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A CANONICAL FORM FOR FULLY INDECOMPOSABLE (0, 1)-MATRICES

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This paper develops another canonical form for (0, 1)-matrices which may be used in the same spirit as the nearly decomposable matrix [5] or the k-nearly decomposable matrix [1]. This form is intrinsic in each fully indecomposable matrix and does not require the replacement of any of its non-zero entries by 0's. In particular

Form. If A is a fully indecomposable $n\times n(0, 1)$-matrix, with $n>1$, there are permutations matrices P and Q so that

$$PAQ = \begin{bmatrix} A_1 & 0 & \cdots & 0 & F_1 \\ F_2 & A_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & F_s & A_s \end{bmatrix} \quad \text{where} \quad s > 1, \text{ each } A_i (i = 1, \ldots, s)$$

is fully indecomposable and each F_i $(i=1, \ldots, s)$ has at least one non-zero entry.

Proof. (The proof has some similarity to that in [5]). As A is fully indecomposable each non-zero entry is on a positive diagonal. Consider $B=(a_{ij}(\text{per } A_{ij}/\text{per } A))$ where A_{ij} is obtained from A by deleting its i row and j column. It is easily seen that B is doubly stochastic and $b_{ij}>0$ if and only if $a_{ij}>0$. Therefore we argue that B has the specified form.

Consider $\gamma(B)=\min_{R,C} \sum_{\substack{i\in R\\ j\in C}} b_{ij}$, $|R|+|C|=n$ where $|K|$ denotes the number of elements in a set K. Suppose $\gamma(B)=\sum_{\substack{i\in R_0\\ j\in C_0}} b_{ij}$. Pick permutation matrices P_1 and Q_1 so that

$$P_1BQ_1 = \begin{bmatrix} B_1 & E_1 \\ E_2 & B_2 \end{bmatrix}$$

where E_1 is in the R_0 rows of B and in the C_0 columns of B. If B_1 is not fully indecomposable there are permutation matrices P_2 and Q_2 so that

$$P_2P_1BQ_1Q_2 = \left[\begin{array}{cc|c} \bar{B}_1 & 0 & \\ E & \bar{B}_2 & E_1' \\ \hline \multicolumn{2}{c|}{E_2'} & B_2 \end{array}\right].$$

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By the minimality of $\gamma(B)$ and the doubly stochasticity of B it follows by row-column sum arguments involving $\gamma(B)=\sigma(E_1)=\sigma(E_2)$ that

$$P_2P_1BQ_1Q_2 = \begin{bmatrix} \bar{B}_1 & 0 & \bar{E}_1 \\ E & \bar{B}_2 & 0 \\ 0 & \bar{E}_2 & B_2 \end{bmatrix}.$$

Here $\sigma(X)=\sum_{i,j} x_{ij}$ where X is a matrix. Hence by relabeling we have

$$P_2P_1BQ_1Q_2 = \begin{bmatrix} B_1 & 0 & E_1 \\ E_2 & B_2 & 0 \\ 0 & E_3 & B_3 \end{bmatrix}.$$

By continuing this argument on main diagonal blocks we may find permutation matrices P and Q so that

$$PBQ = \begin{bmatrix} B_1 & 0 & \cdots & 0 & E_1 \\ E_2 & B_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & E_s & B_s \end{bmatrix} \quad \text{where} \quad s > 1, \text{ each } E_i (i = 1, \ldots, s)$$

has at least one non-zero entry and B_i $(i=1, \ldots, s)$ is fully indecomposable or $B_i=(0)$.

Of course if any $B_i=(0)$ then $\gamma(B)=\sigma(E_i)=1$ which is impossible since B is doubly stochastic and hence $\gamma(B)<1$. Therefore each B_i $(i=1, \ldots, s)$ is fully indecomposable. Now

$$PAQ = \begin{bmatrix} A_1 & 0 & \cdots & 0 & F_1 \\ F_2 & A_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & F_s & A_s \end{bmatrix} \text{ and the form is developed.}$$

We now address ourselves to showing the utility of this new form. For this we list the the following tools.

LEMMA 1. *If* $k_i \geq 3$ *is an integer for* $i=1, \ldots, t$ *and* $t \geq 2$, *then*

$$\prod_{i=1}^{t} k_i \geq 3\left(\sum_{i=1}^{t} k_i - 3\right).$$

Proof.
$$\prod_{i=1}^{t} k_i \geq k_1 \prod_{i=2}^{t} k_i \geq k_1\left(\sum_{i=2}^{t} k_i\right) \geq 3\left(\sum_{i=1}^{t} k_i - 3\right).$$

LEMMA 2. *If* A *is an* $n \times n$ *fully indecomposable* (0, 1)-*matrix then*

$$\text{per } A \geq \sigma(A) - 2n + 2[4].$$

Lemma 2 may be deduced by the use of our form however this inequality has already been easily established in other works. The inequality we choose to argue is given in [2]. The result there is obtained by some rather exhaustive techniques which may be greatly simplified.

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THEOREM. *Let* $\Lambda_n(3)=\{n\times n(0,1)$*-matrix with precisely three* 1*'s in each row and column*$\}$. *Then* $\min_{A\in\Lambda_n(3)}(\text{per } A)\geq 3(n-1)$.

Proof. The proof is by induction on n. For $n=3$, per $A=6$ and the inequality holds. Therefore suppose the inequality holds for $A\in\Lambda_k(3)$ where $k=3,\ldots,n-1$. Let $A\in\Lambda_n(3)$. It is well known that $\min_{A\in\Lambda_n(3)}(\text{per } A)$ is achieved on a fully indecomposable matrix. (The argument is that of Lemma 2 [3, 63].) Hence we may assume A is fully indecomposable. We now argue cases.

Case I. $\gamma(A)=1$. Without loss of generality we may assume

$$A=\begin{bmatrix} A_1 & 0 & \cdots & 0 & F_1 \\ F_2 & A_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & F_s & A_s \end{bmatrix} \text{ as specified in the form.}$$

Suppose for $i=1,\ldots,s$ we have that A_i is $n_i\times n_i$. Then as $\gamma(A)=\sigma(F_1)=\cdots=\sigma(F_s)$ it follows that $n_i\geq 3$ for $i=1,\ldots,s$. Hence

$$\begin{aligned} \text{per } A &\geq \prod_{i=1}^{s} \text{per } A_i \\ &\geq \prod_{i=1}^{s} (\sigma(A_i)-2n_i+2) \\ &\geq \prod_{i=1}^{s} (3n_i-1-2n_i+2) \\ &\geq \prod_{i=1}^{s} (n_i+1) \qquad \text{and as a consequence of Lemma 1} \\ &\geq 3(n-3+s)\geq 3(n-1). \end{aligned}$$

Case II. $\gamma(A)=2$. Suppose each 1 in A lies on at least $n-1$ positive diagonals of A. Then by expanding the permanent along any row we have that per $A\geq 3(n-1)$. If A has some 1, we may assume as a_{11}, on less than $n-1$ positive diagonals we argue as follows.

Consider A_{11}. If A_{11} is fully indecomposable, then by Lemma 2, per $A_{11}\geq [3(n-1)-2]-2(n-1)+2=n-1$ which implies that a_{11} is on at least $n-1$ positive diagonals in A. Hence it must be that A_{11} is partly decomposable and so there exist permutation matrices P and Q so that

$$PAQ=\left[\begin{array}{c|cc} 1 & \multicolumn{2}{c}{E_1} \\ \hline E_2 & A_1 & 0 \\ & X & A_2 \end{array}\right].$$

Continuing to decompose A by $\gamma(A)$, as in the form, allows the assumption, without loss of generality, that

$$A=\begin{bmatrix} A_1 & 0 & \cdots & 0 & F_1 \\ F_2 & A_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & F_s & A_s \end{bmatrix}$$

with $n_1=1$ and $\gamma(A)=\sigma(F_1)=\cdots=\sigma(F_s)$. We now argue two cases:

Case 1. $n_i \geq 2$ for $i=2,\ldots,s$. In this case

$$\prod_{i=2}^{s} \text{per } A_i \geq \prod_{i=2}^{s} (3n_i-2-2n_i+2) = \prod_{i=2}^{s} n_i \geq \sum_{i=2}^{s} n_i = n-1$$

which contradicts a_{11} being on fewer than $n-1$ positive diagonals.

Case 2. Some $n_{i_0}=1$, $i_0 \neq 1$. Suppose $a_{1j_1}=1$ and $a_{1j_2}=1$ are in F_1 with $a_{i_1 1}=1$ and $a_{i_2 1}=1$ in F_2. Define the $(n-1)\times(n-1)$ matrix

$$\bar{A}=\begin{bmatrix} A_2 & 0 & \cdots & 0 & \bar{F}_2 \\ F_3 & A_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & F_s & A_s \end{bmatrix} \quad \text{where} \quad \bar{a}_{i_1-1 j_1-1} = \bar{a}_{i_2-1 j_2-1} = 1$$

are the only 1's in $\bar{F}_2$. Per $\bar{A}$ then represents the sum of all positive diagonal products in A containing a_{11}, all those containing $a_{1j_1}\cdot a_{i_1 1}$, and all those containing $a_{1j_2}\cdot a_{i_2 1}$. (Note that as $n_{i_0}=1$, $\bar{a}_{i_1-1 j_1-1}\cdot \bar{a}_{i_2-1 j_2-1}$ is not on a positive diagonal product of $\bar{A}$ as $\bar{a}_{i_1-1 j_1-1}$, $\bar{a}_{i_2-1 j_2-1}$ would then have to be on a positive diagonal which contains the two 1's in F_3 and hence contains the two 1's in $F_4,\ldots,$ and hence contains the two 1's in $F_{n_{i_0}}$ which is impossible as $n_{i_0}=1$.) Now as $s\geq 3$, $a_{1j_1}a_{i_2 1}$ as well as $a_{1j_2}a_{i_1 1}$ each lies on at least two positive diagonals, namely those which they share with each 1 in F_3. This may be seen by noting that each entry in a fully indecomposable (0, 1)-matrix is on a positive diagonal [5, 68] and hence this property holds for each of $A_1, A_2, A_3, \ldots, A_s$. Therefore it follows that any selection of precisely one 1 in each of $F_1, F_2, F_3, \ldots, F_s$ must lie on a positive diagonal of A. Hence

$$\text{per } A \geq \text{per } \bar{A}+4$$

and since $\bar{A} \in \Lambda_{n-1}(3)$ we have by the induction hypothesis that

$$\text{per } A \geq 3(n-2)+4 > 3(n-1).$$

All cases having been argued, the proof is concluded.

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