A "Cubical" Universe

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§1. Einstein[1] has studied a universe in which the time coordinate t is uncurved and the spatial section is the surface of a sphere

$$x^2 + y^2 + z^2 + w^2 - a^2 = 0 \tag{1}$$

in four dimensions. Some interest attaches to the case where this surface is replaced by

$$f(x, y, z, w) \equiv x^{2n} + y^{2n} + z^{2n} + w^{2n} - a^{2n} = 0 \tag{2}$$

where n is a positive integer.

We can examine the distribution of matter necessary, on Einstein's law of gravitation, to produce such a closed space. As n increases[2], (2) approximates more and more closely to the four-dimensional cube whose sixteen vertices are the points $(\pm a, \pm a, \pm a, \pm a)$. Thus the Gaussian curvature is small except in the neighbourhood of these corners, so that the matter necessary to give rise to this curvature is also largely concentrated at the corners. Hence the interest of this particular model is that it reproduces to some extent an essential feature of actual space in which the matter is more or less symmetrically concentrated into isolated nebulae or galaxies. It is possible to estimate the total mass, so that we can see the effect of varying the degree of concentration of matter in universes of the same general shape and can compare these with the uniform distribution of matter corresponding to (1).

§2. We have to study the cylindrical surface $f = 0$ in the five euclidean dimensions x, y, z, w, t. The spatial section (2) is everywhere convex, lies entirely inside the cube whose vertices are

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[1] Eddington, *Mathematical Theory of Relativity* (1924), § 67.

[2] Such approximations have been discussed in a recent communication to the Society by Professor J. E. A. Steggall.

$(\pm a, \pm a, \pm a, \pm a)$ and touches it at its intersections with the coordinate axes, and passes through the points $(\pm 2^{-1/n} a, \pm 2^{-1/n} a, \pm 2^{-1/n} a, \pm 2^{-1/n} a)$. These we may call the "corners" of the surface since, as n increases, they approach the corners of the cube.

The principal radii of curvature at the point (x, y, z, w, t) are $1/k_i$ $(i = 1, 2, 3, 4)$ given by

$$- k_i (f_x^2 + f_y^2 + f_z^2 + f_w^2 + f_t^2)^{\frac{1}{2}} = \lambda_i, \tag{3}$$

where λ_i is a root of the quartic equation

$$\begin{vmatrix} f_{xx} - \lambda & f_{xy} & f_{xz} & f_{xw} & f_{xt} & f_x \\ f_{yx} & f_{yy} - \lambda & f_{yz} & f_{yw} & f_{yt} & f_y \\ f_{zx} & f_{zy} & f_{zz} - \lambda & f_{zw} & f_{zt} & f_z \\ f_{wx} & f_{wy} & f_{wz} & f_{ww} - \lambda & f_{wt} & f_w \\ f_{tx} & f_{ty} & f_{tz} & f_{tw} & f_{tt} - \lambda & f_t \\ f_x & f_y & f_z & f_w & f_t & 0 \end{vmatrix} = 0. \tag{4}$$

This reduces to

$$\mu \Sigma' x^{4n-2} (y^{2n-2} - \mu)(z^{2n-2} - \mu)(w^{2n-2} - \mu) = 0, \tag{5}$$

writing

$$\lambda = 2n(2n-1)\mu, \tag{6}$$

where the summation Σ' is over the coordinates x, y, z, w alone.

Now we require the Gaussian curvature G given by[1]

$$G = 2(k_1 k_2 + k_1 k_3 + k_1 k_4 + k_2 k_3 + k_2 k_4 + k_3 k_4). \tag{7}$$

From the ratio of the coefficients of μ^2 and μ^4 in (5) we have therefore

$$G = 2(2n-1)^2 \Sigma' x^{4n-2} (y^{2n-2} z^{2n-2} + y^{2n-2} w^{2n-2} + z^{2n-2} w^{2n-2}) / (\Sigma' x^{4n-2})^2 \tag{8}$$

using (3) and (6) and substituting for $\Sigma' f_x^2$. Also, since by (5) $k_4 = 0$, we have $G_{44} = 0$.

Einstein's law of gravitation, neglecting[2] his cosmical constant λ, gives a density of matter (or energy) ρ where, since G_4 is zero,

$$8\pi\rho = 8\pi \; T_4^4 = \tfrac{1}{2} G. \tag{9}$$

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[1] Eddington, *op. cit.*, § 65.

[2] When n is large our space is almost "flat" except at the corners, so that it cannot take account of λ which would require a non-zero curvature everywhere. See § 5 below.

We see then from the form of (8) that given δ, it is possible by choosing n sufficiently large to make ρ as small as we please at distances greater than δ from a "corner." Thus the matter is concentrated at the corners and the concentration increases with n.

§3. To find the total (relative) mass we have to integrate ρ through the three-dimensional volume given by the surface of the four-dimensional figure (2). We employ the substitution

$$x^n = a^n \cos\theta \equiv a^n c_1, \quad y^n = a^n \sin\theta\cos\phi \equiv a^n s_1 c_2,$$
$$z^n = a^n \sin\theta\sin\phi\cos\psi \equiv a^n s_1 s_2 c_3, \quad w^n = a^n \sin\theta\sin\phi\sin\psi \equiv a^n s_1 s_2 s_3, \quad (10)$$

for positive x, y, z, w. We have $0 \leqslant \theta, \phi, \psi \leqslant \pi/2$, and the whole surface consists of sixteen such segments. The volume element appropriate to ρ is given, for example, by

$$dv = (\Sigma' f_x^2)^{\frac{1}{2}}\, dy\, dz\, dw / f_x, \quad (11)$$

i.e.

$$dv = (\Sigma' x^{4n-2})^{\frac{1}{2}} s_1^{-1+3/n} c_1^{-1+1/n} s_2^{-1+2/n} c_2^{-1+1/n} s_3^{-1+1/n} c_3^{-1+1/n} d\theta\, d\phi\, d\psi / n^3 \quad (12)$$

on changing variables. We omit temporarily the factors in a.

We are going to work with large n, so we can write approximately[1]

$$\Sigma' x^{4n-2} = c_1^4 + s_1^4 [c_2^4 + s_2^4 (c_3^4 + s_3^4)]$$
$$= 1 - 2s_1^2 c_1^2 - 2s_1^4 s_2^2 c_2^2 - 2s_1^4 s_2^4 s_3^2 c_3^2 = 1 - \Theta \text{ (say)}, \quad (13)$$

and we have $0 \leqslant \Theta \leqslant 3/4$

Hence the total (relative) mass M is

$$M = \int \rho\, dv$$

taken over the whole space. From (8), (9) and (12) this gives

$$M = \frac{2}{\pi}\frac{(2n-1)^2}{n^3} \int_0^{\pi/2} s_1^{-1+3/n} c_1^{-1+1/n} d\theta \int_0^{\pi/2} s_2^{-1+2/n} c_2^{-1+1/n} d\phi \int_0^{\pi/2} s_3^{-1+1/n} c_3^{-1+1/n}$$
$$\frac{\Sigma' \{x^{4n-2}(y^{2n-2}z^{2n-2} + y^{2n-2}w^{2n-2} + z^{2n-2}w^{2n-2})\}}{(\Sigma' x^{4n-2})^{3/2}} d\psi. \quad (14)$$

By symmetry each of the four terms in Σ' in the numerator will make the same contribution; so we retain only the first and insert a factor 4.

[1] Since, for example, the difference between $c_3^{4-2/n}$ and c_3^4 is appreciable only when this term is small compared with $s_3^{4-2/n}$ or s_3^4 and *vice versa*. The error is of order $1/n$.

Changing variables in this expression and using the approximation (13) we have

$$M = \frac{8}{\pi}\frac{(2n-1)^2}{n^3}\int_0^{\pi/2} s_1^{3-1/n}\,c_1^{3-1/n}\,d\theta\int_0^{\pi/2} s_2^{-1+2/n}\,c_2^{-1+1/r}\,d\phi\int_0^{\pi/2} s_3^{-1+1/n}\,c_3^{-1+1/n}$$

$$[s_2^{2-2/n}\,c_2^{2-2/n}\,(s_3^{2-2/n}+c_3^{2-2/n})+s_2^{4-4/n}\,s_3^{2-2/n}\,c_3^{2-2/n}]\left[1+\frac{3}{2}\Theta+\frac{15}{8}\Theta^2+\frac{35}{16}\Theta^3+\ldots\right]d\psi. \quad (15)$$

It is here legitimate to integrate term by term. The leading term in the triple integral in (15) is then

$$\frac{1}{8}\cdot\frac{\Gamma(2-1/2n)\,\Gamma(2-1/2n)}{\Gamma(4-1/n)}\left[\frac{2\,\Gamma(1)\,\Gamma(1-1/2n)}{\Gamma(2-1/2n)}\cdot\frac{\Gamma(1/2n)\,\Gamma(1-1/2n)}{\Gamma(1)}\right.$$
$$\left.+\frac{\Gamma(2-1/n)\,\Gamma(1/2n)}{\Gamma(2-1/2n)}\cdot\frac{\Gamma(1-1/2n)\,\Gamma(1-1/2n)}{\Gamma(2-1/n)}\right]. \quad (16)$$

Now by taking $\Gamma(2-1/2n)=\Gamma(2)$, etc., and $\Gamma(1/2n)=2n$ we shall be neglecting only terms of order $1/n$ of those retained. We suppose n large and make this approximation. The term (16) then gives $n/8$. Performing similar calculations for the succeeding powers of θ we find for the triple integral in (15) the value

$$\frac{n}{8}\left[1+\frac{3}{4}+\frac{1}{2}+\frac{31}{96}+\frac{33}{160}+\ldots\right]. \quad (17)$$

Extrapolating for the remaining part of the series we obtain an approximation to the sum, which is an absolute constant independent of n, giving for (17) the result

$$3{\cdot}2\,n/8. \quad (18)$$

To this order we obtain for M from (15), after reintroducing the factor a, the value

$$M = 12{\cdot}8a/\pi. \quad (19)$$

To this order the total volume of space V is just the surface of the four-dimensional cube, *i.e.*

$$V = 64\,a^3. \quad (20)$$

The mean density $\bar{\rho}$ is then

$$\bar{\rho} = 1{\cdot}06/M^2. \quad (21)$$

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The "spherical" case ($n = 1$) gives for the corresponding qualities the values[1]

$$M' = 3\pi a / 4; \qquad V' = 2\pi^2 a^3; \qquad \bar{\rho}' = 0{\cdot}66 / M'^2. \qquad (22)$$

§ 4. We can now draw the following conclusions:

(i) When the concentration of matter has proceeded far enough (n sufficiently large) further concentration of the same total (relative) mass does not, to a first approximation, affect the shape and size of the universe. This is shown by the fact that (19), (20), (21) do not contain n, and it is a result that might have been expected.

(ii) From (21), (22) we have if $M = M'$ then $\bar{\rho} = 1{\cdot}6\bar{\rho}'$. Thus for given total (relative) mass the whole change from uniform distribution to infinite concentration at sixteen symmetrically placed points alters the mean density by a factor less than 2.

We have derived these results only for special models of the universe. But the physical nature of the case should allow a more general application. If so the result (ii) is of interest as showing that in estimating the size of the universe it will not affect the order of magnitude to assume a uniform distribution of matter.

§ 5. The components G_{11}, G_{22}, G_{33} of the curvature tensor could, if necessary, be derived from the roots of (5). They will give the stress-system in the matter, but this will not be expected to approximate to any natural system of forces. Indeed it follows from the fact that the relative density ρ is less than the proper density $\rho_0 \,(= G / 8\pi)$ that the stress must be a tension and not a pressure.

The two weaknesses of the model are this negative pressure and the omission of the cosmical constant λ. If we could include λ we should probably simultaneously remove the first defect. Actually it is the existence of λ that makes it particularly necessary to study *closed* spaces.

The difficulties in studying non-homogeneous universes lie in being able to discover a suitable metric and then in being able to carry out the integration for the total mass. As far as I know, no attempt has yet been published, so it seems that in spite of its inadequacy the present model may perhaps serve to indicate how the size of space depends on the distribution of matter in it. We have

[1] These results are for a "spherical" space in which the curvature is due entirely to the matter present, Einstein's cosmical constant being neglected.

compared homogeneous and non-homogeneous universes in which the curvature depended wholly on the matter present. If we could compare cases where it depends partially on λ we should expect similar results still to hold.

As far as the model is successful, it represents space as closed and containing sixteen nebulae. An observer on one of them would see the six nearest in orthogonal directions and eight others symmetrically placed. The remaining one, that diametrically opposite in the four-dimensional representation, will be seen in all directions, providing a continuous (though not uniform) background in the sky.