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CONSTANT MEAN CURVATURE SURFACES IN HOMOGENEOUSLY REGULAR 3-MANIFOLDS

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We establish several theorems concerning properly embedded constant mean curvature surfaces (*cmc*-surfaces) in homogeneously regular 3-manifolds, when the mean curvature H is large.

1. Introduction

Henceforth N will denote an orientable homogeneously regular 3-manifold. This means there is some positive R so that the geodesic balls of N of radius R, centred at any point of N, are embedded, and in these balls, all the sectional curvatures are bounded by some constant; the constant independent of the point of N where the balls are centred.

We shall first prove a diameter estimate for complete immersed (strongly) stable *cmc*-surfaces Σ in N, provided H is large (depending only on N). Here Σ may have boundary and our result says there are positive constants C_1, C_2 such that whenever Σ is a stable complete *cmc*-surface in N with $H \geqslant C_1$, then the intrinsic distance of any point of Σ to $\partial\Sigma$, is at most C_2. In particular, when $\partial\Sigma = \emptyset$, then such a Σ must be compact. The idea behind the proof of this theorem originates in Doris Fisher-Colbries theorem on stable minimal surfaces [1]. For *cmc* surfaces in $\mathbb{R}^3$, it is implicit in Lopez and Ros [3], and in $\mathbb{R}^3$, for any $H \neq 0$, it is proved in [5]. Also see [4], where it is proved in $\mathbb{H}^2 \times \mathbb{R}$, when $H > 1/\sqrt{3}$.

We shall use the diameter estimate theorem to prove a maximum principle at infinity for properly embedded H-surfaces in N, provided H is large. The proof is inspired by the authors' proof, with Antonio Ros, in $\mathbb{R}^3$ for $H \neq 0$. The important difference is the compact case. In $\mathbb{R}^3$, one can not have an H-surface inside the mean convex component determined by another H-surface. One can translate one surface until it touches the other and the usual maximum principle shows this is not possible. In N, one must do something else.

Notice the maximum principle is certainly not true for H small, even in the compact case. For example, consider a surface of revolution M as in Figure 1.

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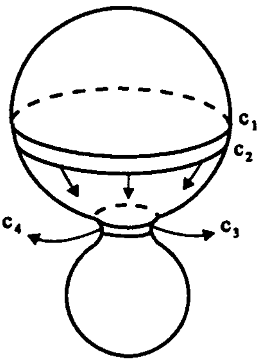

Figure 1

Here C_1 and C_3 are disjoint geodesics and C_2, C_4 are curves of the same geodesic curvature. C_4 is in the component determined by C_2 whose boundary is mean convex (C_2). This gives counterexamples in dimension two, both for curvature zero and curvature non-zero. One can take $N = M \times S^1$ to obtain counterexamples in dimension 3.

We shall also prove that a closed (weakly) stable H-surface in N has genus at most 3 when H is large

2. The diameter estimate theorem for stable H-surfaces in N

Theorem 1. *Let N be a complete Riemannian 3-manifold with uniformly bounded scalar curvature $S(x)$. Let H and $c > 0$, satisfy*

$$3H^2 + S(x) \geqslant c, \qquad \text{for } x \in N.$$

Then if Σ is a stable H-surface immersed in N, one has, for $x \in \Sigma$:

$$d_\Sigma(x, \partial\Sigma) \leqslant \frac{2\pi}{\sqrt{3c}}.$$

Here d_Σ is the intrinsic distance in Σ.

Proof: The stability operator L of Σ is:

$$L = \Delta + |A|^2 + \mathrm{Ric}(n),$$

where A is the second fundamental form of Σ and n a unit normal vector field along Σ. We say that M is *stable* if

$$-\int_M uLu \geqslant 0$$

for any smooth function u with compact support on M. This type of stability is often called strong stability. We rewrite L, introducing the exterior curvature K_e of Σ, the

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intrinsic curvature K_Σ of Σ, the sectional curvature K_s of N of the tangent plane to Σ, and the scalar curvature S of N. We have

$$\begin{aligned} L &= \Delta + |A|^2 + \operatorname{Ric}(n) \\ &= \Delta + (4H^2 - 2K_e) + (S - K_s) \\ &= \Delta + (4H^2 - 2K_e) + S - (K_\Sigma - K_e) \\ &= \Delta + 4H^2 - K_e + S - K_\Sigma \\ &= \Delta + 3H^2 + (H^2 - K_e) + S - K_\Sigma. \end{aligned}$$

Since $H^2 - K_e \geqslant 0$, we have:

$$L - \Delta + K_\Sigma \geqslant 3H^2 + S.$$

Hence if u is a positive function on Σ, we have:

$$L(u) - \Delta(u) + K_\Sigma u \geqslant (3H^2 + S)u \geqslant cu.$$

Since Σ is stable, there is a smooth positive u on Σ with $L(u) = 0$, ([1]). Thus, by the previous inequality:

$$-\Delta u + K_\Sigma u \geqslant cu.$$

Let $B_R(p) = \{q \in \Sigma \mid d_\Sigma(p,q) \leqslant R\}$, and let ds denote the metric of Σ.

Make a conformal change of the metric on $B_R(p)$, $d\widetilde{s} = u ds$, and let γ be a minimising geodesic for the $d\widetilde{s}$ metric from p to $\partial B_R(p)$.

Let $a = \int_\gamma ds \geqslant R$, and $\widetilde{R} = \int_\gamma d\widetilde{s}$. Since γ is a minimising geodesic one has

$$0 \leqslant \int_0^{\widetilde{R}} \left(\left(\frac{d\phi}{d\widetilde{s}} \right)^2 - \widetilde{K}\phi^2 \right) d\widetilde{s},$$

for all ϕ defined on $[0, \widetilde{R}]$, $\phi(0) = \phi(\widetilde{R}) = 0$.

We have

$$\begin{aligned} \widetilde{K} &= \frac{1}{u^2}(K_\Sigma - \Delta \ln u), \quad \Delta = \Delta_{ds}, \\ \Delta \ln u &= \frac{1}{u^2}\left(u\Delta u - |\nabla u|^2\right). \end{aligned}$$

Hence

$$\begin{aligned} \widetilde{K} &= \frac{1}{u^2}\left(K_\Sigma - \frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2}\right), \\ \phi^2 \widetilde{K} u &= \frac{\phi^2}{u^2}\left(K_\Sigma u - \Delta u + \frac{|\nabla u|^2}{u}\right) \\ &\geqslant \frac{\phi^2}{u^2}\left(cu + \frac{|\nabla u|^2}{u}\right). \end{aligned}$$

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In particular $\widetilde{K} > 0$.

Rewriting the stability inequality:

$$0 < \int_0^{\widetilde{R}} \left(\left(\frac{d\phi}{d\widetilde{s}}\right)^2 - \widetilde{K}\phi^2 \right) d\widetilde{s} = \int_0^a \left(\frac{d\phi}{d\widetilde{s}}\right)^2 u\, ds - \int_0^a \widetilde{K}\phi^2 u\, ds$$

$$0 < \int_0^a \widetilde{K}\phi^2 u\, ds < \int_0^a \left(\frac{d\phi}{d\widetilde{s}}\right)^2 u\, ds$$

$$= \int_0^a \left(\frac{d\phi}{ds}\right)^2 \frac{ds}{u}$$

We know

$$\widetilde{K}\phi^2 u \geqslant \frac{\phi^2}{u}\left(c + \frac{|\nabla u|^2}{u^2}\right),$$

so

$$\int_0^a \frac{\phi^2}{u}\left(c + \frac{|\nabla u|^2}{u^2}\right) ds < \int_0^a \left(\frac{d\phi}{ds}\right)^2 \frac{ds}{u}.$$

Now replace ϕ by $\phi\sqrt{u}$:

$$(\phi\sqrt{u}) = (d\phi)\sqrt{u} + \phi\frac{1}{2\sqrt{u}}\, du$$

Denote $. = d/(ds)$.

$$\left(\frac{d(\phi\sqrt{u})}{ds}\right)^2 = u\dot{\phi}^2 + \frac{\phi^2\dot{u}^2}{4u} + \phi\dot{\phi}\dot{u}$$

$$\int_0^a \phi^2\left(c + \frac{|\nabla u|^2}{u^2}\right) ds \leqslant \int_0^a \left(\dot{\phi}^2 + \frac{\phi^2\dot{u}^2}{4u^2} + \frac{\phi\dot{\phi}\dot{u}}{u}\right) ds$$

$$\int_0^a \left(\frac{-3\phi^2\dot{u}^2}{4u^2} + \dot{\phi}^2 - c\phi^2 + \frac{\dot{u}\phi\dot{\phi}}{u}\right) ds \geqslant 0$$

(here we used $|\nabla u|^2 = \dot{u}^2 + u_r^2 \geqslant \dot{u}^2$). Let

$$a = \frac{\sqrt{6}}{2}\frac{\dot{u}}{u}\phi, \quad b = \frac{\sqrt{6}}{3}\dot{\phi}, \quad (a^2 + b^2 \geqslant 2ab),$$

then

$$\frac{3}{4}\frac{\dot{u}^2\phi^2}{u^2} + \frac{\dot{\phi}^2}{3} \geqslant \frac{\dot{u}}{u}\phi\dot{\phi},$$

$$\int_0^a \left(\frac{4}{3}\dot{\phi}^2 - c\phi^2\right) ds \geqslant 0.$$

Integration by parts ($u = \dot{\phi}$, $v = \phi$),

$$\int_0^a \left(\frac{4}{3}\ddot{\phi} + c\phi\right)\phi\, ds \leqslant 0.$$

Choose $\phi = \sin(\pi s a^{-1})$, $s \in [0, a]$,

$$\int_0^a \left[c - \frac{4\pi^2}{3a^2}\right] \sin^2(\pi s a^{-1})\, ds \leqslant 0,$$

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$$c \leqslant \frac{4\pi^2}{3a^2} \quad \text{and } a \geqslant R, \text{ so}$$
$$c \leqslant \frac{4\pi^2}{3R^2}.$$

Hence $d_\Sigma(p, \partial\Sigma) \leqslant (2\pi)/\sqrt{3c}$, and Theorem 1 is proved. ∎

3. Large mean curvature

In this section we shall discuss several properties of *cmc*-surfaces in N, for H sufficiently large.

PROPERTY 1. There is a $c > 0$, such that whenever Σ is a connected *cmc* embedded compact surface in N, with $H \geqslant c$, then Σ separates N into 2 components.

PROOF: Let $x \in \Sigma$ and consider the geodesic γ starting at x, normal to Σ at x, and going into the mean convex side of M at x (locally). Let $\Sigma(t)$ denote the parallel surfaces to Σ, starting at $\Sigma(0) = \Sigma$ (in a neighbourhood of x) and going into the mean convex side of Σ for $t > 0$. These local surfaces are defined for t small, and they are orthogonal to γ where they are defined. The first variation formula for the mean curvature yields:

$$\frac{d}{dt} H_t(x)|_{t=0} = L(1)(x),$$

L the stability operator. We have

$$\begin{aligned} L(1)(x) &= \Delta(1) + \Big(|A|^2(x) + \mathrm{Ric}\big(n(x)\big)\Big) \\ &= 4H(x)^2 - 2K_e(x) + \big(S(x) - K_s(x)\big) \\ &\geqslant 2H(x)^2 + \big(S(x) - K_s(x)\big). \end{aligned}$$

Since N is homogeneously regular, $|S(x) - K_s(x)|$ is bounded (independent of x), so there exist $\delta > 0$, $c > 0$, such that $2H(x)^2 + \big(S(x) - K_s(x)\big) \geqslant \delta$, whenever $H = H(x) \geqslant c$.

Hence the parallel surfaces to Σ along γ at x have strictly increasing mean curvature. This remains true along γ, as long as the parallel surfaces are non-singular along γ; for example, if γ has no focal points of Σ at x. We refer the reader to the paper by Galloway and Rodriguez [**2**] where this type of argument is used.

We claim that this c works in property 1. Suppose not, so Σ is compact, embedded, and $H \geqslant c$. Clearly Σ has a trivial normal bundle in N, Σ has a mean convex side (where the mean curvature vector points) and a concave side (the other side).

Consider all paths β in N starting at a point x of Σ, entering the mean convex side of Σ near x, and meeting Σ again for the first time, at a point $y \in \Sigma$, coming from the concave side of Σ when arriving at y; see Figure 2.

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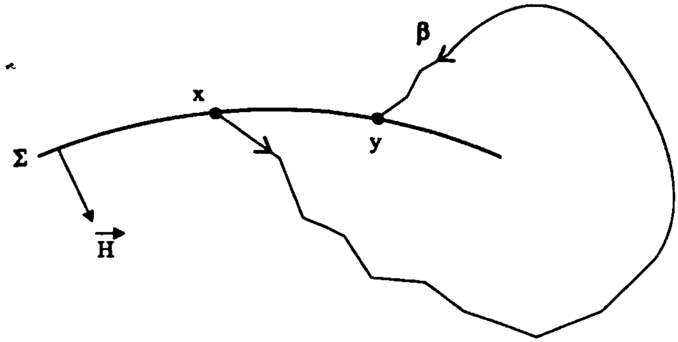

Figure 2

Since Σ is compact and embedded, (and Σ does not separate), the infimum of the lengths of all such paths β is strictly positive. So there exists such a path β that minimises the length of all such paths, going from some point x of Σ to a point y of Σ. Clearly β is a geodesic of N which is orthogonal to Σ at x and y, and β meets Σ exactly at $\{x, y\}$. Also, the fact that β minimises length among such paths implies there are no focal points of Σ at x along β. Thus the parallel surfaces to Σ at x, are defined at every point of β. By our choice of c, their mean curvature is strictly increasing along β, when one goes from x to y.

However the parallel surface to Σ at y, is tangent to Σ at y, locally on the concave side of Σ at y, has mean curvature vector pointing in the same direction as the mean curvature vector of Σ at y, but this parallel surface at y has strictly bigger mean curvature than H. This is a contradiction, and proves property 1. $\square$

PROPERTY 2. Let $\delta > 0$ be less than the injectivity radius of N. There is a constant c (greater than the constant of property 1) such that whenever Σ is a properly embedded H-surface in N with $H \geqslant c$, then

$$d_N(y, \Sigma) \leqslant \delta,$$

for all y in the mean convex component of $N - \Sigma$. In particular, this component W is compact when Σ is compact.

PROOF: Let c_1 be greater that the mean curvature of each geodesic sphere of radius δ, centred at any point of N. N is homogeneously regular so such a c_1 exists. Also choose c_1 larger than the constant of property 1.

Let W be the mean convex component of $N - \Sigma$, and let $y \in W$. If the distance from y to Σ were greater than δ then the geodesic sphere S, of radius δ, centred at y, would be contained in W.

Let β be a path minimising the distance between Σ and S in W. Then β is a geodesic of N, orthogonal to Σ and S at the points $x \in \Sigma$ and $y \in S$, which are the endpoints of β. Since β is minimising, there are no focal points of Σ at x on β. Then the parallel surfaces to Σ along β, exist from x to y. But, as in the proof of property 1, the parallel surface of

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Σ at y has mean curvature strictly bigger than H, hence bigger than δ; a contradiction. This proves the property 2.

REMARK. It is interesting to understand the geometry of such W. It is not hard to see that W is a handlebody of a geodesic graph in N. What type of geodesic graphs are possible? Where are the vertices of such a graph in N? What sort of "balancing" formulas exist? Can the geodesic graph be a triangle? More precisely, can a sequence of $H - tori$ converge to a geodesic triangle as H diverges?

4. THE MAXIMUM PRINCIPLE AT INFINITY

THEOREM 2. *Let N be a orientable homogeneously regular 3-manifold. There is a constant $c > 0$, such that whenever $H \geqslant c$, and M_1, M_2 are properly embedded H-surfaces in N which bound a connected domain W, then the mean curvature vector points out of W along the boundary of W.*

PROOF: Choose c so that the diameter stability estimate holds for $H \geqslant c$ (that is, $3H^2 + S(x) \geqslant c$), and c also large enough so that the parallel surfaces, on the mean convex side, have larger mean curvature (that is, choose x such that $H \geqslant c$ implies $2H^2 + \big(S(x) - K_s(x)\big) > 0$).

Let M_1, M_2 and W satisfy the hypothesis of Theorem 2. Suppose the mean curvature vector of M_1 points into W. We shall show this is impossible.

First suppose M_1 is compact. Since M_2 is proper, there is a minimising geodesic β in W from $x \in M_1$, to $y \in M_2$, β minimises the length of all paths joining a point of M_1 to a point of M_2, in W.

Clearly β is orthogonal to M_1 and M_2 at x and y respectively, and β has no focal points of M_1 at x. Thus the parallel surfaces to M_1 at x, exist along β until y. Since their mean curvature is strictly increasing along β, from x to y, this gives a contradiction, as in the proof of property 1.

Thus we may assume M_1 is not compact. Now the proof proceeds as in the proof of the maximum principle at infinity for H-surfaces in $\mathbb{R}^3$, due to the author and Ros [5]. Since this paper is not yet published, we reproduce the proof here (with minor modifications).

Let $x_1 \in M_1$, $x_2 \in M_2$, and γ be a path in W joining x_1 to x_2. Let $R > 0$ and S be the geodesic disk of M_1 centred at x_1 of radius R, $\Gamma = \partial S$ smooth. Since M_1 is non compact and properly embedded, $\partial S = \Gamma$ leaves any compact set of N for R sufficiently large. Thus $\operatorname{dist}_N(\gamma, \Gamma) \to \infty$, as $R \to \infty$. In particular, for R large, S is not (strongly) stable since the stability diameter estimate fails. So assume R chosen so that S is not stable.

We shall find a smooth stable H-surface $\Sigma \subset W$, $\partial\Sigma = \Gamma$ and Σ homologous to S, $\operatorname{rel}\Gamma$, in W. Then Σ satisfies the stability estimate. But $\Sigma \cap \gamma \neq \emptyset$, since Σ is homologous to S and $S \cap \gamma = \{x_1\}$. This contradicts the fact that $\operatorname{dist}_N(\gamma, \Gamma) \to \infty$ as $R \to \infty$.

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We now show how to find Σ.

Consider bounded open subsets Q of W of finite perimeter, and with $S \subset \partial Q$, $\partial Q \cap M_1 = S$. Let Σ be the free boundary of Q, that is, $\partial Q = S \cup \Sigma$, $\partial\Sigma = \Gamma = \partial S$. Let $A(\Sigma)$ be the area of Σ (the 2-mass of Σ) and $V(Q)$ denote the volume of Q.

Define the functional F on such Q's of finite perimeter by

$$F(Q) = A(\Sigma) + 2HV(Q).$$

A minimum (Q, Σ) of F yields a stable Σ as desired (assuming Σ smooth, $\Sigma - \partial\Sigma \subset$ interior W, $\partial\Sigma = \Gamma$).

Observe first that the mean curvature vector of Σ points out of Q. Suppose not, let $x \in \Sigma$ be a point where $\vec{H}(x)$ points into Q, and let B be a small ball of N centred at x such that $\vec{H}_\Sigma$ points into Q along $\Sigma \cap B$.

We can assume ∂B is mean convex so the domain of B bounded by $\Sigma \cup (\partial B \cap Q)$ is a good barrier for solving the Plateau problem. Let D be a least area surface in this domain with $\partial D = \Sigma \cap \partial B$.

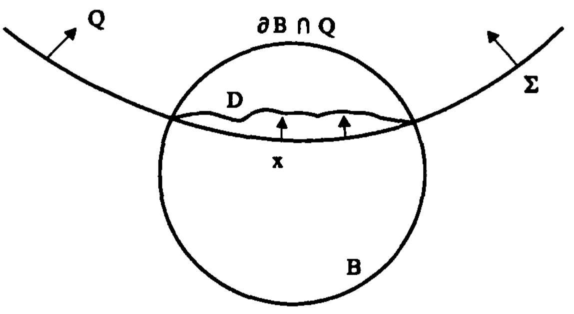

Figure 3

Denote by $\widetilde{Q}$ the domain Q with the domain $\overline{Q}$ removed; $\overline{Q}$ the domain in B bounded by $(\Sigma \cap B) \cup D$. Clearly $F(\widetilde{Q}) < F(Q)$, which contradicts that Q is a minimum of F.

Next we show Σ is stable for the functional G (see below for the definition of G). Suppose Σ were not stable. Then there is a Jacobi function f on Σ, $f > 0$ on int Σ, $f/\partial\Sigma = 0$, and there exists $\lambda < 0$, such that (here L is the stability operator)

$$L(f) + \lambda f = 0 \quad \text{on } \Sigma.$$

So for x in the interior of Σ, $L(f)(x) > 0$. Let $\Sigma(t)$, $t > 0$, t small, be a variation of Σ with compact support whose variation field is the normal field defined by f. Then

$$\left.\frac{d^2G(t)}{dt^2}\right|_{t=0} = -\int_\Sigma fL(f) < 0,$$

and for t small, $t > 0$,

$$G(t) = A\big(\Sigma(t)\big) + 2HVQ(t) < A(\Sigma).$$

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Here $V(Q(t))$ is the algebraic volume between $\Sigma(t)$ and Σ. Since $f > 0$ on interior Σ, and $\Sigma(t)$ is in the mean convex side of Σ (outside Q), the algebraic volume equals the volume of the domain $Q(t)$. Thus

$$F(Q \cup Q(t)) < F(Q),$$

which contradicts that Q minimises F.

It remains to prove a minimum Q of F exists in W as desired.

The minimum of F will be in a compact region of N we now define. Let Σ_{min} be an embedded minimal surface in W, $\partial\Sigma_{\text{min}} = \Gamma$, Σ_{min} minimises area in the homology class of S rel Γ. Let Q_{min} denote the domain in W bounded by $S \cup \Sigma_{\text{min}}$.

Observe that for any domain Q in the class we are considering:

$$F(Q \cap Q_{\text{min}}) \leqslant F(Q).$$

So a minimum of F is contained in Q_{min}.

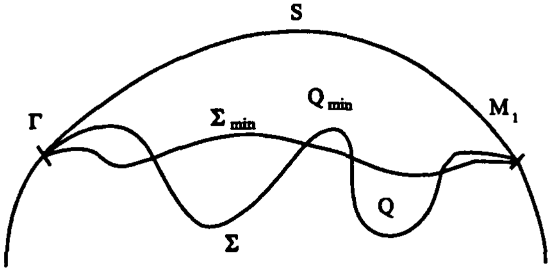

Figure 4

Recall that S is unstable. The same argument we used before with an eigenfunction $f > 0$ on interior Σ, with negative eigenvalue λ, applies to S. This produces a variation $\Sigma_{\text{unst}} \subset W$, $\partial\Sigma_{\text{unst}} = \Gamma$, int $\Sigma_{\text{unst}} \subset \operatorname{int} W$, and $S \cup \Sigma_{\text{unst}}$ bounds a domain $Q_{\text{unst}} \subset W$, with $F(Q_{\text{unst}}) < A(S)$.

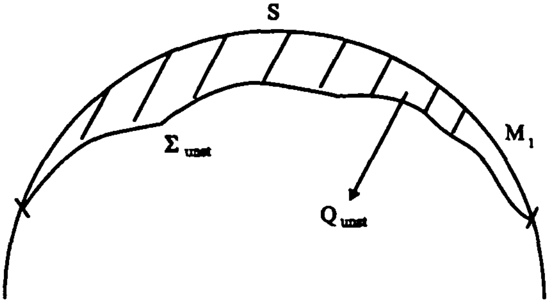

Figure 5

Q_{unst} is foliated by surfaces $\Sigma(\tau)$, $\Sigma(0) = S$, $\Sigma(1) = \Sigma_{\text{unst}}$. The foliation is obtained from the first eigenfunction f of L on S, using the normal variations (in the direction

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of $\vec{H}_{M_1}$) as follows. We can assume 0 is not an eigenvalue of L on S, by perturbing S slightly. Then there is a smooth function v on S satisfying $L(v) = 1$, $v = 0$ on Γ. By the boundary maximum principle, the gradient of f does not vanish on Γ. So, for a small $a > 0$, the function $u = f + av$ satisfies $L(u) \geqslant a > 0$ on $\overline{S}$, $u = 0$ on Γ. Now Σ_{unst} is the graph of u in Q, and $\Sigma_{\text{unst}} \cup S = \partial Q_{\text{unst}}$, Q_{unst} foliated by the surfaces $\Sigma(\tau)$, the graphs of τu, $0 \leqslant \tau \leqslant 1$.

Hence $H_\tau = H\big(\Sigma(\tau)\big)$ is strictly increasing on $\operatorname{int} S$ for τ near 0. So we can assume Σ_{unst} chosen close enough to S so that $H_\tau > H$ in Q_{unst}.

Let X be the unit normal vector field to the foliation $\Sigma(\tau)$, oriented by $\vec{H}$. We have $\operatorname{div} X = -2H_\tau$ in Q_{unst}, hence $\operatorname{div} X < -2H$ for $\tau > 0$.

This last inequality implies that a minimum Q for F, necessarily contains Q_{unst}.

More precisely we have that if for some admissible Q, $Q_{\text{unst}} \not\subset Q$, then $F(Q \cup Q_{\text{unst}}) < F(Q)$.

To see this, since $\operatorname{div} X < -2H$ on Q_{unst}, one has:

$$-2HV(Q_{\text{unst}} - Q) > \int_{Q_{\text{unst}}-Q} \operatorname{div} X = \int_{\partial(Q_{\text{unst}}-Q)} \langle X, \nu\rangle = \int_{S_{\text{unst}}-Q} \langle X, \nu\rangle + \int_{\Sigma\cap Q_{\text{unst}}} \langle X, \nu\rangle.$$

On $S_{\text{unst}} - Q$, $\nu = X$ and $\langle X, \nu\rangle \geqslant -1$ on the other points of the boundary, so

$$2HV(Q_{\text{unst}} - Q) + A(S_{\text{unst}} - Q) < A(\Sigma \cap Q_{\text{unst}}).$$

Hence

$$\begin{aligned} F(Q \cup Q_{\text{unst}}) &= 2H\big(V(Q) + V(Q_{\text{unst}} - Q)\big) + A(S_{\text{unst}} - Q) + A(\Sigma - Q_{\text{unst}}) \\ &< 2HV(Q) + A(\Sigma \cap Q_{\text{unst}}) + A(\Sigma - Q_{\text{unst}}) \\ &= F(Q). \end{aligned}$$

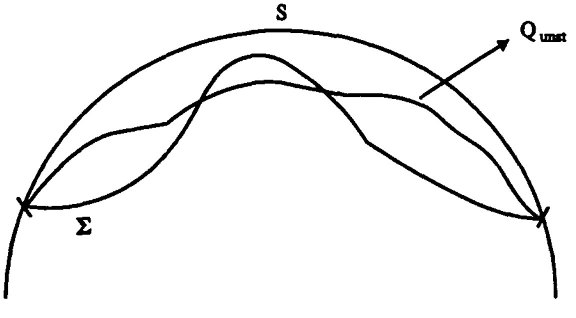

Figure 6

Next consider M_2. Let T be an ε-tubular neighbourhood of M_2 in W such that the parallel surfaces $M_2(t)$ in W, $0 \leqslant t \leqslant \varepsilon$, are smooth and embedded in $T \cap Q_{\min} = E$. Choose Q_{unst} and ε sufficiently small, so that $E \cap Q_{\text{unst}} = \emptyset$.

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We claim that if Q is an admissable domain for F then if $Q \cap E \neq \emptyset$, we have $F(Q - E) < F(Q)$.

There are two cases to consider: the mean curvature vector of M_2 points into W or it points out of W. We shall check the later case and leave the first case to the reader.

By our choice of c and $H \geqslant c$, we know that $H(t) = H\big(M_2(t)\big) < H$ for $0 < t \leqslant \varepsilon$. Let Y be the unit normal vector field to the foliation $M_2(t)$, oriented by the mean curvature vector, so that $\operatorname{div} Y = -2H_t > -2H$, for $t > 0$.

Let $Q(+) = Q \cap E$. By Stokes:

$$-2HV\big(Q(+)\big) < \int_{Q(+)} \operatorname{div} Y = \int_{\partial(Q(+))} \langle Y, \nu \rangle = \int_{Q \cap M_2(\varepsilon)} \langle Y, \nu \rangle + \int_{\Sigma \cap E} \langle Y, \nu \rangle,$$

where ν is the outer conormal to the boundary.

On $M_2(\varepsilon)$, $\nu = -Y$, and $\langle Y, \nu \rangle \leqslant 1$ on $\Sigma \cap E$. Hence

$$-2HV\big(Q(+)\big) + A(M_2(\varepsilon) \cap Q) < A(\Sigma \cap E),$$

and

$$\begin{aligned} F(Q - E) &= 2H\big(V(Q) - V(Q \cap E)\big) + A(\Sigma - E) + A\big(Q \cap M_2(\varepsilon)\big) \\ &< 2HV(Q) + A(\Sigma \cap E) + A(\Sigma - E) \\ &= F(Q). \end{aligned}$$

Thus $F(Q - E) < F(Q)$ whenever $Q \cap E \neq \emptyset$.

Denote by V the closure of the complement in $Q_{\min}$ of Q_{unst} and E. We now show a minimum Q of F exists with the free boundary Σ of Q, contained in V, $\operatorname{int} \Sigma \subset \operatorname{int} W$, $\partial\Sigma = \Gamma$, and Σ a smooth stable H-surface; stable surfaces are smooth.

Let Q_n be a minimising sequence for F. One can approximate Q_n so that (calling the approximation Q_n as well) ∂Q_n is smooth and transverse to the smooth boundary components of V. Then we can construct another minimising sequence $\widetilde{Q}_n$ such that $\widetilde{Q}_n \subset Q_{\min}$, $\widetilde{Q}_n \cap E = \emptyset$, and $Q_{\text{unst}} \subset \widetilde{Q}_n$, for all n. Then geometric measure theory gives a minimum Q of F in V with the free boundary Σ of Q the desired surface. This completes the proof. ∎

Theorem 3. *Let Σ be a closed immersed (weakly) stable H-surface in N. Assume H large so that*

$$4H^2 + S(x) + K_{\text{sect}}(x) \geqslant 0,$$

for all x. Then Σ has genus g at most three.

Proof: The idea of this proof goes back to Lopez and Ros [3], and perhaps earlier. Our point is that the proof works in homogeneously regular 3-manifolds N provided H is large. Now we give the proof.

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Let $\phi\colon \Sigma \to S^2$ be a meromorphic map such that $\deg\phi \leqslant 1 + [(g+1)/2]$, where the bracket denotes the greatest integer function. Composing with a Mobius transformation of S^2, one can suppose $\int_\Sigma \phi = 0$.

Then apply the stability inequality to the three coordinate functions of ϕ to conclude:

$$0 < \int_\Sigma \Big(|\nabla\phi|^2 - \big(|A|^2 + \mathrm{Ric}(n)\big) \Big)$$

we have $|\nabla\phi|^2 = 2\,\mathrm{Jac}(\phi)$, and

$$\begin{aligned} |A|^2 + \mathrm{Ric}(n) &= 4H^2 - K_e + S - K \\ &= 4H^2 + S + K_s - 2K \\ &> -2K. \end{aligned}$$

Hence

$$\begin{aligned} 0 &< 8\pi \deg(\phi) + \int_\Sigma 2K \\ &\leqslant 8\pi\Big(1 + \Big[\frac{g+1}{2}\Big]\Big) + 8\pi(1-g). \end{aligned}$$

Thus $0 < 2 + [(g+1)/2] - g$, and this implies $g \leqslant 3$. ∎

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