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GROUPS WITH ČERNIKOV CONJUGACY CLASSES

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Abstract

The aim of this paper is to prove some embedding theorems for groups with Černikov conjugacy classes. Moreover a characterization of periodic central-by-Černikov groups is given.

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1. Introduction

In [1], we considered groups with polycyclic-by-finite conjugacy classes or PC-groups. A group G is a PC-group if $G/C_G(\langle x^G\rangle)$ is polycyclic-by-finite for each $x \in G$. This definition generalizes one form of the definition of an FC-group and our results in [1] were all similar to known results for FC-groups. Here we consider the dual condition that $G/C_G(\langle x^G\rangle)$ is a Černikov group. Groups satisfying this condition were first studied by Polovickiĭ ([5], [6]) and are usually referred to as CC-groups. The basic result of Polovickiĭ is that G is a CC-group if and only if the normal closure $\langle x^G\rangle$ of each element of G is either a Černikov group or the extension of a Černikov group by an infinite cyclic group. It follows that the periodic CC-groups are precisely the groups which are locally (normal and Černikov) in the sense that they have a local system consisting of normal Černikov subgroups.

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Some of our results hold for all CC-groups but some hold only for periodic CC-groups, the distinction between these two cases being similar to the distinction between FC-groups and locally normal groups.

In PC-groups (and in FC-groups) each ascendant subgroup has ascendancy type at most ω and such subgroups can be characterized by a local subnormality condition. For CC-groups the situation is a little more complicated as even in a Černikov group one may have ascendant subgroups of ascendancy type greater than ω. However we show in Section 2 that serial subgroups of CC-groups are ascendant and have ascendancy type at most 2ω (Theorem 2.5). We also consider locally soluble and locally nilpotent CC-groups determining the possible types of an ascending abelian normal (central) series.

In the theory of FC-groups one of the questions which has received most attention is that of determining when a residually finite periodic FC-group can be embedded in a direct product of finite groups. In Section 3 we consider embedding properties of periodic CC-groups which are residually Černikov. Polovickiĭ [5] showed that Hall's Theorem on countable residually finite periodic FC-groups can be generalized to countable residually Černikov periodic CC-groups. Here we show that a periodic CC-group with trivial centre is isomorphic to a subgroup of a direct product of Černikov groups (Theorem 3.2). It seems likely that further results of this type should be possible as at the present we do not know of any example of a residually Černikov periodic CC-group which cannot be embedded in a direct product of Černikov groups.

In the final section we obtain a characterization of groups G in which $G/Z(G)$ is Černikov. We have not been able to obtain the dual characterization of groups with Černikov derived subgroup. Some of the difficulties associated with that question are discussed in Section 4, but before that it is worth commenting on two of the more fundamental difficulties which arise in CC-groups but not in either FC-groups or PC-groups.

If G is a periodic FC-group and G/H is finite, then there is a finite normal subgroup F of G such that $FH = G$. Similarly, if G is a PC-group and G/H is polycyclic-by-finite then, since G/H is finitely generated, there is a polycyclic-by-finite normal subgroup P of G such that $PH = G$ ([1], Corollary 2.3). However, there is no analogous result for periodic CC-groups. For, let $G = \mathrm{Dr}_{n=1}^{\infty}\langle g_n\rangle$ where $\langle g_n\rangle$ is cyclic of order p^n. If $H = \langle g_n^{-1}g_{n+1}^{p} : n = 1, 2, \dots\rangle$, then $G/H \simeq C_{p^\infty}$ but every subgroup of G of finite rank is finite.

If F is a finite subgroup of the periodic FC-group G, then F^G is finite. Similarly if P is a polycyclic-by-finite subgroup of the PC-group G then P^G is polycyclic-by-finite. Again there is no corresponding result for periodic CC-groups as the following example shows.

EXAMPLE 1.1. Let $Q = \mathrm{Dr}_{n=0}^{\infty} Q_n$, where $Q_n = \langle x_{n,m} : m = 1, 2, \ldots, x_{n,1}^p = 1, x_{n,m+1}^p = x_{n,m} \rangle$ is a quasicyclic p-group. For each $r = 1, 2, \ldots$ define the automorphism α_r of Q by

$$\begin{aligned} x_{0,m}\alpha_r &= x_{0,m}x_{r,m-r} && \text{if } m > r, \\ x_{0,m}\alpha_r &= x_{0,m} && \text{if } m \leq r, \\ x_{r,m}\alpha_r &= (x_{r,m})^{-1} && \\ x_{s,m}\alpha_r &= x_{s,m} && \text{if } s \neq 0, r. \end{aligned}$$

Then $\alpha_r^2 = 1$ and $\alpha_r\alpha_s = \alpha_s\alpha_r$. Thus $A = \langle \alpha_r : r = 1, 2, \ldots \rangle$ is an elementary abelian 2-group. Consider the split extension G of Q by A. Then $\langle \alpha_r^G \rangle = Q_r\langle \alpha_r \rangle$ and each element of Q is contained in a finite normal subgroup of the form $\langle x_{0,m}, x_{1,m-1}, \ldots, x_{m-1,1} \rangle$. Thus each element of G is contained in a Černikov normal subgroup and so G is a periodic CC-group. But Q_0 is a Černikov subgroup of G and $Q_0^G = Q$ has infinite rank.

These difficulties mean that the proofs given here are not the same as those for PC-groups or FC-groups although the results are based on the known results for FC-groups. We shall refer to [8] for results required on FC-groups. For other results and notation we refer to [7].

2. Locally soluble CC-groups and ascendant subgroups

A number of our results in this section are made simpler by considering $D(G)$, the join of all periodic divisible abelian normal subgroups of the CC-group G. It is well-known that $D(G)$ is a periodic divisible abelian normal subgroup of G (see [7, Part 1, Lemma 4.46]).

THEOREM 2.1. (i) *A locally soluble CC-group has an ascending abelian normal series of type at most* ω.

(ii) *A locally nilpotent CC-group has an ascending central series of type at most* 2ω.

PROOF. Let $\overline{G} = G/D(G)$. For each $x \in G$, $\langle x^G \rangle$ is Černikov-by-cyclic and so $\langle \overline{x}^{\overline{G}} \rangle$ is finite-by-cyclic. Hence $\langle \overline{x}^{\overline{G}} \rangle$ is a finite extension of a central cyclic subgroup and so $\overline{G}$ is a PC-group.

(i) By [1, Theorem 3.2], $\overline{G}$ has an ascending abelian normal series of type at most ω and hence so does G.

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(ii) By [1, Theorem 3.2], the locally nilpotent PC-group $\overline{G}$ has an ascending central series of type at most ω. If P is a Černikov abelian normal subgroup, then $\Omega_n(P)$ is a finite normal subgroup of G and so is contained in $Z_\omega(G)$. Hence $D(G) \leq Z_\omega(G)$ and so $Z_{2\omega}(G) = G$.

The following example shows that there are locally nilpotent CC-groups with hypercentrality length 2ω.

EXAMPLE 2.2. Let $c \geq 1$ and let H be a finite p-group which is nilpotent of class c. Consider the wreath product $G_c = C_{p^\infty} \wr H$ and let B be the base group of G_c. Since B is not central in G_c, G_c is not nilpotent. Hence $B = Z_\omega(G_c)$ and G_c is a hypercentral Černikov group with hypercentrality length $\omega + c$. Let $G = \mathrm{Dr}_{c=1}^{\infty} G_c$; then G is hypercentral with hypercentrality length 2ω.

It is well-known that a nilpotent Černikov group is centre-by-finite and so is an FC-group. There is a similar result for nilpotent periodic CC-groups.

THEOREM 2.3. *A nilpotent periodic CC-group G is an FC-group and hence $D(G) \leq Z(G)$.*

PROOF. Let $x \in G \backslash Z(G)$ and consider $N = \langle x^G \rangle$. Then N is a nilpotent Černikov group and so is centre-by-finite. Therefore N' is finite and the Černikov abelian normal subgroup N/N' of G/N' is the union of a countable chain of finite characteristic subgroups. Since $N = \langle x^G \rangle$ it follows that N is finite, as required.

Theorem 2.1(ii) and Example 2.2 suggest that ascendant subgroups may be rather more complicated than in PC-groups and indeed Example 2.2 can be adapted to give ascendant subgroups of ascending type 2ω.

EXAMPLE 2.4. Let $c \geq 1$ and let H be a finite p-group having a subnormal subgroup K of defect c. Consider the wreath product $G_c = C_{p^\infty} \wr H$ and let B be the base group of G_c; then $B = \mathrm{Dr}_{h \in H} P_h$, where $P_h \simeq C_{p^\infty}$. Consider the subgroup $L_c = \langle P_1, K \rangle = K \ltimes P_1^K$ where $P_1^K = \mathrm{Dr}_{k \in K} P_k$. If L_c is subnormal in G_c, then $[B, L_c, \ldots, L_c] \leq L_c \cap B = P_1^K$ and hence $[B, L_c] \leq P_1^K$ [7, Part 1, Lemma 3.13]. But it is clear that K does not act trivially on B/P_1^K and so L_c is not subnormal in G_c. Thus L_c is ascendant in G_c with ascendancy type $\omega + r$ for some integer r. Let

$$L_c = K_0 \lhd K_1 \lhd \cdots \lhd K_\omega \lhd \cdots \lhd K_{\omega+r} = G_c.$$

Assume that $K_n \leq BK$. If $K_{n+1} \nleq BK$, then there is an element $g = bh \in N_{G_c}(K_n) \backslash BK$. Thus $h \in H \backslash K$ and $P_h = P_1^h = P_1^g \leq K_n$. But then L_c is

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subnormal in $\langle P_h, L_c\rangle$ and, as before, $[P_h^K, L_c, \dots, L_c] \le P_h^K \cap L_c = 1$ and hence K acts trivially on P_h^K. This is clearly false and so $K_{n+1} \le BK$. It follows that $K_\omega \le BK$ and $BK_\omega = BK$. Since BK is subnormal with defect c in G_c it follows that K_ω has defect at least c in G_c. Therefore L_c has ascendancy type $\omega + c$ in G_c.

Finally, let $G = \mathrm{Dr}_{c=1}^{\infty} G_c$ and let $L = \mathrm{Dr}_{c=1}^{\infty} L_c$. Then L is an ascendant subgroup of G of type 2ω.

THEOREM 2.5. *Let H be a serial subgroup of the CC-group G. Then H is ascendant in G of type at most 2ω.*

PROOF. Let T be the join of all Černikov normal subgroups of G, so that $T \ge G'$ [7, Part 1, Theorem 4.36]. We show first that if N is any normal subgroup of G contained in T, then HN is a serial subgroup of HT. There is a local system $\{X_i : i \in I\}$ of T consisting of Černikov normal subgroups of G. Then $\{HNX_i : i \in I\}$ is a local system of subgroups of HT. Since H is serial in HX_i and X_i satisfies the minimal condition, H is ascendant in HX_i. It follows that HN is ascendant in HNX_i for each $i \in I$ and hence HN is a serial subgroup of HT [3, Lemma 1].

The subgroup T has an ascending series of normal subgroups of G with finite factors. It follows from the above that H is ascendant in HT and hence in G.

Now let D be the join of all periodic divisible abelian normal subgroups of G. Then HD/D is an ascendant subgroup of the PC-group G/D and so $HD \lhd^{\omega} G$ [1, Theorem 3.3]. The proof of the result is completed by showing that $H \lhd^{\omega} HD$. Let $M/H\cap D$ be a minimal normal subgroup of $HD/H\cap D$ contained in $D/H \cap D$. Then $M/H \cap D$ is finite and so centralized by H. If $S_n/H \cap D$ is the nth term of the $HD/H \cap D$-socle series of $D/H \cap D$, then $\{HS_n : n = 0, 1, 2, \dots\}$ forms an ascending series from H to HD.

3. Embedding in direct products of Černikov groups

One of the problems in considering residually finite PC-groups was that a polycyclic-by-finite subgroup H could not necessarily be avoided by a normal subgroup K with polycyclic-by-finite factor group [1, Example 4.1]. There is no such difficulty with residually Černikov CC-groups.

LEMMA 3.1. *Let H be a Černikov normal subgroup of the residually Černikov CC-group G. Then there is a normal subgroup K of G such that $H\cap K = 1$ and G/K is a Černikov group.*

PROOF. Let S be the socle of H; then S is finite and so there is a normal subgroup K of G such that $K \cap S = 1$ and G/K is Černikov. It follows that $K \cap H = 1$.

Polovickiĭ [5] showed that a countable residually Černikov periodic CC-group can be embedded in a direct product of Černikov groups. The proof uses the above lemma and follows the proof of Hall's Theorem on countable residually finite periodic FC-groups. If G is a periodic CC-group, then $G/Z(G)$ is residually Černikov and hence, if $G/Z(G)$ is countable, then it is isomorphic to a subgroup of a direct product of Černikov groups. It is also possible to give a variation of Gorčakov's proof [2] to consider CC-groups without centre.

THEOREM 3.2. *Let G be a CC-group with $Z(G) = 1$. Then G is isomorphic to a subgroup of a direct product of Černikov groups with trivial centres.*

PROOF. First we show that G is periodic. Let x be any element of G; then $\langle x^G \rangle$ is Černikov-by-cyclic and so it contains only finitely many (finite) minimal normal subgroups of G. Since G is residually Černikov, there is a normal subgroup H of G such that G/H is Černikov and $H \cap \langle x^G \rangle$ intersects each minimal normal subgroup of G trivially. If $a \in H \cap \langle x^G \rangle$, then $[a, G]$ is a Černikov normal subgroup of G which is contained in $H \cap \langle x^G \rangle$, so that $[a, G] = 1$ and $a = 1$. Therefore $H \cap \langle x^G \rangle = 1$ and x has finite order.

Let N be a minimal normal subgroup of G. Then N is finite and there is a normal subgroup K of G such that $K \cap N = 1$ and G/K is Černikov. Choose $M \triangleleft G$ maximal such that $M \geq K$ and $M \cap N = 1$. Then NM/M is the unique minimal normal subgroup of the Černikov group G/M and NM/M is non-central, so that G/M has trivial centre.

For each minimal normal subgroup N_i of G choose M_i as above. Then $\bigcap_{i \in I} M_i$ intersects each minimal normal subgroup of G trivially and so

$$\bigcap_{i \in I} M_i = 1.$$

Therefore we have an embedding

$$\rho : G \to \prod_{i \in I} (G/M_i)$$

where $\rho(g) = (gM_i)_{i \in I}$. We write ρ_i for the natural projection of G onto G/M_i.

Suppose that $\rho_i(g) \notin C_{G/M_i}(N_iM_i/M_i)$. Then there is an element $n_i \in N_i$ such that $[\rho_i(g), n_iM_i] \neq 1$ and hence $[g, n_i] \neq 1$. But $[g, n_i] \in \langle g^G \rangle \cap N_i$ and, since $[g, n_i] \neq 1$, $N_i \leq \langle g^G \rangle$. But $\langle g^G \rangle$ is Černikov and so contains only finitely many minimal normal subgroups of G. Hence

(1) *for each* $g \in G$, $\rho_i(g) \in C_{G/M_i}(N_iM_i/M_i)$ *for all but finitely many* i.

Now consider $x \in G$ and let $X = \langle x^G \rangle$. We show that $\rho(X)$ has only finitely many non-trivial components and so ρ embeds G into $\mathrm{Dr}_{i \in I}(G/M_i)$. Suppose not, then there are infinitely many non-trivial components which we shall denote by $\rho_i(X)$, $i = 1, 2, \ldots$. First, we show that

(2) *there are integers* $1 = s_1 < s_2 < \cdots$ *and elements* $g_1, g_2, \ldots \in G$ *such that*

$$[\rho_{s_n}(g_n), N_{s_n}M_{s_n}/M_{s_n}] \neq 1,$$
$$[\rho_i(g_n), N_iM_i/M_i] = 1 \quad \textit{for all } i \geq s_{n+1}.$$

Suppose that we have constructed $1 = s_1 < \cdots < s_n$ and the elements $g_1, \ldots, g_n$. By (1) there is an s_{n+1} ($< s_n$) such that $\rho_i(g_n) \in C_{G/M_i}(N_iM_i/M_i)$ for all $i \geq s_{n+1}$. But $N_{s_{n+1}}M_{s_{n+1}}/M_{s_{n+1}}$ is non-central and so there is an element g_{n+1} such that $[\rho_{s_{n+1}}(g_{n+1}), N_{s_{n+1}}M_{s_{n+1}}/M_{s_{n+1}}] \neq 1$. Then we can obtain all the s_n and g_n satisfying (2).

We now need to strengthen the last condition on the elements g_n and we show that

(3) *there are integers* $1 = t_1 < t_2 < \cdots$, *where each* t_i *is equal to* s_j *for some* j, *and elements* $h_1 = g_1$, $h_2, h_3, \ldots$ *such that*

$$h_i \in \langle g_1, g_2, \ldots \rangle, \quad [\rho_{t_n}(h_n), N_{t_n}M_{t_n}/M_{t_n}] \neq 1,$$
$$[\rho_{t_m}(h_n), N_{t_m}M_{t_m}/M_{t_m}] = 1, \quad \textit{for all } m \neq n.$$

Suppose that we have defined $1 = t_1 < \cdots < t_n$ and the elements $h_1, \ldots, h_n \in \langle g_1, g_2, \ldots \rangle$ such that

$$[\rho_{t_i}(h_i), N_{t_i}M_{t_i}/M_{t_i}] \neq 1,$$
$$[\rho_{t_j}(h_i), N_{t_j}M_{t_j}/M_{t_j}] = 1, \quad \text{for all } j \neq i,\ 1 \leq i, j \leq n.$$

Then there is an integer k such that $h_1, \ldots, h_n \in \langle g_1, \ldots, g_k \rangle$. Infinitely many of the elements $g_{k+1}, g_{k+2}, \ldots$ have the same action on each of the finite factors $N_{t_i}M_{t_i}/M_{t_i}$ $(i = 1, \ldots, n)$. If these elements are $g_{l(1)}$, $g_{l(2)}, \ldots$, then put $h_{n+1} = g_{l(1)}^{-1}g_{l(2)}$. Then h_{n+1} centralizes each $N_{t_i}M_{t_i}/M_{t_i}$

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$(i = 1, \dots, n)$ but does not centralize $N_{s_{l(2)}} M_{s_{l(2)}} / M_{s_{l(2)}}$. Put $t_{n+1} = s_{l(2)}$ so that h_{n+1} centralizes $N_{t_i} M_{t_i} / M_{t_i}$ for $i = 1, \dots, n$ but not for $i = n+1$. Also $h_1, \dots, h_n$ centralize $N_i M_i / M_i$ for all $i \geq s_{k+1}$ and so certainly centralize $N_{t_{n+1}} M_{t_{n+1}} / M_{t_{n+1}}$. Thus we can define the integers t_n and elements h_n satisfying (3).

Now consider the subgroup $C_G(X)\langle h_{n+1}, \dots \rangle$. This subgroup centralizes $N_{t_n} M_{t_n} / M_{t_n}$, but $C_G(X)\langle h_n, h_{n+1}, \dots \rangle$ does not centralize $N_{t_n} M_{t_n} / M_{t_n}$. Therefore we have an infinite descending chain of subgroups

$$C_G(X)\langle h_1, h_2, \dots \rangle > C_G(X)\langle h_2, h_3, \dots \rangle > \dots$$

and this is contrary to $G/C_G(X)$ being Černikov.

Thus $\rho(X)$ has only finitely many non-trivial components, as required.

As for FC-groups, these results seem to indicate that it is only the centre of a periodic residually Černikov CC-group which can prevent it from being embeddable in a direct product of Černikov groups. However, there is a significant difference in the situation here. It is easy to give examples of residually finite periodic abelian groups which are not subgroups of direct products of finite groups; for example, the torsion subgroup of $\prod_{n=1}^{\infty} C_{p^n}$. But every periodic abelian group can be embedded in a periodic divisible abelian group which is, of course, a direct product of Černikov groups. Thus any example of a periodic residually Černikov CC-group which is not isomorphic to a subgroup of a direct product of Černikov groups would have to be non-abelian, uncountable and have non-trivial centre. We know of no such example, and it seems possible that there may be a far more general result here.

4. Centre-by-Černikov groups

B. H. Neumann [4] considered various conditions equivalent to a group having finite derived subgroup or having finite central factor group. Results of this type were also obtained in [1] for groups in which the derived subgroup or the central factor group is polycyclic-by-finite.

We have been unable to obtain a corresponding characterization of groups with Černikov derived subgroup but can prove a similar result for groups whose central factor group is Černikov.

If $Y \leq X \leq G$ we say that the interval $[X/Y]$ satisfies the minimal condition if the set of subgroups H of G such that $Y \leq H \leq X$ satisfies the minimal condition. The conditions we consider involve the interval $[H^G/H]$

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satisfying the minimal condition or H/H_G being Černikov (where H_G denotes the core of H in G).

LEMMA 4.1. *Let G be a periodic hyper-((locally soluble)-by-finite) group. If, for each cyclic subgroup X of G, the interval $[X^G/X]$ satisfies the minimal condition, then G is a CC-group.*

PROOF. Let x be any element of G. Since $\langle x\rangle$ is finite and $[\langle x^G\rangle/\langle x\rangle]$ satisfies the minimal condition, it follows that $K = \langle x^G\rangle$ satisfies $\mathrm{Min}-n$.

Let N be a minimal normal subgroup of K; then $N = \langle a^K\rangle$ for some a, and since $[\langle a^K\rangle/\langle a\rangle]$ satisfies the minimal condition, N satisfies $\mathrm{Min}-n$. It follows that N is the direct product of finitely many isomorphic simple groups and hence it is finite. Therefore $\langle x^G\rangle$ is hyperfinite. Since $\langle x^G\rangle$ satisfies $\mathrm{Min}-n$, it is a Černikov group [7, Part 1, p. 148].

We cannot omit the periodicity condition from Lemma 4.1. For, let G be the extension of a p^∞-group A by an automorphism α of infinite order. Then $G' = A$ is Černikov and so $[H^G/H]$ satisfies the minimal condition for all $H \le G$. But $\langle\alpha^G\rangle = G$ and so $G/C_G(\langle\alpha^G\rangle)$ is not Černikov. Thus G is not a CC-group.

The other condition does not imply that G is a CC-group, even if G is periodic and soluble. For, let p be an odd prime and let G be the extension of an infinite elementary abelian p-group A by its automorphism α which inverts each element. If $H \le G$, then $H \cap A \lhd G$ and so H/H_G is finite. But G is not a CC-group, since $\langle\alpha^G\rangle = G$.

We shall actually consider a combination of these two conditions, namely that H^G/H_G is Černikov for all $H \le G$. However we need to split the proof of the main theorem into special cases which sometimes only depend on one of these conditions.

LEMMA 4.2. *Let H be a normal subgroup of the periodic CC-group G such that G/H and H' are Černikov. Then G' is Černikov.*

PROOF. Clearly we may assume that H is abelian. If $h \in H$, then $\langle h^G\rangle$ is an abelian Černikov normal subgroup of G. Since a periodic group of automorphisms of a Černikov abelian group is finite [7, Part 1, Corollary to 3.29.2], it follows that $G/C_G(\langle h^G\rangle)$ is finite. Let R/H be the finite residual of G/H; then R/H is divisible abelian and so $R \le C_G(\langle h^G\rangle)$. Therefore $R \le C_G(H)$ and $H \le Z(R)$. Since $R/Z(R)$ is a Černikov group, R' is also Černikov [7, Part 1, Theorem 4.23]. Since G/R is finite there is a Černikov normal subgroup C of G such that $CR = G$. Then $G' \le R'C$ is Černikov.

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COROLLARY 4.3. *If H is a normal subgroup of the periodic CC-group G such that G/H is Černikov, then G'/H' is Černikov.*

PROOF. Apply the lemma to G/H'.

LEMMA 4.4. *Let H be a normal subgroup of the periodic CC-group G such that G/H and $H/Z(H)$ are Černikov. Then $G/Z(G)$ is Černikov.*

PROOF. Let $R/Z(H)$ be the finite residual of the Černikov group $G/Z(H)$. For each $h \in Z(H)$, $\langle h^G \rangle$ is a Černikov subgroup of $Z(H)$ and since $\langle h^G \rangle$ is abelian it must be finite. Thus $G/C_G(\langle h^G \rangle)$ is finite and so $R \leq C_G(\langle h^G \rangle)$. Therefore $R \leq C_G(Z(H))$ and $Z(H) \leq Z(R)$; in particular $R/Z(R)$ is Černikov. But G/R is finite and so there is a normal subgroup $K = \langle k_1, \dots, k_n \rangle^G$ of G such that $KR = G$. Hence $G/C_G(K)$ is Černikov and so $G/(C_G(K) \cap Z(R))$ is Černikov. But $C_G(K) \cap Z(R)$ centralizes $KR = G$ and so $C_G(K) \cap Z(R) \leq Z(G)$.

Our results are all of the type: If G' is not Černikov or $G/Z(G)$ is not Černikov then there is a subgroup H of G such that $[H^G/H]$ does not satisfy the minimal condition or H/H_G is not Černikov.

LEMMA 4.5. *Let G be a periodic CC-group with G' Černikov and $G/Z(G)$ not Černikov. Then there is an abelian subgroup A of G such that A/A_G is not Černikov.*

PROOF. Let $H = C_G(G')$; then G/H is a periodic group of automorphisms of the Černikov group G' and so is Černikov. By Lemma 4.4, $H/Z(H)$ is not Černikov. Now H is nilpotent of class two and so, by Lemma 2.3, H is a periodic FC-group. Since $H/Z(H)$ is infinite, there is an abelian subgroup A of H such that A/A_H is infinite [8, Theorem 7.20].

Since H' is Černikov, $H' \cap A$ is a Černikov subgroup of A and so there is a subgroup X of A such that $X \cap H' = 1$ and A/X is Černikov. Since $[X_H, H] \leq X \cap H' = 1$ we have $X_H = X \cap Z(H)$. If X/X_H is Černikov, then $X/X \cap Z(H)$ is Černikov and hence $A/A \cap Z(H)$ is Černikov. But $H/Z(H)$ is residually finite and so $A/A \cap Z(H)$, as a residually finite Černikov group, is finite. But this is contrary to A/A_H being infinite and so X/X_H is not Černikov and X is the required subgroup.

LEMMA 4.6. *Let G be a periodic CC-group with $G'/G' \cap Z(G)$ not Černikov. Then there is a subgroup X of G such that X/X_G is not Černikov.*

Proof. We show that there are elements $a_1, b_1, a_2, b_2, \ldots$ of G such that

$$[a_i, a_j] = [b_i, b_j] = [a_i, b_j] = 1 \quad \text{if } i \neq j,$$
$$[a_i, b_i] = c_i \notin Z(G),$$
$$\langle a_1, b_1, \ldots, a_{i-1}, b_{i-1}\rangle^G \cap \langle a_i, b_i\rangle^G \leq Z(G).$$

Suppose that we have defined the elements $a_1, b_1, \ldots, a_{i-1}, b_{i-1}$ satisfying the above conditions. Let

$$C = C_G(\langle a_1, b_1, \ldots, a_{i-1}, b_{i-1}\rangle^G),$$

so that G/C is Černikov. Since $\langle a_1, b_1, \ldots, a_{i-1}, b_{i-1}\rangle^G$ is a Černikov normal subgroup of G and $G/Z(G)$ is residually Černikov, it follows from Lemma 3.1 that there is a normal subgroup K of G such that G/K is Černikov and $\langle a_1, b_1, \ldots, a_{i-1}, b_{i-1}\rangle^G \cap K \leq Z(G)$. Let $W = K \cap C \lhd G$; then G/W is Černikov, W centralizes $\langle a_1, b_1, \ldots, a_{i-1}, b_{i-1}\rangle^G$ and $\langle a_1, b_1, \ldots, a_{i-1}, b_{i-1}\rangle^G \cap W \leq Z(G)$.

By Corollary 4.3, G'/W' is Černikov and so $W'/W' \cap Z(G)$ is not Černikov. So there are elements $a_i, b_i \in W$ such that $[a_i, b_i] = c_i \notin Z(G)$. Note that $\langle a_i, b_i\rangle^G \leq W$ and so $\langle a_1, b_1, \ldots, a_{i-1}, b_{i-1}\rangle^G \cap \langle a_i, b_i\rangle^G \leq Z(G)$. Thus we can define the elements a_i, b_i for all $i = 1, 2, \ldots$.

Let $A = \langle a_1, a_2, \ldots\rangle$ and note that $A_G \geq A \cap Z(G)$. Let $\overline{x}$ denote the element $x(A \cap Z(G)) \in G/A \cap Z(G)$ and for $X \geq A \cap Z(G)$, let $\overline{X} = X/A \cap Z(G)$. Then $\overline{A} = \mathrm{Dr}_{i=1}^{\infty}\langle \overline{a}_i\rangle$. Consider the conjugates of A under the elements b_i,

$$A^{b_i} = \langle a_1, \ldots, a_{i-1}, a_i c_i, a_{i+1}, \ldots\rangle \geq A \cap Z(G).$$

Thus $\overline{A^{b_i}} = (\mathrm{Dr}_{j\neq i}\langle \overline{a}_j\rangle) \times \langle \overline{a}_i \overline{c}_i\rangle$. If $\langle \overline{a}_i\rangle \neq \langle \overline{a}_i \overline{c}_i\rangle$ for infinitely many i, say $i(1), i(2), \ldots$, then $\overline{A}/\overline{A} \cap \bigcap_{j=1}^{\infty} \overline{A^{b_{i(j)}}}$ has a factor group isomorphic to

$$\mathop{\mathrm{Dr}}_{j=1}^{\infty}(\langle \overline{a}_{i(j)}\rangle / \langle \overline{a}_{i(j)}\rangle \cap \langle \overline{a}_{i(j)} \overline{c}_{i(j)}\rangle)$$

and so A/A_G is not Černikov.

Therefore, we may assume that $\langle \overline{a}_i\rangle = \langle \overline{a}_i \overline{c}_i\rangle$ for all but finitely many i. But then $a_i c_i = a_i^k z$ for some integer k and some $z \in Z(G)$, and so $c_i \in C_G(a_i)$. Applying the same argument to the subgroup $B = \langle b_1, b_2, \ldots\rangle$ we may also assume that $c_i \in C_G(b_i)$ for all but finitely many i. Thus there is an integer s such that $c_i \in C_G(\langle a_i, b_i\rangle)$ for all $i \geq s$ and so the group $H = \langle a_s, b_s, a_{s+1}, b_{s+1}, \ldots\rangle$ is nilpotent of class two.

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Also $c_i \notin H \cap Z(G)$ and, if we denote $c_i(H \cap Z(G))$ by $\overline{c}_i$, $\langle \overline{c}_s, \overline{c}_{s+1}, \ldots \rangle = \mathrm{Dr}_{i=s}^{\infty} \langle \overline{c}_i \rangle$. By replacing each a_i by an appropriate power we may assume that $\langle \overline{c}_i \rangle$ has prime order p_i. Let $\overline{H} = H/H \cap Z(G)$. Then $\overline{H}$ is nilpotent of class two, $\overline{H}' = \mathrm{Dr}_{i=s}^{\infty} \langle \overline{c}_i \rangle$ and so $\overline{H}/Z(\overline{H})$ is a direct product of groups of prime order.

By Theorem 2.3, $\overline{H}$ is a periodic FC-group with $\overline{H}/Z(\overline{H})$ infinite. Hence there is an abelian subgroup $\overline{X}$ of $\overline{H}$ such that $\overline{X}/\overline{X}_{\overline{H}}$ is infinite ([8], Theorem 7.20). But $\overline{X}_{\overline{H}} \geq \overline{X} \cap Z(\overline{H})$ and so $\overline{X}/\overline{X}_{\overline{H}}$ is a direct product of cyclic groups of prime order and so is not Černikov. If $\overline{X} = X/H \cap Z(G)$, then X/X_H is not Černikov and hence X/X_G is not Černikov.

Theorem 4.7. *Let G be a periodic CC-group such that $G/Z(G)$ is not Černikov. Then G has a subgroup X such that X/X_G is not Černikov.*

Proof. By Lemma 4.6 we may assume that $G'/G' \cap Z(G)$ is Černikov. Applying Lemma 4.5 to $G/G' \cap Z(G)$ we may assume that

$$(G/G' \cap Z(G))/Z(G/G' \cap Z(G))$$

is Černikov. But $Z(G/G' \cap Z(G)) = Z_2(G)/G' \cap Z(G)$ and so $G/Z_2(G)$ is Černikov. If Y is the centre of $Z_2(G)$ then $Z_2(G)/Y$ is not Černikov by Lemma 4.4, and so we may assume that $G = Z_2(G)$ is nilpotent of class two and hence is an FC-group.

We show that there are elements $a_1, b_1, a_2, b_2, \ldots$ of G such that

$$[a_i, a_j] = [b_i, b_j] = [a_i, b_j] = 1 \quad \text{if } i \neq j,$$
$$[a_i, b_i] = c_i \text{ has prime order } p_i,$$
$$\langle c_1, \ldots, c_n \rangle = \langle c_1, \ldots, c_{n-1} \rangle \times \langle c_n \rangle.$$

Suppose that we have defined the elements $a_1, b_1, \ldots, a_{i-1}, b_{i-1}$ satisfying the above conditions. Now $\langle c_1, \ldots, c_{i-1} \rangle$ is a finite subgroup of G' and so there is a Černikov factor group G'/L such that $L \cap \langle c_1, \ldots, c_{i-1} \rangle = 1$. The factor group G/L has Černikov derived subgroup G'/L. If $G/C_G(G/L)$ is not Černikov then by Lemma 4.5 there is an abelian subgroup A/L of G/L such that A/A_G is not Černikov. Therefore we may assume that $G/C_G(G/L)$ is Černikov. Also $G/C_G(\langle a_1, b_1, \ldots, a_{i-1}, b_{i-1} \rangle^G)$ is finite. Let $K = C_G(G/L) \cap C_G(\langle a_1, b_1, \ldots, a_{i-1}, b_{i-1} \rangle^G)$, so that G/K is Černikov and, by Lemma 4.4, $K/Z(K)$ is not Černikov. Therefore we can choose elements $a_i, b_i \in K$ such that $[a_i, b_i] = c_i$ has prime order p_i. Since $c_i \in L$, we have $\langle c_1, \ldots, c_{i-1} \rangle \cap \langle c_i \rangle = 1$. Thus we can define the elements a_i, b_i for all i.

Finally, let $H = \langle a_1, b_1, \ldots \rangle$ so that $H' = \langle c_1, c_2, \ldots \rangle \leq Z(H)$ and $H/Z(H)$ is an infinite abelian group in which each p-component is elementary abelian. By [8, Theorem 7.20], there is an abelian subgroup A of H such that A/A_H is infinite. But $A_H \geq A \cap Z(H)$ and so each p-component of A/A_H is elementary abelian. Hence A/A_H is not a Černikov group and so A/A_G is not Černikov.

COROLLARY 4.8. *Let G be a periodic hyper-((locally soluble)-by-finite) group. Then $G/Z(G)$ is Černikov if and only if H^G/H_G is Černikov for each subgroup H of G.*

PROOF. If $G/Z(G)$ is Černikov then so is G' [7, Part 1, Theorem 4.23]. Let $H \leq G$; then

$$H \cap Z(G) \leq H_G \leq H \leq H^G \leq HG'.$$

It follows that H/H_G is Černikov and hence HG'/H_G is Černikov.

Conversely, suppose that H^G/H_G is Černikov for each subgroup H of G. By Lemma 4.1, G is a CC-group and the result now follows from Theorem 4.7.

There are a number of open questions related to the above result. First, is it sufficient to impose conditions only on abelian subgroups? This is the case in the corresponding result for FC-groups and for PC-groups [1] but is not the case if we consider FC-groups with $|G/Z(G)| < \mathfrak{m}$, where $\mathfrak{m}$ is some uncountable cardinal. Also, in the corresponding results for FC-groups and PC-groups there is the equivalent condition that $|G : N_G(H)|$ is finite or the interval $[G/N_G(H)]$ satisfies the maximal condition. The minimal condition on $[G/N_G(H)]$ seems more difficult to work with.

However, the main gap in the results given above is that we have not been able to prove that if G' is not Černikov then there is a subgroup H such that $[H^G/H]$ does not satisfy the minimal condition. The main case to be dealt with here would be that in which G is nilpotent of class two (and hence an FC-group). We can easily obtain a subgroup A such that $A^G \geq AC$ and C is an infinite abelian group. The difficulty is in obtaining a subgroup C of infinite rank. This problem does not occur in the proof for FC-groups as it is easy to deal with the situation in which C has a section isomorphic to C_{p^∞}. Perhaps we should point out that the argument given in the last paragraph of page 144 of [8] is incorrect as it only refers to subgroups of C isomorphic to C_{p^∞}.

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