Can. J. Math., Vol. XXXIII, No. 6, 1981, pp. 1420–1431

SEPARATING CLOSED SETS BY CONTINUOUS MAPPINGS INTO DEVELOPABLE SPACES

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1. Introduction. A topological space X is called *developable* if it has a development, i.e., a sequence $(\mathscr{A}_n)_{n \in \mathbf{N}}$ of open covers of X such that for each $x \in X$ the collection $\{\mathrm{St}\,(x, \mathscr{A}_n) | n \in \mathbf{N}\}$ is a neighbourhood base of x, where

$$\mathrm{St}\,(x, \mathscr{A}_n) = \cup \{A | A \in \mathscr{A}_n, x \in A\} \text{ [2]}.$$

This class of spaces has turned out to be one of the most natural and useful generalizations of metrizable spaces [**23**]. In [**4**] it was shown that some well known results in metrization theory have counterparts in the theory of developable spaces (i.e., Urysohn's metrization theorem, the Nagata-Smirnov theorem, and Nagata's "double sequence theorem"). Moreover, in [**3**] it was pointed out that subspaces of products of developable spaces (i.e., D-completely regular spaces) can be characterized in much the same way as subspaces of products of metrizable spaces (i.e., completely regular T_1-spaces). In particular it was proved that a topological space X is D-completely regular if and only if every closed subset A of X and every point $x \in X \backslash A$ can be separated by a continuous mapping into a developable T_1-space (see [**5**], [**10**], [**11**] and [**17**] for more results concerning D-completely regular spaces).

These observations suggest the question whether it is possible to work out an interesting concept of normality related to developable spaces. It is the aim of this paper to introduce such a concept. In Section 2 we prove the following theorem:

THEOREM 1. *The following conditions are equivalent for a topological space* (X, τ):

(1) *For every pair A, B of disjoint closed subsets of X there exists a pair F, G of disjoint closed G_δ-sets such that $A \subset F$ and $B \subset G$.*

(2) *For every finite open cover $\mathscr{A}$ of X there exists a finite open F_σ-cover $\mathscr{B}$ of X (i.e., each $B \in \mathscr{B}$ is an open F_σ-set) which refines $\mathscr{A}$.*

(3) *For every finite open cover $\mathscr{A}$ of X there exists a developable topology $\tau' \subset \tau$ and a τ'-open cover $\mathscr{B}$ of X which refines $\mathscr{A}$.*

(4) *For every pair A, B of disjoint closed subsets of X there exists a continuous mapping $f: X \to Y$ into a developable T_1-space Y such that* $\mathrm{cl}\, f[A] \cap \mathrm{cl}\, f[B] = \emptyset$.

Received June 16, 1980 and in revised form January 20, 1981.

We call a topological space *D-normal* if it satisfies one of the equivalent conditions (1)–(4) of Theorem 1. Obviously every normal topological space and every developable space is D-normal. Condition (1) implies that σ-spaces, semistratifiable spaces, and, more generally, perfect spaces are D-normal. In particular, every countable product of the Sorgenfrey line is D-normal [**9**]. On the other hand there exist completely regular T_1-spaces which are not D-normal (see for instance [**21**], Example 94).

In Section 3 we prove several other interesting characterizations of D-normal spaces. The last two sections contain results concerning hereditarily D-normal spaces and subspaces, products, and continuous images of D-normal spaces. Throughout this paper no separation axioms are assumed unless explicitly stated.

2. Proof of theorem 1. We call an open cover $\mathscr{A}$ of a topological space an *F_σ-cover* if every $A \in \mathscr{A}$ is an open F_σ-set. If $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are open covers of a topological space, then $\mathscr{A}_1 \wedge \ldots \wedge \mathscr{A}_n$ denotes the open cover consisting of all finite intersections $A_1 \cap \ldots \cap A_n$, where $A_1 \in \mathscr{A}_1, \ldots, A_n \in \mathscr{A}_n$. To simplify the proof of Theorem 1 we introduce the following notation. If β is a family of open covers of a topological space X and $B \subset X$, we define

$$\operatorname{int}_\beta B = \{x \mid x \in B, \operatorname{St}(x, \mathscr{C}) \subset B \text{ for some } \mathscr{C} \in \beta\}.$$

For an open cover $\mathscr{A}$ of X we define

$$\operatorname{int}_\beta \mathscr{A} = \{\operatorname{int}_\beta A \mid A \in \mathscr{A}\}.$$

The family β is called *kernel-normal* if for each $\mathscr{B} \in \beta$ there exists a $\mathscr{B}' \in \beta$ which refines $\operatorname{int}_\beta \mathscr{B}$. An open cover $\mathscr{A}$ of X is called *kernel-normal* if it is a member of a kernel-normal sequence of open covers of X.

It is easy to see that every normal open cover (as introduced by J. W. Tukey [**22**]) of a topological space is kernel-normal. While it is well known that normality of open covers implies the existence of certain pseudo-metrizable topologies on a given space, the following proposition shows that kernel-normality yields developable topologies.

PROPOSITION 1. *An open cover $(A(i))_{i\in I}$ of a topological space (X, τ) is kernel-normal if and only if there exists a developable topology $\tau' \subset \tau$ and a τ'-open cover $(B(i))_{i\in I}$ of X such that $B(i) \subset A(i)$ for each $i \in I$.*

Proof. Assume first that $\mathscr{A} = (A(i))_{i\in I}$ is kernel-normal. Then there exists a kernel-normal sequence β of open covers of X such that $\mathscr{A} \in \beta$. Denote by μ the collection consisting of all open covers $\mathscr{A}_1 \wedge \ldots \wedge \mathscr{A}_n$, $n \in \mathbf{N}$, where $\mathscr{A}_i \in \beta$ for each $i \in \{1, \ldots, n\}$. It is easy to see that

$$\tau' = \{U \mid U \subset X, \operatorname{int}_\mu U = U\}$$

is a topology on X such that $\tau' \subset \tau$. We claim that (X, τ') is developable.

To prove this assertion it suffices to verify the following fact:

(*) For each $\mathscr{H} \in \mu$ the collection $\mathrm{int}_\mu \mathscr{H}$ is a τ'-open cover of X.

In fact, since μ is countable, (*) implies that $\{\mathrm{int}_\mu \mathscr{H} | \mathscr{H} \in \mu\}$ is a development of (X, τ').

To verify (*) consider an arbitrary $\mathscr{H} \in \mu$. There exist $\mathscr{A}_1, \ldots, \mathscr{A}_n \in \beta$ such that $\mathscr{H} = \mathscr{A}_1 \wedge \ldots \wedge \mathscr{A}_n$. At first we show that $\mathrm{int}_\mu \mathscr{H}$ is a cover of X. Since β is kernel-normal, $\mathrm{int}_\beta \mathscr{A}_j$ is a cover of X for each $j \in \{1, \ldots, n\}$. Thus, given an arbitrary point x of X, there exist $A_1 \in \mathscr{A}_1, \ldots, A_n \in \mathscr{A}_n$ such that

$$x \in \mathrm{int}_\beta A_1 \cap \ldots \cap \mathrm{int}_\beta A_n.$$

For each $j \in \{1, \ldots, n\}$ there is a $\mathscr{C}_j \in \beta$ such that $\mathrm{St}\,(x, \mathscr{C}_j) \subset A_j$. Hence

$$\mathrm{St}\,(x, \mathscr{C}_1 \wedge \ldots \wedge \mathscr{C}_n) \subset A_1 \cap \ldots \cap A_n,$$

i.e., $x \in \mathrm{int}_\mu H$ where $H = A_1 \cap \ldots \cap A_n \in \mathscr{H}$.

Next we prove that for each $H \in \mathscr{H}$, $\mathrm{int}_\mu H$ is τ'-open, i.e., that

$$\mathrm{int}_\mu H \subset \mathrm{int}_\mu (\mathrm{int}_\mu H).$$

Consider an arbitrary $H \in \mathscr{H}$ and a point $x \in \mathrm{int}_\mu H$. There exist $\mathscr{E}_1, \ldots, \mathscr{E}_k \in \beta$ such that

$$\mathrm{St}\,(x, \mathscr{E}_1 \wedge \ldots \wedge \mathscr{E}_k) \subset H.$$

Since β is kernel-normal, there is an $\mathscr{F}_j \in \beta$ for each $j \in \{1, \ldots, k\}$ such that $\mathscr{F}_j$ refines $\mathrm{int}_\beta \mathscr{E}_j$. We claim that

$$\mathrm{St}\,(x, \mathscr{F}_1 \wedge \ldots \wedge \mathscr{F}_k) \subset \mathrm{int}_\mu H.$$

In fact, if $z \in \mathrm{St}\,(x, \mathscr{F}_1 \wedge \ldots \wedge \mathscr{F}_k)$, then there exist $E_1 \in \mathscr{E}_1, \ldots, E_k \in \mathscr{E}_k$ such that

$$x, z \in \mathrm{int}_\beta E_1 \cap \ldots \cap \mathrm{int}_\beta E_k.$$

For each $j \in \{1, \ldots, k\}$ there is a $\mathscr{G}_j \in \beta$ such that $\mathrm{St}\,(x, \mathscr{G}_j) \subset E_j$. Now

$$\mathrm{St}\,(z, \mathscr{G}_1 \wedge \ldots \wedge \mathscr{G}_k) \subset E_1 \cap \ldots \cap E_k \subset \mathrm{St}\,(x, \mathscr{E}_1 \wedge \ldots \wedge \mathscr{E}_k) \subset H,$$

which proves that $z \in \mathrm{int}_\mu H$. Hence we have shown that

$$\mathrm{St}\,(x, \mathscr{F}_1 \wedge \ldots \wedge \mathscr{F}_k) \subset \mathrm{int}_\mu H,$$

i.e., $x \in \mathrm{int}_\mu (\mathrm{int}_\mu H)$, which completes the proof of (*).

Define now $B(i) = \mathrm{int}_\mu A(i)$ for each $i \in I$. Since $\mathscr{A}$ belongs to μ, (*) implies that $(B(i))_{i \in I}$ is a τ'-open cover of X with the desired property.

For the proof of the reverse implication assume that there exists a developable topology $\tau' \subset \tau$ and a τ'-open cover $\mathscr{B}$ of X which refines $\mathscr{A}$. Let $(\mathscr{A}_n)_{n\in\mathbf{N}}$ be a development of (X, τ'). Then $\beta = \{\mathscr{A}\} \cup \{\mathscr{A}_n | n \in \mathbf{N}\}$ is a kernel-normal sequence of τ-open covers of X containing $\mathscr{A}$, which completes the proof.

Remark. As outlined in [**3**], Proposition 1 can be proved more elegantly using nearness structures as introduced by H. Herrlich [**12**]. For the sake of simplicity, however, we have given a direct proof.

PROPOSITION 2. *If μ is a collection of open covers of a topological space X such that*

(i) *$\mathscr{A}, \mathscr{B} \in \mu$ implies $\mathscr{A} \wedge \mathscr{B} \in \mu$,*

(ii) *for each $\mathscr{A} \in \mu$ there exists a sequence $\beta = (\mathscr{A}_n)_{n\in\mathbf{N}}$ in μ and an $n \in \mathbf{N}$ such that $\mathscr{A}_n$ refines $\operatorname{int}_\beta \mathscr{A}$, then every $\mathscr{A} \in \mu$ is kernel-normal.*

Proof. Consider an arbitrary $\mathscr{A} \in \mu$. For technical reasons we define $\mathscr{A}_{(0,n)} = \{X\}$ for each $n \in \mathbf{N}$. Using complete induction and conditions (i) and (ii) it is easy to verify that for each $k \in \mathbf{N}$ there exists an open cover $\mathscr{A}_{(k,k)} \in \mu$ and a sequence $\beta(k) = (\mathscr{A}_{(k,n)})_{n>k}$ in μ such that

(a) $\mathscr{A} = \mathscr{A}_{(1,1)}$ and $\mathscr{A}_{(k,k)} = \mathscr{A}_{(k-1,k)}$ for each $k > 1$;

(b) $\mathscr{A}_{(k,k+1)}$ refines $\operatorname{int}_{\beta(k)} \mathscr{A}_{(k,k)}$ for each $k \in \mathbf{N}$;

(c) $\mathscr{A}_{(k,n)}$ refines $\mathscr{A}_{(k-1,n)}$ for each $k \in \mathbf{N}$ and for each $n \geqq k$.

We define $\xi = (\mathscr{A}_{(k,k)})_{k\in\mathbf{N}}$ and claim that ξ is kernel-normal. To prove this assertion we note that $\operatorname{int}_{\beta(k)} A \subset \operatorname{int}_\xi A$ for every subset A of X and for each $k \in \mathbf{N}$. For if $x \in \operatorname{int}_{\beta(k)} A$, there exists an $n > k$ such that

$$\operatorname{St}(x, \mathscr{A}_{(n,k)}) \subset A.$$

Condition (c) implies that

$$\operatorname{St}(x, \mathscr{A}_{(n,n)}) \subset \operatorname{St}(x, \mathscr{A}_{(n-1,n)}) \subset \ldots \subset \operatorname{St}(x, \mathscr{A}_{(k,n)}).$$

Therefore $x \in \operatorname{int}_\xi A$. In particular, it follows that $\operatorname{int}_{\beta(k)} \mathscr{A}_{(k,k)}$ refines $\operatorname{int}_\xi \mathscr{A}_{(k,k)}$ for each $k \in \mathbf{N}$. By virtue of (a) and (b) $\mathscr{A}_{(k+1,k+1)} = \mathscr{A}_{(k,k+1)}$ refines $\operatorname{int}_{\beta(k)} \mathscr{A}_{(k,k)}$. Hence $\mathscr{A}_{(k+1,k+1)}$ refines $\operatorname{int}_\xi \mathscr{A}_{(k,k)}$ for each $k \in \mathbf{N}$, which proves that ξ is kernel-normal. Since $\mathscr{A} \in \xi$, the proof is complete.

Proof of Theorem 1. (1) implies (2): We use induction to prove that every finite open cover $\mathscr{A} = \{A(1), \ldots, A(k)\}$ of X has a finite open F_σ-refinement. If $k \leqq 2$, the assertion follows directly from the assumption. Now suppose that $k > 2$ and that every open cover of X with cardinality less than k has a finite open F_σ-refinement. We may assume that no proper subcollection of $\mathscr{A}$ covers X. Then there exists an open F_σ-cover $\{F(1), F(2)\}$ such that

$$F(1) \subset A(1) \text{ and } F(2) \subset \cup \{A(i) | i \in \{2, \ldots, k\}\}.$$

Similarly there exists an open F_σ-cover $\{G(1), G(2)\}$ such that

$$G(1) \subset F(1) \cup \cup \{A(i) | i \in \{3, \ldots, k\}\} \text{ and } G(2) \subset A(2).$$

It is easy to see that $\{F(1) \cup G(1), A(3), \ldots, A(k)\}$ is a cover of X. Therefore there exists an open F_σ-cover $\{U(2), \ldots, U(k)\}$ of X such that $U(2) \subset F(1) \cup G(1)$ and $U(i) \subset A(i)$ for each $i \in \{3, \ldots, k\}$. Since $\{F(1), G(2), U(3), \ldots, U(k)\}$ is an open F_σ-refinement of $\mathscr{A}$, the induction is complete.

(2) implies (3): Let μ be the collection of all finite open covers of X. By virtue of Proposition 1 it suffices to show that each $\mathscr{A} \in \mu$ is kernel-normal. Hence, by Proposition 2, it remains to prove that for each $\mathscr{A} \in \mu$ there exists a sequence $\beta = (\mathscr{A}_n)_{n\in\mathbf{N}}$ in μ and an $n \in \mathbf{N}$ such that $\mathscr{A}_n$ refines $\operatorname{int}_\beta \mathscr{A}$.

Consider an arbitrary $\mathscr{A} \in \mu$. Assuming (2) there exists an open refinement $\mathscr{B} = \{B(1), \ldots, B(k)\}$ of $\mathscr{A}$ such that

$$B(i) = \cup \{B(i, n) | n \in \mathbf{N}\} \text{ for each } i \in \{1, \ldots, k\},$$

where every $B(i, n)$ is a closed subset of X. For each $n \in \mathbf{N}$ we construct an open cover $\mathscr{A}_n$ of X as follows. If

$$x \in X(n) = \cup \{B(i, n) | i \in \{1, \ldots, k\}\},$$

define

$$A(x, n) = \cap \{B(i) | i \in \{1, \ldots, k\}, x \in B(i, n)\} \\ \setminus \cup \{B(i, n) | i \in \{1, \ldots, k\}, x \notin B(i, n)\}.$$

Every $A(x, n)$ is an open subset of X containing x. Therefore

$$\mathscr{A}_n = \{B(i) \setminus X(n) | i \in \{1, \ldots, k\}\} \cup \{A(x, n) | x \in X(n)\}$$

is an open cover of X which can easily be seen to be finite, i.e., $\mathscr{A}_n \in \mu$ for each $n \in \mathbf{N}$. It suffices to prove that $\mathscr{A}_1$ refines $\operatorname{int}_\beta \mathscr{A}$, where $\beta = (\mathscr{A}_n)_{n\in\mathbf{N}}$.

Obviously $\mathscr{A}_1$ refines $\mathscr{B}$ and $\operatorname{int}_\beta \mathscr{B}$ refines $\operatorname{int}_\beta \mathscr{A}$. Therefore it remains to prove that $\mathscr{B} = \operatorname{int}_\beta \mathscr{B}$, i.e., that $B(i) \subset \operatorname{int}_\beta B(i)$ for each $i \in \{1, \ldots, k\}$. Consider a fixed $B(i_0) \in \mathscr{B}$ and a point $x \in B(i_0)$. There exists an $n \in \mathbf{N}$ such that $x \in B(i_0, n)$. We claim that $\operatorname{St}(x, \mathscr{A}_n) \subset B(i_0)$. To prove this assertion we consider an arbitrary point $z \in X(n)$ such that $x \in A(z, n)$. For each $i \in \{1, \ldots, k\}$ such that $x \in B(i, n)$ we have $z \in B(i, n)$, for otherwise we would have

$$x \in B(i, n) \subset \cup \{B(i, n) | i \in \{1, \ldots, k\}, z \notin B(i, n)\} \subset X \setminus A(z, n),$$

which is impossible. Therefore

$$A(z, n) \subset \cap \{B(i) | i \in \{1, \ldots, k\}, z \in B(i, n)\} \\ \subset \cap \{B(i) | i \in \{1, \ldots, k\}, x \in B(i, n)\} \subset B(i_0).$$

Since

$$\text{St}(x, \mathscr{A}_n) = \bigcup \{A(z, n) \mid z \in X(n), x \in A(z, n)\},$$

it follows that $\text{St}(x, \mathscr{A}_n) \subset B(i_0)$, i.e., $x \in \text{int}_\beta B(i_0)$.

(3) implies (4): Consider a pair A, B of disjoint closed subsets of X. Assuming (3) there exists a developable topology $\tau' \subset \tau$ and a τ'-open cover of X which refines $\{X \backslash A, X \backslash B\}$. Let $f: (X, \tau') \rightarrow Y$ be the T_0-identification of (X, τ') (i.e., Y is the quotient space of (X, τ') with respect to the equivalence relation "$x \pi y$ if and only if x has the same τ'-neighbourhoods as y"). Then $f: (X, \tau) \rightarrow Y$ is a continuous mapping into a developable T_1-space [**24**] such that

$$\text{cl} f[A] \cap \text{cl} f[B] = \emptyset.$$

Since it is evident that (4) implies (1), the proof is complete.

Following N. C. Heldermann a topological space is called *D-regular* if it has a base of open F_σ-sets [**11**] (see also [**5**]). Our next result is an immediate consequence of Theorem 1. It will be applied in [**5**].

COROLLARY 1. *Every D-regular Lindelöf space is D-normal.*

Proof. Let X be a D-regular Lindelöf space and consider a finite open cover $\mathscr{A} = \{A(1), \ldots, A(k)\}$ of X. Then there exists a countable open F_σ-cover $\mathscr{F}$ of X which refines $\mathscr{A}$. Define

$$B(i) = \bigcup \{F \mid F \in \mathscr{F}, F \subset A(i)\}$$

for each $i \in \{1, \ldots, k\}$. Since every countable union of F_σ-sets is an F_σ-set, $\{B(1), \ldots, B(k)\}$ is an open F_σ-cover which refines $\mathscr{A}$. Hence, by virtue of Theorem 1, X is D-normal.

3. D-normal spaces and open covers. One of the most useful properties of normal topological spaces is the fact that every point-finite open cover has a shrinking. Moreover every locally finite open cover of a normal space has a locally finite cozero-set refinement. Our next theorem provides similar characterizations of D-normal spaces. We call a subset B of a topological spaces X *D-open* if there exists a continuous mapping $f: X \rightarrow Y$ into a developable space Y and an open subset U of Y such that $B = f^{-1}[U]$. An open cover of X is called D-open if it consists of D-open sets. Using Theorem 1, we can now prove the following:

THEOREM 2. *The following properties of a topological space X are equivalent:*

(1) *X is D-normal.*

(2) *For every point-finite open cover $(A(i))_{i \in I}$ of X there exists a D-open cover $(B(i))_{i \in I}$ of X such that $B(i) \subset A(i)$ for each $i \in I$.*

(3) *For every point-finite open cover* $(A(i))_{i\in I}$ *of* X *there exists an open* F_σ*-cover* $(B(i))_{i\in I}$ *of* X *such that* $B(i) \subset A(i)$ *for each* $i \in I$.

(4) *Every locally finite open cover of* X *has a locally finite* D*-open refinement.*

(5) *Every locally finite open cover of* X *has a locally finite open* F_σ*-refinement.*

(6) *Every locally finite open cover of* X *is kernel-normal.*

Proof. (1) implies (2): Let $\mathscr{A} = (A(i))_{i\in I}$ be a point-finite open cover of X. Assume that $I = \{i|\ 0 \leqq i < \alpha\}$ is well-ordered. Using transfinite induction we will find a D-open set $B(i)$ for each $i \in I$ such that

$$(*)_i \quad X\backslash(\cup\{B(j)|\ j < i\} \cup \cup \{A(k)|\ k > i\}) \subset B(i) \subset A(i).$$

If $i = 0$, there exists a D-open set $B(0)$ such that

$$X\backslash\cup\{A(k)\ k| > 0\} \subset B(0) \subset A(0)$$

according to condition (4) of Theorem 1. Assume now that $i > 0$ and that for every $j < i$ an open F_σ-set $B(j)$ is defined such that $(*)_j$ holds. Using the fact that $\mathscr{A}$ is point-finite it is easy to see that

$$\{B(j)|\ j < i\} \cup \{A(k)|\ k \geqq i\}$$

is an open cover of X. Therefore there exists a D-open set $B(i)$ such that

$$X\backslash(\cup\{B(j)|\ j < i\} \cup \cup \{A(k)|\ k > i\}) \subset B(i) \subset A(i),$$

i.e., $(*)_i$ is satisfied. Using the point-finiteness of $\mathscr{A}$ again one can show that the collection $(B(i))_{i\in I}$ covers X.

Clearly (2) implies (3) and (4), and (3) and (4) imply (5). By virtue of Proposition 1 and Theorem 1, (6) implies (1). Therefore it remains to prove that (5) implies (6):

Let μ be the collection of all locally finite open covers of X. By virtue of Proposition 2 it suffices to show that for each $\mathscr{A} \in \mu$ there exists a sequence $\beta = (\mathscr{A}_n)_{n\in\mathbf{N}}$ in μ such that $\mathscr{A}_n$ refines $\operatorname{int}_\beta \mathscr{A}$ for some $n \in \mathbf{N}$. Consider an arbitrary $\mathscr{A} \in \mu$. Assuming (5) there exists a locally finite open F_σ-refinement $\mathscr{B} = (B(i))_{i\in I}$ of $\mathscr{A}$. For each $i \in I$ let $(B(i, n))_{n\in\mathbf{N}}$ be a sequence of closed subsets of X such that

$$B(i) = \cup\{B(i, n)|\ n \in \mathbf{N}\}.$$

If $n \in \mathbf{N}$ and $x \in X(n) = \cup\{B(i, n)|\ i \in I\}$, define

$$A(x, n) = \cap\{B(i)|\ i \in I, x \in B(i, n)\} \\ \cap (X\backslash\cup\{B(i, n)|\ i \in I, x \notin B(i, n)\}).$$

Since $\mathscr{B}$ is locally finite, every $A(x, n)$ is an open set containing x. It is easy to see that for each $n \in \mathbf{N}$ the collection

$$\mathscr{A}_n = \{B(i)\backslash X(n)|\ i \in I\} \cup \{A(x, n)|\ x \in X(n)\}$$

is a locally finite open cover of X, i.e., $\beta = (\mathscr{A}_n)_{n \in \mathbf{N}}$ is a sequence in μ. Since $\mathscr{A}_1$ refines $\operatorname{int}_\beta \mathscr{A}$, the proof is complete.

An open cover $\mathscr{B} = (B(i))_{i \in I}$ of a topological space X is called *dissectable* if there exists a function

$$D: I \times \mathbf{N} \rightarrow \mathscr{P}(X)$$

with the following properties:

(i) $B(i) = \bigcup \{D(i, n) | \, n \in \mathbf{N}\}$ for each $i \in I$;

(ii) $T(D, n) = \bigcup \{D(i, n) | \, i \in I\}$ is closed for each $n \in \mathbf{N}$;

(iii) for each $n \in \mathbf{N}$ and for each $x \in T(D, n)$ the set

$$\bigcap \{B(i) | \, i \in I, x \in D(i, n)\}$$
$$\bigcap (X \backslash \bigcup \{D(i, n) | \, i \in I, x \notin D(i, n)\})$$

is a neighbourhood of x. A function $D: I \times \mathbf{N} \rightarrow \mathscr{P}(X)$ satisfying (i)–(iii) is called a *dissection* of $\mathscr{B}$. Every open cover of a developable space is dissectable [**4**]. In [**6**] it was shown that every countable open cover of a perfect space (i.e., closed sets are G_δ's) is dissectable. Using Theorem 2, we can now prove the following:

THEOREM 3. *The following properties of a topological space X are equivalent*:

(1) *X is D-normal.*

(2) *Every countable point-finite open cover of X is kernel-normal.*

(3) *For every countable point-finite open cover $(A(n))_{n \in \mathbf{N}}$ of X there exists a dissectable open cover $(B(n))_{n \in \mathbf{N}}$ of X such that $B(n) \subset A(n)$ for each $n \in \mathbf{N}$.*

Proof. (1) implies (2): Let $\mathscr{A} = (A(n))_{n \in \mathbf{N}}$ be a point-finite open cover of X. By virtue of Theorem 2 there exists a D-open cover $(B(n))_{n \in \mathbf{N}}$ of X such that $B(n) \subset A(n)$ for each $n \in \mathbf{N}$. Hence, for each $n \in \mathbf{N}$ there exists a continuous mapping $f_n: X \rightarrow Y_n$ into a developable space Y_n and an open subset $U(n)$ of Y_n such that $f^{-1}[U(n)] = B(n)$. Denote by τ' the initial topology on X with respect to $(f_n: X \rightarrow Y_n)_{n \in \mathbf{N}}$. Since it is well known that developability is invariant under the formation of initial topologies with respect to countable families of mappings, the space (X, τ') is developable. Therefore $\mathscr{A}$ is kernel-normal (according to Proposition 1).

That (2) implies (3) follows from Proposition 1 and the fact that every open cover of a developable space is dissectable [**4**]. Clearly (3) implies condition (2) of Theorem 1, which completes the proof.

If $\mathscr{A}$ is an open cover of a topological space X, a continuous mapping $f: X \rightarrow Y$ is called an *$\mathscr{A}$-mapping* if there exists an open cover $\mathscr{B}$ of Y such that $\{f^{-1}[B] | \, B \in \mathscr{B}\}$ refines $\mathscr{A}$. In [**18**] C. M. Pareek has characterized those topological spaces which admit an $\mathscr{A}$-mapping onto a developable space for every open cover $\mathscr{A}$. Using the fact that the T_0-identification of an arbitrary topological space yields an open mapping, we obtain from Theorem 2, Theorem 3, and Proposition 1 the following characterization of D-normal spaces in terms of $\mathscr{A}$-mappings.

THEOREM 4. *For a topological space X the following properties are equivalent*:

(1) *X is D-normal.*

(2) *For every locally finite open cover $\mathscr{A}$ of X there exists an $\mathscr{A}$-mapping from X onto a developable T_1-space.*

(3) *For every countable point-finite open cover $\mathscr{A}$ of X there exists an $\mathscr{A}$-mapping from X onto a developable T_1-space.*

4. Hereditarily D-normal spaces. A topological space is called *hereditarily D-normal* if every subspace is D-normal. There exist D-normal spaces which are not hereditarily D-normal. In fact, even subspaces of normal topological spaces need not be D-normal. If $P(\omega_i + 1)$ is the subspace of the ordinal $\omega_i + 1$ obtained by deleting all those points to which a sequence converges, then $P(\omega_1 + 1) \times P(\omega_2 + 1)$ is a normal P-space (i.e., a normal space in which every G_δ-set is open) and the subspace of $P(\omega_1 + 1) \times P(\omega_2 + 1)$ obtained by deleting the point (ω_1, ω_2) is a non-normal P-space [**16**]. It follows from Theorem 1 that every D-normal P-space is normal, hence this subspace cannot be D-normal. On the other hand, the following theorem implies that every perfect space is hereditarily D-normal.

THEOREM 5. *For a topological space X the following properties are equivalent:*

(1) *X is hereditarily D-normal.*

(2) *Every open subspace of X is D-normal.*

(3) *For every pair A, B of separated subsets of X there exists a pair F, G of separated G_δ-sets such that $A \subset F$, $B \subset G$, and*

$$(\mathrm{cl}\, F \backslash F) \cup (\mathrm{cl}\, G \backslash G) \subset \mathrm{cl}\, A \cap \mathrm{cl}\, B.$$

(4) *For every pair A, B of arbitrary subsets of X there exists a pair D, E of F_σ-sets such that $A \subset D$, $B \subset E$,*

$$\mathrm{int}\, D \cup E = X = D \cup \mathrm{int}\, E, \textit{ and}$$

$$D \cap E \cap (\mathrm{cl}\, A \cup \mathrm{cl}\, B) \subset \mathrm{cl}\, A \cap \mathrm{cl}\, B.$$

Proof. Using condition (1) of Theorem 1 it is easy to see that (1), (2) and (3) are equivalent. The proof that (3) and (4) are equivalent is similar to the proof of the main theorem of [**13**].

https://doi.org/10.4153/CJM-1981-108-1 Published online by Cambridge University Press

The following results show that hereditarily D-normality can replace the much stronger property of complete normality in some nice theorems of M. Katětov [**15**]. They can be proved by modifying the methods of Katětov using condition (3) of Theorem 5 and the main result of [**7**].

THEOREM 6. *Let X be a T_1-space and Y be an arbitrary topological space such that $X \times Y$ is hereditarily D-normal. Then every countable subset of X is closed or Y is perfect.*

THEOREM 7. *A countably compact Hausdorff space X is metrizable if and only if $X \times X \times X$ is hereditarily D-normal.*

THEOREM 8. *Let X_n be a Hausdorff space for each $n \in \mathbf{N}$ containing at least two points. Then the product space $\prod_{n \in \mathbf{N}} X_n$ is perfect if and only if it is hereditarily D-normal.*

Proof. Since every perfect space is hereditarily D-normal, we only have to prove the reverse implication. Assume that $\prod_{n \in \mathbf{N}} X_n$ is hereditarily D-normal. For each $k \in \mathbf{N}$ the space $\prod_{n>k} X_n$ contains a countable non-closed subset (since it contains the Cantor space). Hence, by virtue of Theorem 6, the space $X_1 \times \ldots \times X_k$ is perfect. Therefore $\prod_{n \in \mathbf{N}} X_n$ is perfect by a theorem of R. W. Heath and E. Michael [**9**].

5. Some properties of D-normal spaces. A subset B of a topological space X is said to be *normally situated in X* if for every open subset U of X containing B there exists an open subset O of X such that $B \subset O \subset U$ and $O = \cup \{O(i) | \ i \in I\}$, where $(O(i))_{i \in I}$ is a family, locally finite in O, of open F_σ-sets of X. It is easy to see that every normally placed subset (in the sense of [**20**]) of a D-normal space is normally situated. Every normally situated subspace of a normal topological space is normal [**19**].

PROPOSITION 3. *Every normally situated subspace of a D-normal topological space is D-normal.*

The proof of Proposition 3 is based on the following easy to prove lemmas:

LEMMA 1. *Every F_σ-subset of a D-normal space is D-normal.*

LEMMA 2. *Let $(A(i))_{i \in I}$ be a locally finite open F_σ-cover of a topological space X such that $A(i)$ is D-normal for every $i \in I$. Then X is D-normal.*

Proof of Proposition 3. Let S be a normally situated subspace of a D-normal space X. Consider a pair A, B of disjoint closed subsets of S. There are closed subsets D, E of X such that $D \cap S = A$ and $E \cap S = B$. Since $S \subset X \backslash (D \cap E)$, there exists an open set O such that $S \subset O \subset X \backslash (D \cap E)$ and $O = \cup \{O(i) | \ i \in I\}$, where $(O(i))_{i \in I}$ is a family, locally finite in O, of open F_σ-sets of X. By virtue of Lemma 1 and Lemma 2 O is D-normal. Therefore there exist disjoint closed G_δ-sets F, G of O such that $D \cap O \subset F$ and $E \cap O \subset G$. Hence $F \cap S$ and $G \cap S$ are disjoint closed G_δ-sets in S such that $A \subset F \cap S$ and $B \subset G \cap S$, which proves that S is D-normal.

Arbitrary coproducts of D-normal spaces are D-normal. This fact is an immediate consequence of Lemma 2. On the other hand even the product of two paracompact Hausdorff spaces need not be D-normal. In fact, in [**1**] K. Alster and R. Engelking provided an example of a paracompact Hausdorff P-space X such that $X \times X$ is a non-normal P-space. Since D-normal P-spaces are normal, $X \times X$ cannot be D-normal. That quotients of D-normal spaces need not be D-normal is a consequence of a result of J. R. Isbell, who proved in [**14**] that every topological space is a quotient of a paracompact Hausdorff space.

Following J. Chaber a topological space is called *subnormal* if for every pair A, B of disjoint closed subsets there exists a pair F, G of disjoint G_δ-sets such that $A \subset F$ and $B \subset G$ [**8**]. Obviously every D-normal space is subnormal, but there exist subnormal Hausdorff spaces which are not D-normal (see for instance [**17**] for a subparacompact, hence subnormal, metacompact Hausdorff space which is not D-normal). However, the following statements can easily be verified.

PROPOSITION 4. (1) *Every closed image of a D-normal space is subnormal.*
(2) *Every closed-and-open image of a D-normal space is D-normal.*

Remark. Our Theorem 1 suggests the question whether there exists a single developable T_1-space $\mathbf{D}$ such that a topological space X is D-normal if and only if every pair of disjoint closed subsets of X can be separated by a continuous mapping f from X into $\mathbf{D}$. Quite recently N. C. Heldermann (using essentially different methods) has answered this question in the affirmative [**10**].

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