13}$$

Therefore, the complete QCD generating functional is:

$$Z[\psi, A, \varphi] = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\varphi\mathcal{D}\bar{\varphi}\;\; \exp\left\{i\int d^4x[\mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{source}}]\right\}, \tag{6.14}$$

where:

$$\mathcal{L}_{\text{source}} = A_\mu J^\mu + \bar{\chi}\varphi + \bar{\varphi}\chi + \bar{\psi}\eta + \bar{\eta}\psi , \tag{6.15}$$

with $\chi,\ \bar{\chi}$ and $\eta,\ \bar{\eta}$ are respectively source functions for the ghost and fermion fields. The generating functional can now be written in the familiar form as in the case of scalar fields in Eq. (4.85), as the Lagrangian can be decomposed into:

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{I}} , \tag{6.16}$$

where, as one can see in Eq. (5.15), that $\mathcal{L}_{\text{free}}$ has three parts. Therefore, the generating functional reads:

$$Z[\psi, A, \varphi] = \exp\left\{i\int d^4x \mathcal{L}_I\left(\frac{\delta}{i\delta J^a_\mu}, \frac{\delta}{i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}, \frac{\delta}{i\delta\chi}, \frac{\delta}{i\delta\bar{\chi}}\right)\right\} Z_0[J,\ \chi,\ \bar{\chi},\ \eta,\ \bar{\eta}] , \tag{6.17}$$

where Z_0 is the generating function for free fields:

$$Z_0 \equiv Z_0^g[J]Z_0^{FP}[\chi,\ \bar{\chi}]Z_0^q[\eta,\ \bar{\eta}] , \tag{6.18}$$

with:

$$\begin{aligned}
Z_0^g[J] &= \int \mathcal{D}A\;\; \exp\left\{i\int d^4x\left(\mathcal{L}^g_{\text{free}} + AJ\right)\right\} ,\\
Z_0^q[\eta,\ \bar{\eta}] &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\;\; \exp\left\{i\int d^4x\left(\mathcal{L}^q_{\text{free}} + \bar{\psi}\eta + \bar{\eta}\psi\right)\right\} ,\\
Z_0^{FP}[\chi,\ \bar{\chi}] &= \int \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\;\; \exp\left\{i\int d^4x\left(\mathcal{L}^{FP}_{\text{free}} + \bar{\chi}\varphi + \bar{\varphi}\chi\right)\right\} .
\end{aligned} \tag{6.19}$$

6.2 Feynman rules from the path integral

6.2.1 Free-field propagators

The propagator for a free field ϕ is defined as:

$$D(x,y) \equiv \langle 0|\mathcal{T}\phi(x)\phi(0)|0\rangle = (-i)^2 \left.\frac{\delta^2 \ln Z_0}{\delta J(x) J(y)}\right|_{J=0} \tag{6.20}$$

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Following closely the derivation of the scalar propagator in the case of scalar $\lambda\phi^4$ theory discussed in previous chapter, and by using the generalized Gaussian integral, one can rewrite:

$$
\begin{aligned}
Z_0^g[J] &= \exp\left\{i\int d^4x d^4y\ J^{a\mu}(x)D^{ab}_{\mu\nu}(x-y)J^{b\nu}(y)\right\}\ ,\\
Z_0^q[\eta,\bar{\eta}] &= \exp\left\{i\int d^4x d^4y\ \bar{\eta}(x)S(x-y)\eta(y)\right\}\ ,\\
Z_0^{FP}[\chi,\bar{\chi}] &= \exp\left\{i\int d^4x d^4y\ \bar{\chi}(x)D^{ab}(x-y)\chi(y)\right\}\ ,
\end{aligned}
\tag{6.21}
$$

where $D^{ab}_{\mu\nu}$, S, D^{ab} are respectively the gluon, fermion and Faddeev–Popov ghost propagators, which obey respectively the conditions:

$$
\begin{aligned}
\int d^4y\ K^{ac}_{\mu\nu}(x-y)D^{cb}_{\nu\lambda}(y-z) &= g_{\mu\lambda}\delta^{ab}\delta^4(x-z)\ ,\\
\int d^4y\ K^{ac}(x-y)D^{cb}(y-z) &= \delta^{ab}\delta^4(x-z)\ ,\\
\int d^4y\ \Omega(x-y)S(y-z) &= \delta^4(x-z)\ ,
\end{aligned}
\tag{6.22}
$$

where:

$$
\begin{aligned}
K^{ab}_{\mu\nu} &= \delta^{ab}\left[-g_{\mu\nu}\left(\frac{\partial}{\partial x_\mu}\right)^2+\left(1-\frac{1}{\alpha_G}\right)\partial_\mu\partial_\nu\right]\ ,\\
K^{ab} &= \delta^{ab}\left(\frac{\partial}{\partial x_\mu}\right)^2\ ,\\
\Omega &= -i\gamma^\mu\partial_\mu+m\ .
\end{aligned}
\tag{6.23}
$$

Solving these equations give the Feynman rules (visualized in Appendix E) in the momentum space after Fourier transform:

$$
\begin{aligned}
D^{ab}_{\mu\nu}(x) &= (-i)\delta^{ab}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+i\epsilon'}\left(g_{\mu\nu}-(1-\alpha_G)\frac{k_\mu k_\nu}{k^2}\right)\ ,\\
D^{ab}(x) &= (-i)\delta^{ab}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+i\epsilon'}\ ,\\
S(x) &= (i)\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{\hat{k}-m+i\epsilon'}\ .
\end{aligned}
\tag{6.24}
$$

6.2.2 Vertices

Perturbative series can be generated by expanding the exponential in Eq. (6.17):

$$
Z[J,\ldots] = \left\{1+i\int d^4x\ \mathcal{L}^I\left(\frac{\delta}{i\delta J^a_\mu},\ldots\right)+\cdots\right\}Z_0[J,\ldots]\ . \tag{6.25}
$$

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Let, for instance, the three-gluon vertex at order g:

$$\Gamma^{a_1a_2a_3}_{3\mu_1\mu_2\mu_3}(x_1,x_2,x_3) = (-i)^2\frac{\delta^3}{\delta J_1\delta J_2\delta J_3}\int d^4x\,\mathcal{L}_I^{3g}\left(\frac{\delta}{i\delta J^{a\mu}}\right)Z_0^g[J]\bigg|_{J=0}. \quad (6.26)$$

From the three-gluon terms of the Lagrangian $\mathcal{L}_{gg}$, one can deduce:

$$\begin{aligned}Z_{3g}[J] &\equiv \int d^4x\,\mathcal{L}_{\rm I}^{3g}\left(\frac{\delta}{i\delta J^{a\mu}}\right)Z_0^G[J]\bigg|_{J=0}\\ &= \int d^4x\left(\frac{-g}{2}\right)f^{abc}\left(\partial_\mu\frac{\delta}{i\delta J^{a\nu}}-\partial_\nu\frac{\delta}{i\delta J^{a\mu}}\right)\frac{\delta}{i\delta J^{b\mu}}\frac{\delta}{i\delta J^{c\nu}}Z_0^g[J] \quad (6.27)\end{aligned}$$

After some algebra, one obtains:

$$\begin{aligned}Z_{3g}[J] = -i\frac{g}{2}f^{abc}\int d^4x d^4y_1 d^4y_2 d^4y_3\big[\partial_\mu D^{aa_1}_{\nu\lambda_1}(x-y_1)-\partial_\nu D^{aa_1}_{\mu\lambda_1}(x-y_1)\big]\\ \times D^{ba_2\mu}_{\lambda_2}(x-y_2)D^{ca_3\nu}_{\lambda_3}(x-y_3)J^{a_1\lambda_1}(y_1)J^{a_2\lambda_2}(y_2)J^{a_3\lambda_3}(y_3)Z_0^g[J]\,, \quad (6.28)\end{aligned}$$

which gives:

$$\begin{aligned}\Gamma^{a_1a_2a_3}_{3\mu_1\mu_2\mu_3}(x_1,x_2,x_3) = gf^{abc}\int d^4x\big[\partial_\mu D^{aa_1}_{\nu\mu_1}(x-x_1)-\partial_\nu D^{aa_1}_{\mu\mu_1}(x-x_1)\big]\\ \times D^{ba_2\mu}_{\mu_2}(x-x_2)D^{ca_3\nu}_{\mu_3}(x-x_3) + \text{permutations}\,. \quad (6.29)\end{aligned}$$

Taking the Fourier transform, one then deduces:

$$\begin{aligned}\Gamma^{a_1a_2a_3}_{3\mu_1\mu_2\mu_3}(x_1,x_2,x_3) = \int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\exp\left[i\sum_{i=1}^3 k_ix_i\right]\prod_{i=1}^3 D_{\mu_i\lambda_i}\\ \times gf^{a_1a_2a_3}[(k_1-k_2)^{\lambda_3}g^{\lambda_1\lambda_2}+(k_2-k_3)^{\lambda_1}g^{\lambda_2\lambda_3}+(k_3-k_1)^{\lambda_2}g^{\lambda_3\lambda_1}]\,,\end{aligned} \quad (6.30)$$

with:

$$k_1+k_2+k_3=0 \quad (6.31)$$

and:

$$D_{\mu\nu}(k)\equiv\frac{1}{k^2}\left[g_{\mu\nu}-(1-\alpha_G)\frac{k_\mu k_\nu}{k^2}\right]. \quad (6.32)$$

Equation (6.30) gives the Feynman rule for the three-gluon vertex to order g, which is given in Appendix E. One can extend the previous analysis to derive the different Feynman rules listed in Appendix E.

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6.3 Quantization of QED

QED is a particular aspect of the more general non-Abelian case discussed previously. Under the $U(1)$ gauge transformation, one has instead of Eq. (6.4):

$$A_\mu^\theta(x) = A_\mu(x) - \frac{1}{g}\partial_\mu \theta(x) , \tag{6.33}$$

and the response matrix M_G in Eq. (6.8) will be independent of A_μ for any choice of the gauge, and then the Faddeev–Popov factor det M_G plays no physical role and can be dropped in the generating functional in Eq. (6.10).

6.4 Qualitative feature of quantization

For a qualitative physical picture of the quantization procedure, one can notice that the gluon fields A_a^μ have four Lorentz degrees of freedom, while a massless spin-1 gluon has only two physical polarizations. In QED, the gauge-fixing term is enough for making a consistent quantization, as the $U(1)$ gauge symmetry guarantees that the extra degrees of freedom do not generate physical amplitudes, and the physical results is independent of the gauge parameter α_G. In QCD, life is more complicated. For instance, if one tries to evaluate the cross-section of the scattering process $\bar{q}q \to gg \to \bar{q}q$, one notices that, due to the propagation of the longitudinal and scalar gluon polarizations along the internal gluon lines, the contribution of the higher-order diagrams shown in Fig. 6.1, violates unitarity. In QED, the analogous process $e^+e^- \to \gamma\gamma \to e^+e^-$ does not have these drawbacks as unphysical contributions from the longitudinal and scalar components of the photons vanish

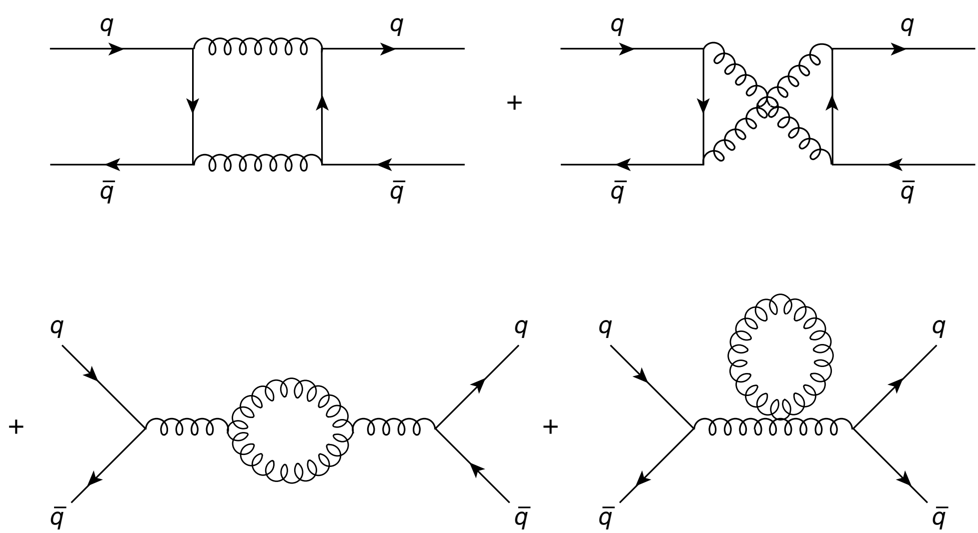

Fig. 6.1. Gluon contributions to the $\bar{q}q \to \bar{q}q$ process.

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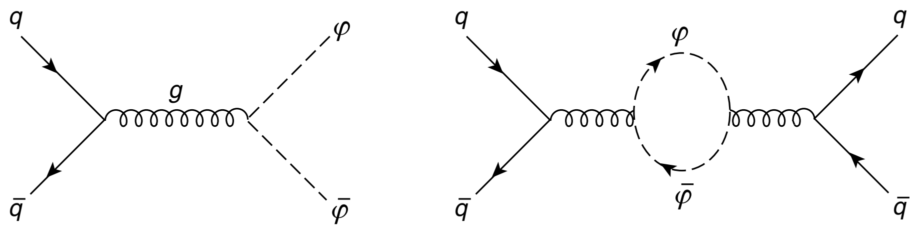

Fig. 6.2. Ghost contributions to the $\bar{q}q \to \bar{q}q$ process.

due to gauge invariance and to the conservation of the electromgnetic current. In order to recover such a property in QCD, one can introduce unphysical scalar fields with negative norms (ghosts) which eliminate the contributions of such unwanted terms from the diagrams depicted in Fig. 6.2.

More generally, the introduction of the Faddeev–Popov ghosts, in addition to the gauge-fixing term, guarantees a consistent quantization of the theory.

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