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DEFINING FAMILIES FOR INTEGRAL DOMAINS OF REAL FINITE CHARACTER

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Throughout this paper R and D will denote integral domains with the same quotient field K. A set of integral domains $\{D_i\}_{i \in I}$ with quotient field K will be said to have FC ("finite character" or "finiteness condition") if $0 \neq \xi \in K$ implies ξ is a unit of D_i for all but finitely many i. If $\bigcap_{i \in I} D_i$ also has quotient field K, then $\{D_i\}$ has FC if and only if every non-zero element in $\bigcap_{i \in I} D_i$ is a non-unit in at most finitely many D_i. A non-empty set $\{V_i\}_{i \in I}$ of rank one valuation rings with quotient field K will be called a *defining family of real R-representatives for D* if $\{V_i\}_{i \in I}$ has FC, $R \not\subset \bigcap_{i \in I} V_i$, and $D = R \cap (\bigcap_{i \in I} V_i)$. D will be called an *R-domain of real finite character* if there exists a defining family of real R-representatives for D.

The concept of an R-domain of real finite character is a continuation of the line of thought begun in [**7**, § 5] and continued in [**4**]. When $R = K$, the reference to R will be omitted; and in this case the domains of real finite character are the rings originally studied by Ribenboim in [**9**] and Griffin in [**3**].

A domain V will be called an *irredundant* real R-representative for D if there exists a defining family of real R-representatives $\{V_i\}_{i \in I}$ for D such that $V \in \{V_i\}_{i \in I}$ and such that $D \neq R \cap \{V_i | i \in I \text{ and } V_i \neq V\}$. We prove in § 1 that any defining family of real R-representatives for D contains every irredundant real R-representative and that the set of all irredundant real R-representatives itself constitutes a defining family of real R-representatives for D. The corresponding theorem for domains of real finite character (i.e. in the case that $R = K$) was proved by Brewer and Mott in [**2**, p. 38, Theorem 14].

In § 2 we give an example to show that there exists a domain D of real finite character such that no irredundant real representative is a quotient ring of D, thus answering a question raised in [**2**, p. 40].

In addition to the above notation, we use P_i to denote the *centre* of V_i on D, i.e. P_i is the intersection of the maximal ideal of V_i with D. We also say that V_i is *explicit on D* if V_i is a quotient ring of D, i.e. if $V_i = D_{P_i}$. If V is a valuation ring, $V(\xi)$ will denote the value of ξ under the valuation canonically associated with V; if V is of rank 1, we also tacitly assume that this value group is imbedded in the real numbers. Finally, we use $\subset$ for containment and $<$ for proper containment.

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1. Irredundant representatives. The following theorem is the key to the results of this section.

1.1. THEOREM. *Let R be a domain with quotient field K, let V be a rank one valuation ring of K such that $R \not\subseteq V$, and assume $D = R \cap V$ also has quotient field K. Then either V is discrete or D contains elements of arbitrarily small positive V-value.*

Proof. Let P denote the centre of V on D. If V is a rational valuation ring (i.e. the value group of V is order-isomorphic to a subgroup of the rationals), then [**4**, Lemma 1.3] implies that $V = D_P$ and hence that either V is discrete or D contains elements of arbitrarily small positive V-value. Therefore we assume that V is not rational.

$R \not\subseteq V$ implies there exists $a \in R$ such that $V(a) < 0$. Moreover, every element in V can be written in the form p/q with $p, q \in P$, so that the V-values of elements in P generate the value group of V. Hence there exists $b \in P \subset R$ such that $V(a)$ and $V(b)$ are rationally independent (i.e., $V(a)$ and $V(b)$ have the property that if m and n are integers such that $mV(a) + nV(b) = 0$, then $m = 0 = n$). Note that if m and n are positive integers, then $a^m b^n \in R$; so $a^m b^n \in P$ if and only if $mV(a) + nV(b) > 0$. The following lemma shows that the set $\{mV(a) + nV(b) | m, n \geqq 1\}$ contains arbitrarily small elements >0, which then proves the theorem.

Let $\mathbf{R}$ denote the real numbers, $\mathbf{Z}$ the integers, and $\mathbf{Z}_+$ the non-negative integers. If α, β are rationally independent real numbers, then for any real number of the form $\gamma = m\alpha + n\beta$, $m, n \in \mathbf{Z}$, the m, n are uniquely determined by γ; and we write $m(\gamma)$, $n(\gamma)$ for these integers.

LEMMA. *If $\alpha < 0$, $\beta > 0$ are rationally independent real numbers, then $G = \{m\alpha + n\beta | m, n \in \mathbf{Z}_+\}$ is dense in* $\mathbf{R}$.

Proof. Let $G' = \{m\alpha + n\beta | m, n \in \mathbf{Z}\}$ and $H = \{m\alpha + n\beta | m \in \mathbf{Z}, n \in \mathbf{Z}_+\}$. It follows from the Archimedian property of $\mathbf{R}$ that, for fixed real numbers $\gamma_1 > \gamma_2$ and for a given $n_0 \in \mathbf{Z}$, $\{\gamma \in G' | \gamma_1 > \gamma > \gamma_2 \text{ and } n(\gamma) = n_0\}$ is finite. Therefore if $n_0 < 0$, one also concludes that $F = \{\gamma \in G' | \gamma_1 > \gamma > \gamma_2$ and $0 > n(\gamma) > n_0\}$ is finite.

First we shall establish that H is dense in $\mathbf{R}$. Since G' is dense in $\mathbf{R}$ [**8**, p. 150, Example 51], it suffices to show that for any $\gamma_1 > \gamma_2 \in G'$, there exists $\gamma \in H$ such that $\gamma_1 \geqq \gamma \geqq \gamma_2$. If $n(\gamma_1) \geqq 0$ or $n(\gamma_2) \geqq 0$, we are done; so assume $n(\gamma_1) < 0$, $n(\gamma_2) < 0$. Let $n_0 = n(\gamma_1) + n(\gamma_2)$. By the finiteness of the set F, we conclude there exists $\gamma \in G'$ such that $\gamma_1 > \gamma > \gamma_2$ and either (i) $n(\gamma) \geqq 0$, or (ii) $n(\gamma) \leqq n_0$. If (i) holds, then γ has the required properties; while if (ii) holds, then $\gamma_2 + (\gamma_1 - \gamma)$ does. Thus, H is dense in $\mathbf{R}$.

Now let $r_1 > r_2$ be real numbers. Since $\alpha < 0$ and $\beta > 0$, there exist only finitely many $\gamma \in H$ such that $m(\gamma) < 0$ and $r_1 > \gamma$. Since there exist infinitely many $\gamma \in H$ such that $r_1 > \gamma > r_2$, it follows that there exists one such γ with $m(\gamma) \geqq 0$. This element is then in G, so G is dense in $\mathbf{R}$.

It follows immediately from 1.1 that (in the notation of 1.1) if the centre of V on D is finitely generated, then V is discrete and hence noetherian. This is a special case of [**4**, Theorem 1.17], which was originally proved using 1.1 and then later was done more directly.

The following simple localization lemma will be useful.

1.2. LEMMA [**4**, Lemma 1.1]. *Let $\{D_\alpha\}$ be a set of overrings contained in the quotient field K of the integral domain D, and let S be a multiplicative system of D. If $\{D_\alpha\}$ has* FC *and $D = \bigcap_\alpha D_\alpha$, then $D_S = \bigcap_\alpha (D_\alpha)_S$.*

1.3. THEOREM. *Let $D < R$ be domains with quotient field K and suppose that $D = R \cap V$ for some rank one valuation ring V of K. If $V_1, \ldots, V_n$ are rank one valuation rings of K such that $D = R \cap V_1 \cap \ldots \cap V_n$, then $V \in \{V_i\}$.*

Proof. For some $t < n$ we have $D < R \cap V_1 \cap \ldots \cap V_t$ but $D = R \cap V_1 \cap \ldots \cap V_{t+1}$. Thus, if $R' = R \cap V_1 \cap \ldots \cap V_t$ and $W = V_{t+1}$, it will suffice to show that $D = R' \cap V = R' \cap W$ implies $V = W$. If V is a rational valuation ring and has centre P on D, then [**4**, Lemma 1.3] implies that $V = D_P$. By 1.2 we would then have $R_P' = K$ and $V = W_P = W$. Thus, we may assume that V is not rational and hence in particular is not discrete. Suppose $V \neq W$. Then we can choose an element $x \in W \setminus V$ and write x in the form a/b with $a, b \in D$. Thus $W(a) \geqq W(b) \geqq 0$, while $0 \leqq V(a) < V(b)$. Note that for $y \in R'$, we have $ya \in D$ if and only if $yb \in D$; for, $ya \in D \Rightarrow ya \in V \Rightarrow yb \in V \Rightarrow yb \in D$, and $yb \in D \Rightarrow yb \in W \Rightarrow ya \in W \Rightarrow ya \in D$. Choose $y \in R' \setminus D$. Then $V(y) < 0$, so $-V(y^n) > V(a)$ for some positive integer n. Theorem 1.1 implies that there exists $d \in D$ such that $V(b) > -V(dy^n) > V(a)$. Hence $z = dy^n$ is an element of R' such that $bz \in V$ and $az \notin V$. It follows that $bz \in D$ and $az \notin D$, and this contradiction completes the proof.

1.4. COROLLARY. *Let D be an R-domain of real finite character. Then any irredundant real R-representative for D occurs in every defining family of real R-representatives; and the set of all irredundant real R-representatives is itself a defining family of real R-representatives for D.*

Proof. If V is an irredundant real R-representative for D, then for some $y \in R \setminus D$, we have $D = D[y] \cap V$. If $\{V_\alpha\}$ is a defining family of real R-representatives for D, then $D = R \cap (\bigcap_\alpha V_\alpha)$; and all but a finite number of the V_α contain y and hence contain $D[y]$. If $V_1, \ldots, V_n$ are the V_α which do not contain y, then $D = D[y] \cap V = D[y] \cap V_1 \cap \ldots \cap V_n$. Hence by 1.3, $V \in \{V_i\} \subset \{V_\alpha\}$.

Let now $\mathscr{I}$ denote the set of all irredundant real R-representatives for D. To prove $D = R \cap \{V | V \in \mathscr{I}\}$, it suffices to show for every $z \in R \setminus D$, there exists $V \in \mathscr{I}$ such that $z \notin V$; for this implies $R \cap \{V | V \in \mathscr{I}\} \subset D$, and the reverse inclusion is immediate. But if $\mathscr{F}$ is a family of real R-representatives for D, then by the FC there exist only finitely many elements $(\geqq 1)$ of $\mathscr{F}$

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which do not contain z; and by deleting from $\mathscr{F}$, if necessary, finitely many of these, we again obtain a defining family in which one of the representatives not containing z is irredundant.

Note that the second assertion of 1.4 is very superficial and requires nothing more than the finiteness condition for a defining family.

A word on terminology. The irredundant real representatives of a domain of real finite character are essential in the sense that they are a subset of every defining family of real representatives and thus provide a suitable generalization of the notion of essential representative of a Krull domain. The essential representatives of a Krull domain, however, may also be characterized as those representatives which are explicit, while the irredundant representatives of a domain of real finite character need not all be explicit (see [**7**, p. 330, Example 5.3] or [**3**, p. 84, Example 1]). Thus, contrary to the way these terms are frequently used, we feel that if the word "essential" is to be used at all for domains of real finite character, then it should refer to the *irredundant* real representatives; or one can avoid the issue completely by using the terms "irredundant" and "explicit". Incidentally, the "explicit" terminology stems from Krull [**5**, p. 559].

2. The example. We give now an example of a domain of real finite character for which *no* irredundant real representative is explicit. It is easily seen (see 2.3) that any localization at a minimal prime ideal of a domain D of real finite character is an irredundant real representative for D, and it follows that our example must have no minimal prime ideals.

We first need some preliminaries.

2.1. *Remarks.* (i) Let v be a valuation of a field k having value group a subgroup of the reals, let $\{X_i\}_{i\in I}$ be a set of indeterminates, and let $\{\gamma_i\}_{i\in I}$ be any set of real numbers. Then v can be extended to a valuation v' of $k(\{X_i\}_{i\in I})$ by defining $v'(X_i) = \gamma_i$ and if $f = M_1 + \ldots + M_t$ is any polynomial with monomial terms $M_1, \ldots, M_t$, then $v'(f) = \inf\{v'(M_i)\}$ [**1**, p. 160, Lemma 1]. We shall refer to an extension v' obtained in this way as an *extension defined by taking* infs.

(ii) If S is a subset of the real numbers and $\{\gamma_i\}_{i\in I}$ are real numbers, we shall say that the γ_i, $i \in I$, are *rationally independent with respect to* S if for any $s \in S$ and any $\gamma_1, \ldots, \gamma_t \in \{\gamma_i\}_{i\in I}$, $r_0 s + r_1\gamma_1 + \ldots + r_t\gamma_t = 0$, r_i rational numbers, implies $r_0 s = r_1 = \ldots = r_t = 0$. In particular, if S is countable, then we can always find an infinite set of γ_i which are rationally independent with respect to S. If $S = \{0\}$, we merely call the γ_i rationally independent.

(iii) Let K be a field and V be a valuation ring of K. V is said to be *well-centred* on a subring R of K if $V(R)^+ = V(K)^+$, where $V(A)^+$ denotes $\{a \in A \mid V(a) \geqq 0\}$. It is immediate that if V is explicit on a subring R of V, then V is well-centred on R.

Consider then a rank 1 valuation V of $k(X)$, k a field, X an indeterminate, such that $V(X)$ is rationally independent with respect to $V(k)$ and V is non-trivial on k. Then V is not well-centred on $k[X]$; for there exists $\alpha \in k$ such that $V(\alpha) > V(X)$, and then $V(\alpha) - V(X) \in V(k(X))^+$ but $V(\alpha) - V(X) \notin V(k[X])^+$. In particular, this shows that V is not explicit on any subring of $k[X]$.

2.2. LEMMA. *Let D be a domain of real finite character, and let $\{V_i\}_{i\in I}$ be a defining family of real representatives for D. Assume moreover that D and I are countable. Then there exists a domain $D' \supset D$ of real finite character having a defining family of real representatives $\{V_j'\}_{j\in J}$ such that*

(i) *J contains I and is countable, and D' is countable;*

(ii) *for each $i \in I$, V_i' extends V_i, and V_i' is the only element of $\{V_j'\}_{j\in J}$ with this property;*

(iii) *no V_i', $i \in I$, is explicit on D';*

(iv) *for $\alpha, \beta \in I$, if P_α, P_β are the centres of V_α, V_β on D and P_α', P_β' the centres of V_α', V_β' on D', then $P_\alpha < P_\beta$ implies $P_\alpha' < P_\beta'$.*

Proof. Let $\{X_i\}_{i\in I}$ be a set of indeterminates, and let K denote the quotient field of D. Since D and I are countable, the union of the value groups of the V_i is a countable set; and hence we can choose γ_i, $i \in I$, to be a collection of positive real numbers rationally independent with respect to this set. By taking infs, we extend any $V \in \{V_i\}$ to a valuation V' of $K(\{X_i\})$ as follows: Let P, P_i be the respective centres of V, V_i on D, and define $V'(X_i) = \gamma_i$ if $P_i \subset P$, $V'(X_i) = 0$ if $P_i \not\subset P$. Now let $D' = K[\{X_i\}] \cap \{V_i'\}_{i\in I}$. $K[\{X_i\}]$ is a countable Krull domain and hence has a countable family of discrete rank 1 representatives. Moreover, $\{V_i'\}_{i\in I}$ has FC. For by the FC of the $\{V_i\}$, every element of K and every X_i has non-zero value for at most finitely many V_i'; so it follows that any polynomial in $K[\{X_i\}]$ has non-zero value for at most finitely many V_i'. Note also that the quotient field of D' is $K(\{X_i\})$ since $D[\{X_i\}] \subset D'$.

Thus, D' is a countable domain of real finite character and has a countable defining family of real representatives consisting of the essential representatives of $K[\{X_i\}]$ together with the $\{V_i'\}_{i\in I}$, and this defining family satisfies (i). Moreover, since each essential representative of $K[\{X_i\}]$ is trivial on K, no one of these is the extension of a V_i. Thus, (ii) is also satisfied. By 2.1(iii), V_i', $i \in I$, is not explicit on any subring of $K(\{X_\alpha | \alpha \in I \text{ and } \alpha \neq i\})[X_i]$ and hence is in particular not explicit on D'.

Now let P_α', P_β' be as in (iv). Since $V_\beta'(X_\beta) > 0$ and $V_\alpha'(X_\beta) = 0$, it suffices to show $P_\alpha' \subset P_\beta'$. Suppose then there exists $f \in D'$ such that $V_\beta'(f) = 0$ and $V_\alpha'(f) > 0$. f is a polynomial in $K[\{X_i\}]$; and since the V_i' are defined by taking infs of the terms of f, we may assume f is a monomial, i.e. $f = \xi Y$, where $\xi \in K$ and $Y = X_{i_1}^{m_1} \ldots X_{i_t}^{m_t}$. Since $V_\beta'(\xi Y) = 0$ and since the non-zero real numbers from the set $\{V_\beta'(X_{i_1}), \ldots, V_\beta'(X_{i_t})\}$ were chosen rationally independent with respect to $V_\beta'(\xi)$, we conclude that $V_\beta'(Y) = 0$. It then follows from the definition of the V_i' that $V_i'(Y) > 0$ implies $P_i \not\subset P_\beta$. But since $\xi Y \in D'$, $V_i'(\xi) \geqq 0$ except possibly for those finitely many i such that $V_i'(Y) > 0$, and for these i we can then choose $r_i \in P_i \backslash P_\beta$. Taking r to be a sufficiently high power of the product of these r_i, we get $r\xi \in D$ and $r \notin P_\beta$. Since $P_\beta' \cap D = P_\beta$, it follows that $(r\xi)Y \notin P_\beta'$ and $\in P_\alpha'$, i.e. we have reduced to the case that $\xi \in D$.

Thus, assume $\xi \in D$. As observed above, $V_\beta'(\xi Y) = 0$ implies $V_\beta'(Y) = 0$; and from the definition of V_β' and the assumption that $P_\alpha \subset P_\beta$, it follows that $V_\alpha'(Y) = 0$ also. But then $V_\alpha'(\xi Y) > 0$ implies $V_\alpha(\xi) > 0$; and since $\xi \in D$, this implies $\xi \in P_\alpha$. Therefore

$$\xi \in P_\alpha \subset P_\beta \Rightarrow V_\beta'(\xi) > 0 \Rightarrow V_\beta'(\xi Y) > 0,$$

a contradiction.

2.3. LEMMA. *Let $\{V_i\}_{i \in I}$ be a defining family of real representatives for a domain D, and let P_i denote the centre of V_i on D. If P is a non-zero prime ideal of D, then $P \supset P_i$ for some i; and if P is a minimal prime, then $D_P = V_i$ for some $i \in I$.*

Proof. If K denotes the quotient field of D, then $D_P = K \cap \{V_i | P_i \subset P\}$ by 1.2. Since $D_P \neq K$, this proves the first assertion. If P is now assumed minimal, then furthermore $D_P = \cap \{V_i | P_i = P\}$. Thus, by the FC, D_P is a finite intersection of valuation rings, each of which dominates D_P. It follows from [**6**, Theorem (11.11)] that each of these is explicit on D_P and hence equals D_P.

2.4. *Example* of a domain D^* of real finite character having a defining family $\mathscr{F}^*$ of real representatives such that no element of $\mathscr{F}^*$ is explicit on D^*.

Let D_0 be, say, a p-adic valuation ring of the rationals, and construct a chain of domains $D_0 < D_1 < \ldots$ such that D_{i+1} is obtained from D_i by means of 2.2. Let $D^* = \cup D_i$ (the construction may be carried out in a field containing the rationals and having uncountable transcendence degree over the rationals). Let $\mathscr{F}_i$ denote the defining family of real representatives for D_i given by 2.2, let $K_i =$ the quotient field of D_i, let $K^* =$ the quotient field of D^*, and let $\mathscr{F}^* = \{$subrings V^* of $K^* |$ there exists n such that $V^* \cap K_i \in \mathscr{F}_i$ for all $i \geqq n\}$. It follows immediately from the definitions and 2.2(ii) that $\mathscr{F}^*$ is a defining family of real representatives for D^*.

We claim that $V^* \in \mathscr{F}^*$ implies V^* is not explicit on D^*. Since V^* is a rank 1 valuation ring, it suffices to show the centre P^* of V^* on D^* is not minimal. There exists an i such that $V^* \cap K_i \in \mathscr{F}_i$. Then $V^* \cap K_{i+1} \in \mathscr{F}_{i+1}$; and since $V^* \cap K_{i+1}$ is not explicit on D_{i+1} by 2.2(iii), it follows from 2.3 that the centre of $V^* \cap K_{i+1}$ on D_{i+1} properly contains the centre of some $W_{i+1} \in \mathscr{F}_{i+1}$. If W^* denotes the uniquely determined element of $\mathscr{F}^*$ which intersects K_{i+1} in W_{i+1}, then by 2.2(iv) the centre of W^* on D^* is properly contained in the centre of V^* on D^*.

Even though nothing can be said about the explicitness of the irredundant representatives of a domain of real finite character, one might hope that distinct irredundant representatives at least have distinct centres. However, this also is false as the following example shows. (It can be easily seen that 2.4 does not provide such an example.)

2.5. *Example*. Let B be a rank one discrete valuation ring with quotient field L, and let X and Y be indeterminates. Let e and π be positive real numbers, $e < 3$, $\pi > 3$, such that 1, e, and π are rationally independent. We extend the valuation ring B to valuation rings V and W of $L(X, Y)$ by setting $V(X) = \pi = W(Y)$ and $V(Y) = e = W(X)$. Let $R = L[X, Y]$, and consider $D = R \cap V \cap W$. Then D is a domain of real finite character; and since $B[X, Y] \subset D$, the quotient field of D is $L(X, Y)$. Moreover, if t is a generator for the maximal ideal of B, then $X/t^3 \in (R \cap V)\backslash W$ and $Y/t^3 \in (R \cap W)\backslash V$. Hence V and W are irredundant representatives for D. But if $f \in L[X, Y]$ is such that $V(f) \geqq 0$ and $W(f) \geqq 0$ and f is a nonunit in one of V or W, then the constant term of f must be a nonunit in B and f must have strictly positive value in both V and W. Hence V and W have the same centre on D.

We conclude with a simple example which shows that the uniqueness theorem for the irredundant rank one valuation rings defining a domain of real finite character does not generalize to the case of a domain which is an intersection of a family of 1-dim quasi-local integrally closed domains with FC.

2.6. *Example*. Let X, Y, and Z be indeterminates over a field k, and let $W = k[X, Y, Z]_{(Z)}$. Then W is a rank one discrete valuation ring and $W = k(X, Y) + M$, where M is the maximal ideal of W. Let $V_1 = k(X) + M$, $V_2 = k(Y) + M$ and $D = V_1 \cap V_2$. Then V_1 and V_2 are 1-dim quasi-local integrally closed domains, $V_1 \cap V_2$ is irredundant, but $D = V_1 \cap V_2 = k + M$ is again a 1-dim quasi-local integrally closed domain. Hence there is no uniqueness property for the "irredundant representatives" of D as an intersection of 1-dim quasi-local integrally closed domains.

References

1. N. Bourbaki, *Algebre commutative*, Chapters 5 and 6 (Hermann, Paris, 1964).

2. J. Brewer and J. Mott, *Integral domains of finite character*, J. Reine Angew. Math. *241* (1970), 34–41.

3. M. Griffin, *Families of finite character and essential valuations*, Trans. Amer. Math. Soc. *130* (1968), 75–85.

4. W. Heinzer and J. Ohm, *Noetherian intersections of integral domains*, Trans. Amer. Math. Soc. *167* (1972), 291–308.

5. W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche*, Math. Z. *41* (1936), 545–577.

6. M. Nagata, *Local rings* (Interscience, New York, 1962).
7. J. Ohm, *Some counterexamples related to integral closure in* $D[[X]]$, Trans. Amer. Math. Soc. *122* (1966), 321–333.
8. L. Pontrjagin, *Topological groups* (Princeton Univ. Press, Princeton, 1939).
9. P. Ribenboim, *Anneaux normaux réels à caractère fini*, Summa Brasil. Math. *3* (1956), 213–253.

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