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RINGS WITH INVOLUTION IN WHICH EVERY TRACE IS NILPOTENT OR REGULAR

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A theorem of Marshall Osborn [**15**] states that a simple ring with involution of characteristic not 2 in which every non-zero symmetric element is invertible must be a division ring or the 2×2 matrices over a field. This result has been generalized in several directions. If R is semi-simple and every symmetric element (or skew, or trace) is invertible or nilpotent, then R must be a division ring, the 2×2 matrices over a field, or the direct sum of a division ring and its opposite [**6**; **8**; **13**; **16**]. On the other hand, if R has no nilpotent ideals and the non-zero symmetrics (or skews) are not zero-divisors, then it has been shown that R must be a domain, an order in 2×2 matrices, or a subdirect sum of a domain and its opposite [**6**; **10**]. It was thus natural to raise the following question [**6**]: If R is a ring with involution in which every symmetric element is nilpotent or a non zero-divisor, and the nil ideals of R are suitably restricted, must R be a domain, or contained in 2×2 matrices or the direct sum of a domain and its opposite?

It has so far been shown [**1**] that this is indeed the case when the ring is Noetherian and has no nilpotent ideals. In this paper we prove that, if R has no nil right ideals, a necessary and sufficient condition for the conclusion to hold is that every left zero-divisor in R is also a right zero-divisor. This includes the Noetherian case as a corollary.

We note that this result is perhaps the best possible, since W. S. Martindale has constructed an example of a prime, semi-simple ring with involution, in which every symmetric element is nilpotent or regular, which is not a domain but cannot be imbedded in 2×2 matrices.

Throughout, R will denote an associative ring with involution $*$ (an anti-automorphism of period 2). Let $S = \{x \in R | x^* = x\}$ denote the set of symmetric elements, and $T = \{x + x^* | x \in R\}$ the set of *traces*. T is clearly a subset of S, and $2S \subseteq T$. An element $x \in R$ is said to be *regular* if it is not a zero-divisor in R. We will call an ideal I of R a $*$-*ideal* if $I^* \subseteq I$. If I is a $*$-ideal of R define

$$T_I = \{x + x^* | x \in I\}.$$

Note that $T_I \subseteq T \cap I$.

The first lemma is trivial but will be used frequently.

LEMMA 1. *If every element of T is nilpotent, then R has a nil right ideal unless R is commutative, $2R = 0$, and $*$ is the identity.*

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Proof. If $T = 0$, it is easy to verify that for $a, b, c \in R$, $a^* = -a$, $ab = -ba$, and $2abc = 0$. Thus $(2R)^3 = 0$, and so $2R = 0$. But then $a^* = a$, $ab = ba$, and we are done.

So, assume that $T \neq 0$. Choose $a \in S$ so $a^2 = 0$, $a \neq 0$. Then $ax + x^*a \in T$, so $(ax + x^*a)^n = 0$, for some n. Multiplying on the right by ax, we see $(ax)^{n+1} = 0$. But then aR is a nil right ideal of R.

We point out that even if *every* symmetric element of R is nilpotent, it is an open question as to whether R must be nil [**16**]. In order to avoid this problem, we will simply assume that R has no nil right ideals. Note that by Lemma 1, this implies that if the involution is not trivial, R must contain a non-nilpotent trace.

LEMMA 2. *Let R be a prime ring with $*$, and let $a \in T$. If $aT_I b = 0$, where b is any element of R and I is a $*$-ideal of R, then $a = 0$ or $b = 0$.*

Proof. Follow exactly the proof of Theorem 6 of [**3**]; it is only necessary to choose the x and y in I.

LEMMA 3. *Suppose that R is a ring with no nil right ideals and that $a \in R$, $a^2 \neq 0$. Let M be the maximal nil ideal of the ring aRa. Then $M \neq aRa$, $A = aRa/M$ has no nil right ideals, and $aMa = 0$.*

Proof. If $M = aRa$, then aRa is nil. But then a^2R would be a nil right ideal of R, and so $a^2 = 0$, a contradiction. Thus $M \neq aRa$.

Let ρ be a nil right ideal of aRa, and choose $x \in \rho$. Then $xara \in \rho$, and so $(xara)^n = 0$, all $r \in R$. But then $(axar)^{n+1} = 0$; that is, $axaR$ is a nil right ideal of R. Thus $axa = 0$, and so $a\rho a = 0$. In particular $aMa = 0$. Also, $\rho^3 = 0$ since $\rho \subseteq aRa$.

Now if $\bar{\rho}$ were a nil right ideal of A, its inverse image ρ in aRa would be a nil right ideal of aRa. Thus $\rho^3 = 0$ by the above. But this implies $\rho \subseteq M$, and so $\bar{\rho} = 0$ in A.

LEMMA 4. *Let R be a prime ring with $*$ and with no nil right ideals in which every trace is nilpotent or regular. Then $t \in T$ nilpotent implies $t^2 = 0$.*

Proof. Assume that $a \in T$, a nilpotent, but $a^2 \neq 0$. Consider the ring $A = aRa/M$ as in Lemma 3. Since $a \in S$, $*$ is an involution on aRa and $M^* \subseteq M$. Thus aRa/M has an induced involution.

Now every trace in A is the image of a trace in aRa. But no element of aRa is regular; thus every trace in aRa (and so in A) is nilpotent. By Lemma 1, there are no traces in A (since A has no nil right ideals) and so $aTa \subseteq M$. Since $aMa = 0$ by Lemma 3, we have $a^2Ta^2 = 0$, and so $a^3Ta^2 = 0$. But $a^3 \in T$, so by Lemma 2 we have $a^3 = 0$. Thus any nilpotent element of T has cube 0.

Let $x \in R$. Then $a^2x + x^*a^2 \in T$, so is nilpotent or regular. Since $xax^* \in T$, $a^2xax^*a^2 = 0$, and so $(a^2x + x^*a^2)ax^*a^2 = 0$. We claim that $(a^2x)^4 = 0$. If

$ax^*a^2 = 0$, this is certainly true, so assume $ax^*a^2 \neq 0$. Then $a^2x + x^*a^2$ is not regular, and so $(a^2x + x^*a^2)^3 = 0$. But $(a^2x + x^*a^2)^3 = (a^2x)^3 + za^2$, for some $z \in R$. Multiplying on the right by a^2x, we have $0 = (a^2x)^4$. Thus a^2R is a nil right ideal of R, a contradiction.

For the next lemma, we need some terminology. If $a, b \in R$, then we define $a \circ b = ab + ba$. If A and B are subsets of R, then $A \circ B$ means the additive subgroup of R generated by all $a \circ b$, with $a \in A$, $b \in B$. Similarly, $[a, b] = ab - ba$, the Lie product. For more details, see [**5**, Chapters 1 and 2]. The lemma extends results in [**5**] and [**11**].

LEMMA 5. *Suppose that R is a prime ring with $*$, $a \in T$ with $a^2 = 0$, and I is a $*$-ideal of R with $a(T_I \circ T_I)a = 0$. Then $a = 0$, unless $2R = 0$ and R is an order in a simple ring Q of dimension $\leqq 4$ over its center.*

Proof. First assume that the characteristic of R is not 2. Then $U = T_I \circ T_I$ is a Jordan ideal of S, and so by the proof of Theorem 2.6 of [**5**], there exists a $*$-ideal J of R such that $U \supseteq J \cap S$. Since $J \cap S \subseteq T_J$, we have $aT_Ja = 0$, and so $a = 0$ by Lemma 2.

Now consider the case when $2R = 0$. Then $T_I \circ T_I = [T_I, T_I]$, the Lie product. Assume that R is not an order in Q as above. Let $W = \{v \in T_I | ava = 0\}$; W is a Lie ideal of T_I. Now the sub-ring T_I' generated by T_I contains a non-zero $*$-ideal of R (by [**11**, Lemma 22]), so any element centralizing T_I would centralize an ideal of R, and so all of R, since R is prime. But $a \notin Z$, the center of R, since $a^2 = 0$; thus there exists $v \in T_I$ with $av \neq va$, or $av + va \neq 0$. Now

$$a[(av + va)y^* + y(av + va)]a = a[v, ya + ay^*]a \in a[T_I, T_I]a$$

for all $y \in I$. Thus $(av + va)y^* + y(av + va) \in W$, all $y \in I$. By [**11**, Lemma 23], this implies $W \supseteq T_J$, for some $\neq 0$ $*$-ideal J of R. But then $aT_Ja = 0$, and so again $a = 0$ by Lemma 2.

The next three lemmas are similar to the arguments in [**6**]. For Lemmas 6 and 7, we use the following construction of Martindale [**12**]. If R is a prime ring, then the central closure of R is a prime ring $B \supset R$, with center a field C, such that $B = RC$. Moreover, given $b \neq 0$ in B, there exists an ideal $U \neq 0$ of R with $0 \neq bU \subset R$.

As in [**2**], the involution $*$ on R may be extended to an involution on B, which we shall also call $*$.

LEMMA 6. *Let R be a prime ring with no nil right ideals with $*$ which is not a domain or an order in a simple ring of dimension $\leqq 4$ over its center, and let B be the central closure of R. If every trace in R is regular or nilpotent, then every trace in B is regular or nilpotent.*

Proof. Assume that $t_0 = u + u^* \in B$ is not regular. We will show that $t_0^2 = 0$. Now $t_0x = 0$, some $x \neq 0 \in B$. Since there exists an ideal U of R

such that $xU \neq 0 \subseteq R$, t_0 also annihilates a non-zero element of R. As in the proof of Theorem 4 of [**5**], there exists a $*$-ideal V of R such that $0 \leqq uV \subseteq R$, $0 \neq u^*V \subseteq R$, $0 \neq t_0V \subseteq R$, $0 \neq Vt_0 \subseteq R$, and $0 \neq t_0Vt_0 \subseteq R$.

Now if $T_V = 0$, then by Lemma 1, V is commutative, which implies R is commutative (since R is prime). Then R is a domain, a contradiction. We may therefore assume that $T_V \neq 0$. Let $v \in T_V$. Then $t_0vt_0 \in T$ and is a zero-divisor in R, so $(t_0vt_0)^2 = 0$ by Lemma 4. If $s \in S \cap V$ and $t \in T$, then $sts \in T_V$, and so $(t_0stst_0)^2 = 0$. This implies that

$$(st_0s)t(st_0^2s)t(st_0s) = 0, \quad \text{for all } t \in T.$$

Say $s_0 \in V \cap S$ and s_0 is regular. Then $s_0ts_0 \in T_V$ is regular or nilpotent. If $s_0t_0s_0$ is regular, then $ts_0t_0^2s_0t = 0$, for all $t \in T$, by the previous paragraph. Since at least one trace t is regular, $s_0t_0^2s_0 = 0$. Thus $t_0^2 = 0$ since s_0 is regular, and we would be done. We may therefore assume that $(s_0t_0s_0)^2 = 0$, for all regular s_0. Thus, $t_0s_0^2t_0 = 0$ for all regular $s_0 \in V \cap S$.

Now if $s \in T_V$ is not regular, then $s^2 = 0$ by Lemma 4, so certainly $t_0s^2t_0 = 0$. Thus $t_0s^2t_0 = 0$ for every $s \in T_V$. Linearizing on s, we see that $t_0(sr + rs)t_0 = 0$, all $r, s \in T_V$; that is, $t_0(T_V \circ T_V)t_0 = 0$. Now let $a = t_0vt_0$, for any $v \in T_V$. Then $a \in T$ and is not regular, so $a^2 = 0$. Thus $a(T_V \circ T_V)a = 0$, where $a \in T$ and $a^2 = 0$. By Lemma 5, if follows that $a = 0$. Thus $t_0(T_V)t_0 = 0$.

Let $x \in V$. Then $x + x^* \in T_V$, so $t_0(x + x^*)t = 0$, and thus $t_0xt_0 = -t_0x^*t_0$. Also, $xt_0x^* = x(u + u^*)x^* \in T_V$, and so $t_0(xt_0x^*)t_0 = 0$. Combining these statements, we have $t_0xt_0xt_0 = 0$, and so $(t_0x)^3 = 0$, for all $x \in V$. But then t_0V is a nilpotent right ideal of R, which is a contradiction.

LEMMA 7. *Let R be a prime ring with no nil right ideals satisfying a generalized polynomial identity. If every element of T is nilpotent or regular, then R is a domain or an order in the 2×2 matrices over a field F.*

Proof. By a theorem of Martindale [**12**], the central closure B of R is a primitive ring with a minimal one-sided ideal. By Lemma 6, B satisfies the same hypotheses as R. The rest of the proof follows exactly the case where every skew-trace is regular (Theorem 4 of [**6**]).

LEMMA 8. *Let R be a ring with no nil right ideals in which every trace is nilpotent or regular. If R is not prime, then either*

(a) *R is a subdirect sum of a domain and its opposite, or*

(b) *$2R = 0$, $*$ is the identity, and R is commutative.*

Proof. We assume that case (b) does not hold. Thus by Lemma 1, R must contain a regular trace. Since R is not prime, there exist non-zero ideals A and B of R with $AB = (0)$.

Let $C = A \cap A^*$. We claim that $C = (0)$. For, let $x \in C$. Then $x^* \in C$, and $(x + x^*)B \subseteq CB = (0)$. Thus $x + x^*$ is nilpotent. Since every trace in C is nilpotent, by Lemma 1, $2C = 0$, $*$ is the identity, and C is commutative. Also, $T_C = 0$. Since R is semi-prime and C is a commutative ideal, C is in the center of R. Let t be a regular element of T and choose $c \in C$. Then $tc \neq 0$ if $c \neq 0$, and $tc \in T_C$, a contradiction unless $c = 0$. Thus $C = A \cap A^* = (0)$.

We next show that A is a domain. For, say $a_1a_2 = 0$, with $a_1, a_2 \neq 0$ in A. Then $a_1a_2^* \in A \cap A^* = (0)$, and so $a_1(a_2 + a_2^*) = 0$. This implies $x = a_2 + a_2^*$ is nilpotent, say $x^n = 0$. But $a_2a_2^* = 0 = a_2^*a_2$ implies that $x^n = a_2^n + (a_2^*)^n = 0$, and so $a_2^n \in A \cap A^* = (0)$. Thus $a_2^n = 0$; that is, every zero-divisor in A is nilpotent. Thus if a is a zero-divisor, aA is a nil right ideal of A, a contradiction. Thus A is a domain.

As in the proof of Theorem 5 of [**6**], if $u^2 = 0$ in R then $uA = 0$ and $uA^* = 0$. Thus $u(a + a^*) = 0$, all $a \in A$. Since $a + a^* \in T$, $(a + a^*)^n = 0 = a^n + (a^*)^n$, some n. As above, this implies $a^n = 0$. But A is a domain, a contradiction. Thus R contains no nilpotent elements.

This means that every element of T is regular. If every element of S were regular, the theorem would follow from the result of Lanski [**10**]. We may thus assume that for some $s \neq 0$ in S, s is a zero-divisor. Let $V = \{x \in R | sx = 0\}$. If $sx = 0$, then $(xs)^2 = 0$ so $xs = 0$. Thus V is an ideal of R. Since $s^* = s$, $V^* = V$. However, $V \cap T = (0)$. As before, this implies $2V = 0$, $*$ is the identity on V, and V is in the center of R. But then for $v \in V$, $t \in T$, we have $vt \neq 0$ but also $vt \in V \cap T = (0)$, a contradiction. Thus the theorem is proved.

We are now able to combine the various lemmas to obtain the desired results.

THEOREM 1. *Let R be a ring with no nil right ideals and with $*$ satisfying a generalized polynomial identity. Assume that every trace in R is nilpotent or regular. Then R is one of the following:*

(1) *a domain,*

(2) *a subdirect sum of a domain and its opposite,*

(3) *an order in $\mathscr{M}_2(F)$, the 2×2 matrices over a field F,*

(4) *commutative of characteristic 2 with the trivial involution.*

Proof. This follows from Lemmas 7 and 8.

The following corollary is actually implicit in the results of [**6**] (and in fact is true if we only assume that R has no nilpotent ideals). We state it here only for completeness.

COROLLARY 1 (Herstein and Montgomery). *Let R be a ring with $*$ and with no nil right ideals in which every non-zero trace is regular. Then R must be one of the four possibilities in Theorem 1.*

Proof. If R is not prime, then we are done by Lemma 8. If R is prime but not a domain, we may assume that R satisfies a generalized polynomial identity. For if every symmetric element were regular, we would be done by Lanski's result [**10**]. We may thus assume that R contains a symmetric zero-divisor, say a. Since $aTa \subset T$ and every non-zero trace is regular, we must have that

$aTa = 0$. Hence if $r \in R$, $a(r + r^*)a = 0$, and so $ara = -ar^*a$. Now if $r_1, r_2 \in R$, then

$$a(r_1ar_2)a = -a(r_1ar_2)^*a = -ar_2^*ar_1^*a = -ar_2ar_1a.$$

Consequently, R satisfies the generalized polynomial identity $p(x, y) = axaya + ayaxa$. Now apply Lemma 7.

We now come to the main result of this paper.

THEOREM 2. *Let R be a ring with $*$ and with no nil right ideals in which every trace is nilpotent or regular. Then R is one of the four possibilities in Theorem* 1 *if and only if R satisfies either one of the following conditions:*

(A) *every left zero-divisor in R is also a right zero-divisor;*

(B) *whenever $xx^* = 0$, for any $x \in R$, there exists $y \neq 0 \in R$ such that $yx = 0$.*

Proof. It is trivial that if R is one of the four possibilities in Theorem 1, then R satisfies conditions (A) and (B). Also, it is clear that condition (A) implies condition (B). Thus it remains to show that if R satisfies condition (B), then R must be either

(1) a domain,

(2) a subdirect sum of a domain and its opposite,

(3) an order in $\mathscr{M}_2(F)$, or

(4) commutative with the trivial involution.

Now if R is not prime, then by Lemma 8 R must be (2) or (4). We may therefore assume that R is prime. As before, we will show that R satisfies a generalized polynomial identity.

First, say that there exists some $x \in R$ so x is a left zero-divisor but $x^*x \neq 0$. Let $a = x^*x$ and let $t \in T$. Then $xTx^* \subseteq T$ and is not regular, so $(xtx^*)^2 = 0$, all $t \in T$ by Lemma 4. Thus $xtx^*xtx^* = 0$, and so $atata = 0$, all $t \in T$. Linearizing on t, and then multiplying on the right by ta, we see $atasata = 0$, for all $s, t \in T$. Now $ata \in T$, and so $ata = 0$ by Lemma 2. Since $aTa = 0$, as in the proof of Corollary 1 we see that R satisfies a generalized polynomial identity.

We may thus assume that for every $x \in R$ which is a left zero-divisor, $x^*x = 0$. But now by our basic assumption (using $*$), there exists some $y \neq 0 \in R$ such that $xy = 0$. By repeating the argument in the above paragraph on the right, we see that we would be done unless $xx^* = 0$. We may therefore assume that for every $x \in R$ which is a left zero-divisor, we have $xx^* = 0$. Now if every non-zero trace were regular, we would be done by Corollary 1. So choose $s \in T$, $s \neq 0$ with $s^2 = 0$. Then for any $r \in S$, $x = sr$ is a left zero-divisor, and so $xx^* = sr^2s = 0$. Linearizing on r, we have $s(ru + ur)s = 0$, for all $r, u \in S$, or $s(S \circ S)s = 0$. But then $s = 0$ by Lemma 5, a contradiction, unless R is an order in a simple ring Q of dimension $\leqq 4$ over its center. Thus R is either (1) or (3), and we are done.

COROLLARY 2. *If R is a semi-prime Goldie ring with $*$ in which every trace is nilpotent or regular, then R must be one of the four possibilities in Theorem* 1.

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Proof. Since a semi-prime Goldie ring is an order (left and right) in a semi-simple Artinian ring [**4**, p. 176], every left zero-divisor in R is a right zero-divisor. In addition, any nil right ideal in a Goldie ring is nilpotent [**9**], and so R being semi-prime implies that R has no nil right ideals. The corollary now follows from Theorem 2.

COROLLARY 3 (Chacron and Chacron [**1**]). *Let R be a Noetherian ring with $*$ in which every non-zero element of the form $x + x^*$ or xx^* is regular or nilpotent. If N is the maximal nilpotent ideal of R, then R/N must be one of the possibilities* (1), (2), *or* (3), *in Theorem* 1.

Proof. This follows from Corollary 2.

We close by giving Martindale's example of a prime, semi-simple ring R with * in which every symmetric element is nilpotent or regular, but which is not a domain and cannot be imbedded in the 2×2 matrices over a field.

Let F be any field, and consider the polynomial ring $F[x, y]$ in the non-commuting indeterminates x and y. Let

$$R = F[x, y]/(x^2).$$

$F[x, y]$ has an involution $*$ by reversing the order in any monomial in x and y, by fixing the elements of F. Since (x^2) is a $*$-ideal, R has an induced involution.

Now if $\bar{z}\bar{w} = 0$ in R, then $\bar{z} = \bar{z}_1\bar{x}$ and $\bar{w} = \bar{x}\bar{z}_1$, for some $\bar{x}_1$ and $\bar{w}_1$ in R. (This has been verified independently by P. M. Cohn in Proc. Amer. Math. Soc. *40* (1973), 91–92, and by D. Estes.) It follows that if $\bar{s}$ is a symmetric zero-divisor in R, then $\bar{s}^2 = 0$. Thus every symmetric element of R is regular or nilpotent, and R is the desired ring.

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