GENERIC DERIVATIONS ON ALGEBRAICALLY BOUNDED STRUCTURES

ANTONGIULIO FORNASIERO AND GIUSEPPINA TERZO

Abstract. Let $\mathbb{K}$ be an algebraically bounded structure, and let T be its theory. If T is model complete, then the theory of $\mathbb{K}$ endowed with a derivation, denoted by T^δ, has a model completion. Additionally, we prove that if the theory T is stable/NIP then the model completion of T^δ is also stable/NIP. Similar results hold for the theory with several derivations, either commuting or non-commuting.

§1. Introduction. Let $\mathbb{K}$ be a structure expanding a field of characteristic 0. Recall that $\mathbb{K}$ is *algebraically bounded* if the model-theoretic algebraic closure and the field-theoretic algebraic closure coincide in every structure elementarily equivalent to $\mathbb{K}$. Algebraically closed, real closed, p-adically closed, pseudo-finite fields, and algebraically closed valued fields are examples of algebraically bounded structures; for more details, examples, and main properties, see [10] and Section 2. Van den Dries in his paper introduced a notion of dimension for any definable set with parameters, which is relevant in our context.

Let L be the language of $\mathbb{K}$, and let T be its theory. In order to study derivations on $\mathbb{K}$, we denote by δ a new unary function symbol, and by T^δ the L^δ-theory expanding T by saying that δ is a derivation. Let $\mathbb{K}$ be algebraically bounded. We remark that being algebraically bounded is not a first-order notion (since an ultraproduct of algebraically bounded structures is not necessarily algebraically bounded). We define an L^δ-theory T^δ_g extending T^δ, with three equivalent axiomatizations (see Sections 3 and 7); one of them is given by T^δ, plus the following axiom scheme:

> For every $X \subseteq \mathbb{K}^n \times \mathbb{K}^n$ which is L-definable with parameters, if the dimension of the projection of X onto the first n coordinates, which we denote by $\Pi_n(X)$, is n, then there exists $\bar{a} \in \mathbb{K}^n$ such that $\langle \bar{a}, \delta\bar{a}\rangle \in X$.

One of the main result of the paper is the following:

THEOREM 1.1. *If T is model complete, then T^δ_g is the model completion of T^δ.*

We also endow $\mathbb{K}$ with several derivations $\delta_1, \dots, \delta_m$ and we consider both the case when they commute and when we don't impose any commutativity. We obtain two theories:

$T^{\bar{\delta}}$: the expansion of T saying that the δ_i are derivations which commute with each other.

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$T^{\bar{\delta},nc}$**:** the expansion of T saying that the δ_i are derivations without any further conditions.

Both theories have a model completion (if T is model complete) (see Sections 4 and 5). For convenience, we use $T_g^{\bar{\delta},?}$ to denote either of the model completions, both for commuting derivations and the non-commuting case. Many of the model-theoretic properties of T are inherited by $T_g^{\bar{\delta},?}$:

THEOREM 1.2 (Section 6). *Assume that T is stable/NIP. Then* $T_g^{\bar{\delta},?}$ *is stable/NIP.*

In a work in preparation, we will prove that if T is simple, then $T_g^{\bar{\delta},?}$ is simple (see [33, 36] for particular cases); we will also characterize when $T_g^{\bar{\delta},?}$ is ω-stable. Moreover, we will prove that $T_g^{\bar{\delta},?}$ is uniformly finite. Finally, if $\mathbb{K}$ extends either a Henselian valued field with a definable valuation or a real closed field, then, under some additional assumptions, T is the open core of $T_g^{\bar{\delta},?}$.

On the other hand, in [15] we show that exponential fields do not admit generic derivations (notice that exponential fields are not algebraically bounded).

1.1. A brief model theoretic history. From a model theoretic point of view, differential fields have been studied at least since Robinson [45] proved that the theory of fields of characteristic 0 with one derivation has a model completion, the theory DCF_0 of differentially closed fields of characteristic 0.

Blum gave a simpler sets of axioms for DCF_0, saying that $\mathbb{K}$ is a field of characteristic 0, and, whenever p and q are differential polynomials in one variable, with q not zero and of order strictly less than the order of p, then there exists a in $\mathbb{K}$ such that $p(a) = 0$ and $q(a) \neq 0$ (see [2, 46] for more details). Pierce and Pillay [39] gave yet another axiomatization for DCF_0, which has been influential in the axiomatizations of other structures (see Section 7).

The theory DCF_0 (and its models) has been studied intensively, both for its own sake, for applications, and as an important example of many "abstract" model theoretic properties: it is ω-stable of rank ω, it eliminates imaginaries, it is uniformly finite, etc. For some surveys, see [2, 6, 20, 31, 35].

Models of DCF_0, as fields, are algebraically closed fields of characteristic 0; their study has been extended in several directions. An important extension, which however goes beyond the scope of this article, is Wood's work [55] on fields of finite characteristic.

From now on, all fields are of characteristic 0. More close to the goal of this article is the passage from one derivation to several commuting ones: McGrail [32] axiomatized $\mathrm{DCF}_{0,\mathrm{m}}$ (the model completion of the theory of fields of characteristic 0 with m commuting derivations). While the axiomatization is complicate (see Section 5 for an easier axiomatization, and [27, 38] for alternative ones), from a model theoretic point of view $\mathrm{DCF}_{0,\mathrm{m}}$ is quite similar to DCF_0: its models are algebraically closed (as fields), it is ω-stable of rank ω^m, it eliminates imaginaries, it is uniformly finite, etc.

Moosa and Scanlon followed a different path in [36], where they studied a general framework of fields with non-commuting operators; for this introduction, the relevant application is that they proved that the theory of m non-commuting derivations has a model completion (see [36] and Section 4), which we denote by

$\mathrm{DCF}_{0,\mathrm{m},\mathrm{nc}}$. Here the model theory is more complicate: $\mathrm{DCF}_{0,\mathrm{m},\mathrm{nc}}$ is stable, but not ω-stable; however, it still eliminates imaginaries and it is uniformly finite.

Surprisingly, we can give three axiomatizations for $\mathrm{DCF}_{0,\mathrm{m},\mathrm{nc}}$ which are much simpler than the known axiomatizations for $\mathrm{DCF}_{0,\mathrm{m}}$ (including the one given in this article), see Sections 4 and 7. We guess that the reason why this has not been observed before is that people were deceived by the rich algebraic structure of $\mathrm{DCF}_{0,\mathrm{m}}$.

Indeed, from an algebraic point of view, $\mathrm{DCF}_{0,\mathrm{m}}$ has been studied extensively (see [25] for a starting point) and is much simpler than $\mathrm{DCF}_{0,\mathrm{m},\mathrm{nc}}$. The underlying combinatorial fact is that the free commutative monoid on m generators Θ, with the partial ordering given by $\alpha \preceq \beta\alpha$ for every $\alpha, \beta \in \Theta$, is a well-partial-order (by Dickson's Lemma); this fact is a fundamental ingredient in Ritt–Raudenbush Theorem, asserting that there is no infinite ascending chain of radical differential ideals in the ring of differential polynomials with m commuting derivations with coefficients in some differential field; moreover, every radical differential ideal is a finite intersection of prime differential ideals. Since in models of $\mathrm{DCF}_{0,\mathrm{m}}$ there is a natural bijection between prime differential ideals and complete types, this in turns implies that $\mathrm{DCF}_{0,\mathrm{m}}$ is ω-stable as we mentioned before.

Very different is the situation for the free monoid on m generators Γ, with the same partial ordering. Γ is well-founded, but (when m is at least 2) not a well-partial-order. Given an infinite anti-chain in Γ, it is easy to build an infinite ascending chain of radical differential ideals (in the corresponding ring of non-commuting differential polynomials), and therefore Ritt–Raudenbush does not hold in this situation.

Some limited form of non-commutativity was considered already in [38, 53, 56], where the derivations live in a finite-dimensional Lie algebra.

People have extended DCF_0 in another direction by considering fields which are not algebraically closed: Singer, and later others [4, 5, 41, 44, 52] studied real closed fields with one generic derivation, and [43] extended to m commuting derivations (see also [14] for a different approach); [8, 17–19] studied more general topological fields with one generic derivation. In [42] the author studied fields with m independent orderings and one generic derivation and in [14] they studied o-minimal structures with several commuting generic "compatible" derivations. In her Ph.D. thesis, Borrata [3] studied ordered valued fields and "tame" pairs of real closed fields endowed with one generic derivation.

The results in [8, 17, 18, 42, 43] extend the one in [52] and are mostly subsumed in this article (because the structures they study are mostly algebraically bounded).

Tressl in [54] studied generic derivations on fields that are "large" in the sense of Pop, and Mohamed in [33] extended his work to operators in the sense of [36]. They assume that their fields are model-complete in the language of rings with additional constants, and therefore they are algebraically bounded (see [24, Theorem 5.4]).[1] Thus, our results extend their result on the existence of a model companion for large fields with finitely many derivations (either commuting as in [54] or non-commuting as in [33]). Moreover, in this paper we consider fields which are not pure fields, such as algebraically closed valued fields (see Section 2 for more examples).

[1] There is a slight misstatement in their theorem, in that $\mathbb{K}$ must be in the language rings with constants, and not only a "pure" field as defined in their paper; besides, their proof allows adding constants to the language in characteristic 0.

It turns out that, while in practice many of the fields studied in model theory are both large and algebraically bounded (and therefore their generic derivations can be studied by using either our framework or the one of Tressl et al.), there exist large fields which are not algebraically bounded (the field $\mathbb{C}((X, Y))$ is large but not algebraically bounded, see [12, Example 8]), and there exist algebraically bounded fields which are not large (see [23]). Tressl and Leòn Sánchez [29, 30] later introduce the notion of "differentially large fields".

Often the fields considered have a topology (e.g., they are ordered fields or valued fields): however, the theories described above do not impose any continuity on the derivation (and the corresponding "generic" derivations are not continuous at any point). In [47, 48] and [1] the authors consider the case of a valued field endowed with a "monotone" derivation (i.e., a derivation δ such that $v(\delta x) \geq v(x)$; in particular, δ is continuous) and prove a corresponding Ax–Kochen–Ersov principle.

§2. Algebraically boundedness and dimension. We fix an L-structure $\mathbb{K}$ expanding a field of characteristic 0.

We recall the following definition in [10], as refined in [23]:

Definition 2.1. Let F be a subring of $\mathbb{K}$. We say that $\mathbb{K}$ is algebraically bounded over F if, for any formula $\phi(\bar{x}, y)$, there exist finitely many polynomials $p_1, \dots, p_m \in F[\bar{x}, y]$ such that for any $\bar{a}$, if $\phi(\bar{a}, \mathbb{K})$ is finite, then $\phi(\bar{a}, \mathbb{K})$ is contained in the zero set of $p_i(\bar{a}, y)$ for some i such that $p_i(\bar{a}, y)$ doesn't vanish. $\mathbb{K}$ is *algebraically bounded* if it is algebraically bounded over $\mathbb{K}$.

Since we assumed that $\mathbb{K}$ has characteristic 0, in the above definition we can replace "$p_i(\bar{a}, y)$ doesn't vanish" with the following:
"$p_i(\bar{a}, b) = 0$ and $\frac{\partial p_i}{\partial y}(\bar{a}, b) \neq 0$".

Fact 2.2 [23], see also [13].

(1) $\mathbb{K}$ *is algebraically bounded iff it is algebraically bounded over* $\operatorname{dcl}(\emptyset)$.
(2) $\mathbb{K}$ *is algebraically bounded over F iff the model theoretic algebraic closure coincide with the field theoretic algebraic closure over F in every elementary extension of* $\mathbb{K}$ (*it suffices to check it in the monster model*).

Remark 2.3. Junker and Koenigsmann in [24] defined $\mathbb{K}$ to be "very slim" if in the monster model the field-theoretic algebraic closure over the prime field coincide with the model-theoretic algebraic closure: thus, $\mathbb{K}$ is very slim iff $\mathbb{K}$ is algebraically bounded over $\mathbb{Q}$.

Let $F := \operatorname{dcl}(\emptyset)$ and we consider $\mathbb{K}$ algebraically bounded over F.

When we refer to the algebraic closure, unless specified otherwise, we will mean the T-algebraic closure; similarly, acl will be the T-algebraic closure, and by "algebraically independent" we will mean according to T (or equivalently algebraically independent over F in the field-theoretic meaning).

From the assumptions it follows that $\mathbb{K}$ is *geometric*: that is, in the monster model $\mathbb{M} \succ \mathbb{K}$, the algebraic closure has the exchange property, and therefore it is a matroid; moreover, T is Uniformly Finite, that is it eliminates the quantifier $\exists^\infty$. In fact, a definable set $X \subseteq \mathbb{K}$ is infinite iff for all $a \in K$ there exist $x, y, x', y' \in X$ such that $x \neq x'$ and $a = (y - y')/(x' - x)$; (see [13, 23]).

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Moreover, $\mathbb{K}$ is endowed with a dimension function dim, associating to every set X definable with parameters some natural number, satisfying the axioms in [10]. This function dim is invariant under automorphisms of the ambient structure: equivalently, dim is "code-definable" in the sense of [5].

We will also use the rank, denoted by rk, associated with the matroid acl: $\mathrm{rk}(V/B)$ is the cardinality of a basis of V over B. Thus, if $X \subseteq \mathbb{M}^n$ is definable with parameters $\bar{b}$,

$$\dim(X) = \max\bigl(\mathrm{rk}(\bar{a}/\bar{b}) : \bar{a} \in X\bigr).$$

2.1. Examples. Some well known examples of fields which are algebraically bounded structures as pure fields are: algebraically closed fields, p-adics and more generally Henselian fields (see [24, Theorem 5.5]), real closed fields, pseudo-finite fields; curve-excluding fields in the sense of [22] are also algebraically bounded. Other examples of algebraically bounded structures which are not necessarily pure fields are:

- Algebraically closed valued fields.
- Henselian fields (of characteristic 0) with arbitrary relations on the value group and the residue field (see [10]).
- All models of "open theories of topological fields", as defined in [8].
- The expansion of an algebraically bounded structure by a generic predicate (in the sense of [7]) is still algebraically bounded (see [7, Corollary 2.6]).
- The theory of fields with several independent orderings and valuations has a model companion, whose models are algebraically bounded (see [9], [21, Corollary 3.12]).

Johnson and Ye in a recent paper [22] produced examples of an infinite algebraically bounded field with a decidable first-order theory which is not large (in the Pop sense), and of a pure field that is algebraically bounded but not very slim.

2.2. Assumptions. Our assumptions for the whole article are the following:

- $\mathbb{K}$ is a structure expanding a field of characteristic 0.
- L is the language of $\mathbb{K}$ and T is its L-theory.
- $F \coloneqq \mathrm{dcl}(\emptyset) \subseteq \mathbb{K}$.
- $\mathbb{K}$ is algebraically bounded over F.
- dim is the dimension function on $\mathbb{K}$ (or on any model of T), acl is the T-algebraic closure, and rk the rank of the corresponding matroid.

§3. Generic derivation. We fix a derivation $\eta : F \to F$ (if F is contained in the algebraic closure of $\mathbb{Q}$ in $\mathbb{K}$, that we denote by $\overline{\mathbb{Q}}$, then η must be equal to 0). We denote by T^δ the expansion of T, saying that δ is a derivation on $\mathbb{K}$ extending η.

In the most important case, $F = \overline{\mathbb{Q}}$ and therefore $\eta = 0$, and T^δ is the expansion of T saying that δ is a derivation on $\mathbb{K}$.

3.1. Model completion. A. Robinson introduced the notion of model completion in relation with solvability of systems of equations. For convenience we recall the definition:

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Definition 3.1. Let U and U^* be theories in the same language L.
U^* is a model completion of U if the following hold:

(1) If $A \models U^*$, then $A \models U$.
(2) If $A \models U$, then there exists a $B \supset A$ such that $B \models U^*$.
(3) If $A \models U$, $A \subset B$, $A \subset C$, where $B, C \models U^*$, then B is elementary equivalent to C over A.

We give the following general criteria for model completion. In our context we use (3).

Proposition 3.2. *Let U and U^* be theories in the same language L such that $U \subseteq U^*$. The following are equivalent*:

(1) *U^* is the model completion of U and U^* eliminates quantifiers.*
(2) (a) *For every $A \models U$, for every $\sigma_1, \dots, \sigma_n \in U^*$, there exists $B \models U$ such that $A \subseteq B$ and $B \models \sigma_1 \wedge \cdots \wedge \sigma_n$.*
  (b) *For every L-structures A, B, C such that $B \models U$, $C \models U^*$, and A is a common substructure, for every quantifier-free $L(A)$-formula $\phi(\bar{x})$, for every $\bar{b} \in B^n$ such that $B \models \phi(\bar{b})$, there exists $\bar{c} \in C^n$ such that $C \models \phi(\bar{c})$.*
(3) (a) *For every $A \models U$, for every $\sigma_1, \dots, \sigma_n \in U^*$, there exists $B \models U$ such that $A \subseteq B$ and $B \models \sigma_1 \wedge \cdots \wedge \sigma_n$.*
  (b) *For every L-structures A, B, C such that $B \models U$, $C \models U^*$, and A is a common substructure, for every quantifier-free $L(A)$-formula $\phi(x)$, and for every $b \in B$ such that $B \models \phi(b)$, there exists $c \in C$ such that $C \models \phi(c)$.*
(4) *For all models A of $U_\forall$ we have*:
  (a) $\mathrm{Diag}(A) \cup U^*$ *is consistent,*
  (b) $\mathrm{Diag}(A) \cup U^*$ *is complete,*
  where $\mathrm{Diag}(A)$ *is the L-diagram of A.*
(5) (*Blum criterion*)
  (a) *Any model of $U_\forall$ can be extended to some model of U^*.*
  (b) *For any $A, A(b) \models U_\forall$ and for all $C^* \models U^*$, where C^* is $|A|^+$-saturated, there exists an immersion of $A(b)$ in C^*.*
(6) *U^* is the model completion of $U_\forall$.*

Proof. First we prove that (1) is equivalent to (6): if U^* is the model completion of $U_\forall$, trivially U^* is a model completion of U and by [46, Theorem 13.2], we have that U^* eliminates quantifiers.

For the converse, we have trivially that any models of U^* is a model of $U_\forall$. Moreover, if $A \models U_\forall$ then there exists $C \models U$ such that there exists an immersion of A in C. But by (6) there exists $B \models U_\forall$ such that there exists an immersion of C in B, and so an immersion of A in B. It is trivial to verify (3) in Definition 3.1. (1) is equivalent to (4): see [46]. Also for the equivalence between (5) and (6) see [46].

It remains to prove the equivalence between (1) and (2). (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is easy. For (3) $\Rightarrow$ (1), in order to obtain that U^* is the model completion of U, we prove that $\mathrm{Diag}(A) \cup U^*$ is consistent, but it is enough to see that it is finitely consistent. By (a) we have finite consistency. To prove that U^* eliminates quantifiers it is equivalent to prove that $\mathrm{Diag}(A) \cup U^*$ is complete, which follows easily from (b). $\dashv$

3.2. The axioms. We introduce the following notation:
Let $\delta : \mathbb{K} \to \mathbb{K}$ be some function, $n \in \mathbb{N}$, $a \in \mathbb{K}$ and $\bar{a}$ tuple of $\mathbb{K}^m$. We denote by:

$$\mathrm{Jet}^\infty_\delta(a) := \langle \delta^i a : i \in \mathbb{N}\rangle, \quad \mathrm{Jet}^n_\delta(a) := \langle \delta^i a : i \leq n\rangle, \quad \mathrm{Jet}(a) := \mathrm{Jet}^n_\delta(a) \text{ for some } n,$$
$$\mathrm{Jet}^\infty_\delta(\bar{a}) := \langle \delta^i \bar{a} : i \in \mathbb{N}\rangle, \quad \mathrm{Jet}^n_\delta(\bar{a}) := \langle \delta^i \bar{a} : i \leq n\rangle, \quad \mathrm{Jet}(\bar{a}) := \mathrm{Jet}^n_\delta(\bar{a}) \text{ for some } n.$$

Definition 3.3. Let $X \subseteq \mathbb{K}^n$ be L-definable with parameters. We say that X is **large** if $\dim(X) = n$.

Two possible axiomatizations for the model completion T^δ_g are given by T^δ and either of the following axiom schemas:

(`Deep`) For every $Z \subseteq \mathbb{K}^{n+1}$ $L(\mathbb{K})$-definable, if $\Pi_n(Z)$ is large, then there exists $c \in \mathbb{K}$ such that $\mathrm{Jet}^n_\delta(c) \in Z$.

(`Wide`) For every $W \subseteq \mathbb{K}^n \times \mathbb{K}^n$ $L(\mathbb{K})$-definable, if $\Pi_n(W)$ is large, then there exists $\bar{c} \in \mathbb{K}^n$ such that $\langle \bar{c}, \delta\bar{c}\rangle \in W$.

Definition 3.4. We denote by

$$T^\delta_{\mathrm{deep}} := T^\delta \cup (\mathtt{Deep}), \qquad T^\delta_{\mathrm{wide}} := T^\delta \cup (\mathtt{Wide}).$$

We will show that both T^δ_{deep} and T^δ_{wide} are axiomatizations for the model completion of T^δ. Notice that the axiom scheme (`Wide`) deals with many variables at the same time, but has only one iteration of the map δ, while (`Deep`) deals with only one variable at the same time, but many iteration of δ.

Theorem 3.5. *Assume that the theory T is model complete. Then the model completion T^δ_g of T^δ exists, and the theories T^δ_{deep} and T^δ_{wide} are two possible axiomatizations of T^δ_g.*

3.3. Proof preliminaries. In order to prove the main result we first introduce the following notation: given a polynomial $p(\bar{x}, y)$ we write

$$p(\bar{a}, b) =^y 0 \iff p(\bar{a}, b) = 0 \wedge \frac{\partial p}{\partial y}(\bar{a}, b) \neq 0.$$

Let S be a field of characteristic 0 and ε be a derivation on it. Let I be a set of indexes (possibly, infinite). Denote $\bar{x} := \langle x_i : i \in I\rangle$, and $\bar{y} := \langle y_i : i \in I\rangle$.

Definition 3.6. There exists a unique derivation $S[\bar{x}] \to S[\bar{x}, \bar{y}]$, $p \mapsto p^{[\varepsilon]}$ such that:

$$\forall a \in S \quad a^{[\varepsilon]} = \varepsilon a, \quad \forall i \in I \quad x_i^{[\varepsilon]} = y_i;$$

such derivation extends uniquely to a derivation $S(\bar{x}) \to S(\bar{x})[\bar{y}]$, $q \mapsto q^{[\varepsilon]}$.
Moreover, the map $S(\bar{x}) \to S(\bar{x})$ defined by

$$q^\varepsilon := q^{[\varepsilon]}(\bar{x}, 0)$$

is the unique derivation on $S(\bar{x})$ such that:

$$\forall a \in S \quad a^\varepsilon = \varepsilon a, \qquad \forall i \in I \quad x_i^\varepsilon = 0;$$

when $p \in S[\bar{x}]$, p^ε is the polynomial obtained by p by applying ε to each coefficient.

REMARK 3.7. For every $q \in S(\bar{x})$ and $\bar{a} \in S^n$,

$$q^{[\varepsilon]}(\bar{x}, \bar{y}) = q^{\varepsilon}(\bar{x}) + \sum_{i \in I} \frac{\partial q}{\partial x_i}(\bar{x}) y_i; \tag{1}$$

$$\varepsilon(q(\bar{a})) = q^{[\varepsilon]}(\bar{a}, \varepsilon \bar{a}). \tag{2}$$

If moreover S' is a field containing S and $\varepsilon' : S(\bar{x}) \to S'$ is a derivation extending ε, then

$$\varepsilon'(q) = q^{[\varepsilon]}(\bar{x}, \varepsilon'(\bar{x})) = q^{\varepsilon}(\bar{x}) + \sum_{i \in I} \frac{\partial q}{\partial x_i}(\bar{x}) \varepsilon'(x_i). \tag{3}$$

We will often also use the following fundamental fact, without further mentions.

FACT 3.8. *Let S' be a field containing S (as in Definition 3.6). Let $\bar{a} := \langle a_i : i \in I \rangle$ be a (possibly, infinite) tuple of elements of S' which are algebraically independent over S, and $\bar{b} := \langle b_i : i \in I \rangle$ be a tuple of elements of S' (of the same length as $\bar{a}$). Then, there exists a derivation ε' on S' extending ε and such that $\varepsilon'(\bar{a}) = \bar{b}$. If moreover $\bar{a}$ is a transcendence basis of S' over S, then ε' is unique.*

PROOF. W.l.o.g., $\bar{a}$ is a transcendence basis of S' over S. By [57, Chapter II, Section 17, Theorem 39], there exists a unique derivation $\varepsilon'' : S(\bar{a}) \to S'$ extending ε and such that $\varepsilon''(\bar{a}) = \bar{b}$; we can also prove it directly, by defining, for every $q \in S(\bar{x})$,

$$\varepsilon''(q(\bar{a})) := q^{[\varepsilon]}(\bar{a}, \bar{b}).$$

Given $c \in S'$, let $p(y) \in S(\bar{a})[y]$ be the (monic) minimal polynomial of c over S. Let $S'' := S(c) \subseteq S'$. Let

$$d := -p^{\varepsilon''}(c)/p'(c) \in S'. \tag{4}$$

Any derivation on ε' on S' extending ε'' must satisfy $\varepsilon'(c) = d$, and by [57, Chapter II, Section 17, Theorem 39] again, there exists a unique derivation $\varepsilon''' : S'' \to S'$ extending ε'' and such that $\varepsilon'''(c) = d$.

By iterating the above construction, we find a unique derivation ε' on S' extending ε''. ⊣

We need the following preliminary lemmas.

LEMMA 3.9. *Let $\alpha(x, \bar{y})$ be an L-formula and $(B, \delta) \models T^{\delta}$. Then there exists a function $\alpha^{[\eta]}$ definable in T such that $\delta a = \alpha^{[\eta]}(a, \bar{b}, \delta\bar{b})$, for every $a, \bar{b} \in B$ with $B \models \alpha(a, \bar{b})$ and $|\alpha(a, B)| < \infty$.*

PROOF. Let $\alpha(x, \bar{y})$ be an L-formula. Since $\mathbb{B}$ is algebraically bounded over F and of characteristic 0, there exist polynomials $p_1(x, \bar{y}), \dots, p_k(x, \bar{y}) \in F[x, \bar{y}]$ associated with the formula $\alpha(x, \bar{y})$ and formulas $\beta_i(x, \bar{y}) =$ "$p_i(x, \bar{y}) =^x 0$" such that $T \vdash (\alpha(x, \bar{y})) \wedge |\alpha(x, \cdot)| < \infty) \to \bigvee_{i=1}^k \beta_i(x, \bar{y})$. Now we can associate to each polynomial p_i the partial function

$$f_i(x, \bar{y}, \delta \bar{y}) := \frac{\frac{\partial p_i}{\partial \bar{y}} \cdot \delta \bar{y} + p^{\eta}}{\frac{\partial p_i}{\partial x}},$$

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where p^η is the polynomial defined in 3.6 obtained by p by applying η to each coefficients.

So now we have a total T-definable function $f(x,\bar{y},\delta\bar{y})$ whose graph is defined in the following way:

$$\begin{aligned} z = f(x,\bar{y},\delta y) \ \Leftrightarrow \ \big(\beta_1(x,\bar{y}) \wedge z = f_1(x,\bar{y},\delta y)\big) \vee \big(\neg\beta_1(x,\bar{y}) \wedge \beta_2(x,\bar{y}) \wedge z = \\ = f_2(x,\bar{y},\delta y)\big) \vee \\ \vee \cdots \vee \big(\neg\beta_1(x,\bar{y}) \wedge \cdots \wedge \neg\beta_{k-1}(x,\bar{y}) \wedge \beta_k(x,\bar{y}) \wedge z = f_k(x,\bar{y},\delta y)\big) \vee \\ \vee \ \big(\neg\beta_1(x,\bar{y}) \wedge \cdots \wedge \neg\beta_{k-1}(x,\bar{y}) \wedge \neg\beta_k(x,\bar{y}) \wedge z = 0\big). \end{aligned}$$

$\dashv$

COROLLARY 3.10. *For any T-definable function $f(\bar{x})$ there exists a T-definable function $f^{[\eta]}$ such that $\delta(f(\bar{x})) = f^{[\eta]}(\bar{x}, \mathrm{Jet}(\bar{x}))$.*

LEMMA 3.11. *Let $t(\bar{x})$ be an L^δ-term. Then there is a T-definable function $f(\bar{x},\bar{y})$ such that $t(\bar{x}) = f(\bar{x}, \mathrm{Jet}(\bar{x}))$.*

PROOF. We prove by induction on the complexity of the term $t(\bar{x})$. If $t(\bar{x})$ is a variable it is trivial. Suppose that $t(\bar{x}) = h(s(\bar{x}))$. By induction there exists a T-definable function g such that $s(\bar{x}) = g(\bar{x}, \mathrm{Jet}(\bar{x}))$. If the function h is in L we can conclude. Otherwise $h = \delta$ and we obtain $t(\bar{x}) = \delta(g(\bar{x}, \mathrm{Jet}(\bar{x})))$. By Corollary 3.10 we conclude the proof. $\dashv$

LEMMA 3.12. *Let $\phi(\bar{x})$ be a quantifier free L^δ-formula. Then there exists an L-formula ψ such that $T^\delta \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x}, \mathrm{Jet}(\bar{x}))$.*

PROOF. Follows from Lemma 3.11. $\dashv$

3.4. Proof of Theorem 3.5. We can finally prove that both T^δ_{deep} and T^δ_{wide} axiomatize T^δ_g. The proof is in three steps: firstly we show that $T^\delta_{\mathrm{wide}} \vdash T^\delta_{\mathrm{deep}}$, and later we prove that the conditions (3) of Proposition 3.2 hold for $U = T^\delta$ and, more precisely, (a) holds for U^* equal to T^δ_{wide} (i.e., that every model of T^δ can be embedded in a model of T^δ_{wide}), and (b) for U^* equal T^δ_{deep} (i.e., if $B \models T^\delta$ and $C \models T^\delta_{\mathrm{deep}}$ have a common substructure A, then every quantifier-free $L^\delta(A)$-formula with one free variable having a solution in B also has a solution in C).

LEMMA 3.13. $T^\delta_{\mathrm{wide}} \vdash T^\delta_{\mathrm{deep}}$.

PROOF. Let $Z \subseteq \mathbb{K}^{n+1}$ be $L(\mathbb{K})$-definable such that $\Pi_n(Z)$ is large. Define

$$W := \{\langle \bar{x}, \bar{y}\rangle \in \mathbb{K}^n \times \mathbb{K}^n : \langle \bar{x}, y_n\rangle \in Z \wedge \bigwedge_{i=1}^{n-1} y_i = x_{i+1}\}.$$

Clearly, $\Pi_n(W) = \Pi_n(Z)$, and therefore $\Pi_n(W)$ is large. By (`Wide`), there exists $\bar{c} \in \mathbb{K}^n$ such that $\langle \bar{c}, \delta\bar{c}\rangle \in W$. Then, $\mathrm{Jet}^n_\delta(c_1) \in Z$. $\dashv$

LEMMA 3.14. *Let $(A,\delta) \models T^\delta$. Let $Z \subseteq A^n \times A^n$ be L-definable with parameters in A, such that $\Pi_n(Z)$ is large. Then, there exists $\langle B, \varepsilon\rangle \supseteq \langle A, \delta\rangle$ and $\bar{b} \in B^n$ such that $B \succeq A$, $\langle B, \varepsilon\rangle \models T^\delta$, and $\langle \bar{b}, \varepsilon\bar{b}\rangle \in Z_B$ (the interpretation of Z in B).*

PROOF. Let $B \succ A$ (as L-structures) be such that B is $|A|^+$-saturated. By definition of dimension, there exists $\bar{b} \in \Pi_n(Z_B)$ which is algebraically independent over A. Let $\bar{d} \in B^n$ be such that $\langle \bar{b}, \delta\bar{b} \rangle \in Z_B$. Let ε be any derivation on B which extends δ and such that, by Fact 3.8, $\varepsilon(\bar{b}) = \bar{d}$. $\dashv$

LEMMA 3.15. *Let $\langle B, \delta \rangle \models T^\delta$, $\langle C, \delta \rangle \models T^\delta_{\text{deep}}$, and $\langle A, \delta \rangle$ be an L^δ-substructures of both models, such that B and C have the same $L(A)$-theory. Let $b \in B$ be such that $\langle B, \delta \rangle \models \theta(b)$, where $\theta(x)$ is a quantifier free L^δ-formula with parameters in A. Then, there exists $c \in C$ such that $\langle C, \delta \rangle \models \theta(c)$.*

PROOF. By Lemma 3.12 there exist $n \in \mathbb{N}$ and an $L(A)$-formula ψ such that $\theta(x) = \psi(\mathrm{Jet}^n_\delta(x))$.

Let $Y^B := \psi(B) = \{\bar{d} \in B^{n+1} : B \models \psi(\bar{d})\}$, and $Y^C := \psi(C)$.

Let d be the smallest integer such that $\delta^d(b)$ is algebraically dependent over $\mathrm{Jet}^{d-1}(b) \cup A$ (or $d = +\infty$ if $\mathrm{Jet}^\infty_\delta(b)$ is algebraically independent over A). We distinguish two cases:

(1) $d \geq n$: in this case, $\Pi_n(Y^C)$ is large because $\mathrm{Jet}^{n-1}(b) \in \Pi_n(Y^B)$, therefore, by (Deep), there exists $c \in C$ such that $C \models \theta(\mathrm{Jet}^n_\delta(c))$.

(2) $d < n$: this means that $\delta^d b \in \mathrm{acl}(\mathrm{Jet}^{d-1}(b))$, so there exists a polynomial $p(\bar{y}, x) \in A[\bar{y}, x]$ such that $p(\mathrm{Jet}^{d-1}(b), \delta^d b) =^x 0$. By Lemma 3.9 there exist $L(A)$-definable functions $f_{d+1}, f_{d+2}, \dots, f_n$ such that $\delta^i = f_i(\mathrm{Jet}^d(b))$ where $i = d+1, d+2, \dots, n$. Let

$$Z^B := \{\bar{y} \in B^{d+1} : p(\bar{y}) =^{y_{d+1}} 0 \wedge \theta(\bar{y}, f_{d+1}(\bar{y}), \dots, f_n(\bar{y}))\}.$$

Notice that $\Pi_d(Z^C)$ is large, because $\mathrm{Jet}^{d-1}(b) \in \Pi_d(Z^B)$, and therefore by axiom (Deep) there exists $c \in C$ such that $\mathrm{Jet}^d(c) \in Z^C$ and so $\mathrm{Jet}^n(c) \in Y^C$. $\dashv$

3.5. Corollaries.

COROLLARY 3.16. *Assume that T eliminates quantifiers. Then, T^δ_{deep} and T^δ_{wide} are axiomatizations for the model completion T^δ_g of T^δ.*

Moreover, T^δ_g admits elimination of quantifiers, and for every L^δ-formula $\alpha(\bar{x})$ there exists a quantifier-free L-formula $\beta(\bar{y})$ such that

$$T^\delta_g \models \forall \bar{x} \left(\alpha(\bar{x}) \leftrightarrow \beta(\mathrm{Jet}(\bar{x})) \right).$$

Finally, T^δ_g is complete.

COROLLARY 3.17. *Assume that T is model complete. Then, T^δ_{deep} and T^δ_{wide} are axiomatizations for the model completion T^δ_g of T^δ.*

The next corollary is without any further assumptions on T.

COROLLARY 3.18. *T^δ_{deep} and T^δ_{wide} are equivalent consistent theories (which we denote by T^δ_g).*

Moreover, for every L^δ-formula $\alpha(\bar{x})$ there exists an L-formula $\beta(\bar{y})$ such that

$$T^\delta_g \models \forall \bar{x} \left(\alpha(\bar{x}) \leftrightarrow \beta(\mathrm{Jet}\, \bar{x}) \right).$$

Finally, T^δ_g is complete.

§4. Several non-commuting derivations. We analyze first the case when there are several not commuting derivations $\delta_1, \dots, \delta_k$ because it is simpler in terms of axiomatization, as we observed in Section 1.1, and later in Section 5 we examine the harder case of commuting derivations.

Let $\bar{\delta} := \langle \delta_1, \dots, \delta_k \rangle$. Let $\eta_1, \dots, \eta_k$ be derivations on F. We denote by $T^{\bar{\delta},nc}$ the $L^{\bar{\delta}}$-expansion of T saying that each δ_i is a derivation and that δ_i extends η_i for $i \le k$.

Theorem 4.1. *Assume that T is model complete. Then, $T^{\bar{\delta},nc}$ has a model completion $T_g^{\bar{\delta},nc}$.*

To give the axioms for $T_g^{\bar{\delta},nc}$ we need some more definitions and notations. We fix $\langle \mathbb{K}, \bar{\delta} \rangle \models T^{\bar{\delta},nc}$.

Let Γ be the free non-commutative monoid generated by $\bar{\delta}$, with the canonical partial order $\preceq$ given by $\beta \preceq \alpha\beta$, for all $\alpha, \beta \in \Gamma$. We fix the total order on Γ, given by

$$\theta \le \theta' \Leftrightarrow |\theta| < |\theta'| \ \vee \ \big(|\theta| = |\theta'| \ \wedge \ \theta <_{lex} \theta'\big),$$

where $<_{lex}$ is the lexicographic order, and $|\theta|$ is the length of θ as a word in the alphabet $\bar{\delta}$.

Remark 4.2. $\preceq$ is a well-founded partial order on Γ, but it is not a well-partial-order (i.e., there exist infinite anti-chains).

Remark 4.3.

(1) As an ordered set, $\langle \Gamma, \le \rangle$ is isomorphic to $\langle \mathbb{N}, \le \rangle$.
(2) $\emptyset$ (i.e., the empty word, corresponding to the identity function on $\mathbb{K}$) is the minimum of Γ.
(3) If $\alpha \preceq \beta$, then $\alpha \le \beta$.
(4) If $\alpha \le \beta$, then $\gamma\alpha \le \gamma\beta$ and $\alpha\gamma \le \beta\gamma$, for all $\gamma \in \Gamma$.

For every variable x and every $\gamma \in \Gamma$ we introduce the variable x_γ. Given $V \subseteq \Gamma$, we denote $x_V := \langle x_\gamma : \gamma \in V \rangle$ and $a^V := \langle \gamma a : \gamma \in V \rangle$. We remark that a^V is an analogue of the notion of Jet in one derivation, i.e., $\mathrm{Jet}^n(a) = a^{\{0,1,\dots,n\}}$. Moreover, we denote Π_A the projection from $\mathbb{K}^B$ to $\mathbb{K}^A$ (for some $A, B \subseteq \Gamma$ and $B \supseteq A$), mapping $\langle a_\mu : \mu \in B \rangle$ to $\langle a_\mu : \mu \in A \rangle$.

We give now two alternative axiomatizations for $T_g^{\bar{\delta},nc}$.

(`nc-Deep`) Let $\mathcal{V} \subset \Gamma$ be finite and $\preceq$-initial. Let $\mathcal{P} \subseteq \mathcal{V}$ be the set of $\preceq$-maximal elements of $\mathcal{V}$, and $\mathcal{F} := \mathcal{V} \setminus \mathcal{P}$. Let $Z \subseteq \mathbb{K}^{\mathcal{V}}$ be $L(A)$-definable. If $\Pi_{\mathcal{F}}(Z)$ is large, then there exists $c \in \mathbb{K}$ such that $c^{\mathcal{V}} \in Z$.

(`nc-Wide`) Let $W \subseteq \mathbb{K}^n \times \mathbb{K}^{k \times n}$ be $L(\mathbb{K})$-definable, such that $\Pi_n(W)$ is large. Then, there exists $\bar{c} \in \mathbb{K}^n$ such that $\langle \bar{c}, \delta_1 \bar{c}, \dots, \delta_k \bar{c} \rangle \in W$.

Definition 4.4. We denote by

$$T_{deep}^{\bar{\delta},nc} := T^{\delta} \cup (\texttt{nc-Deep}), \qquad T_{wide}^{\bar{\delta},nc} := T^{\delta} \cup (\texttt{nc-Wide}).$$

THEOREM 4.5.

(1) *$T^{\bar\delta,nc}_{deep}$ and $T^{\bar\delta,nc}_{wide}$ are consistent and equivalent to each other.*

(2) *If T is model-complete, then the model completion $T^{\bar\delta,nc}_g$ of $T^{\bar\delta,nc}$ exists, and the theories $T^{\bar\delta,nc}_{deep}$ and $T^{\bar\delta,nc}_{wide}$ are two possible axiomatizations of $T^{\bar\delta,nc}_g$.*

(3) *If T eliminates quantifiers, then $T^{\bar\delta,nc}_g$ eliminates quantifiers.*

(4) *For every $L^{\bar\delta}$-formula $\alpha(\bar x)$ there exists an L-formula $\beta(\bar x)$ such that*
$$T^{\bar\delta,nc}_g \models \forall \bar x\, \big(\alpha(\bar x) \leftrightarrow \beta(\bar x^{\Gamma})\big).$$

For the proof, we proceed as in Section 3.4, i.e., it is in three steps:

LEMMA 4.6. $T^{\bar\delta,nc}_{wide} \vdash T^{\bar\delta,nc}_{deep}$.

PROOF. Let $Z, \mathcal{F}, \mathcal{P}, \mathcal{V}$ be as in (`nc-Deep`).

CLAIM 1. *W.l.o.g., we may assume that $\mathcal{P}$ is equal to the set of $\preceq$-minimal elements of $\Gamma \setminus \mathcal{F}$.*

In fact, let $\mathcal{P}'$ be the set of $\preceq$-minimal elements of $\Gamma \setminus \mathcal{F}$; notice that $\mathcal{P} \subseteq \mathcal{P}'$. We can replace $\mathcal{V}$ with $\mathcal{V}' := \mathcal{V} \cup \mathcal{P}'$, and Z with $Z' := \Pi^{-1}(Z)$, where the function Π is defined as:
$$\begin{aligned}\Pi : \mathbb{K}^{\mathcal{V}'} &\longrightarrow \mathbb{K}^{\mathcal{V}}\\ \bar x &\longmapsto \langle x_\mu : \mu \in \mathcal{V}\rangle.\end{aligned}$$

Then, $\Pi_{\mathcal{F}}(Z') = \Pi_{\mathcal{F}}(Z)$, and if $a^{\mathcal{V}'} \in Z'$, then $a^{\mathcal{V}} \in Z$.

We introduce variables $x_0, x_1, \dots, x_k$ and corresponding variable $x_{i,\gamma}$, which for readability we denote by $x(i,\gamma)$ such that $0 \le i \le k, \gamma \in \Gamma$. For brevity, we denote
$$\bar x := \langle x(i,\gamma) : 0 \le i \le k, \gamma \in \mathcal{V}\rangle \qquad \text{and} \qquad \bar x_i := \langle x(i,\gamma) : \gamma \in \mathcal{V}\rangle, \quad i = 0,\dots,k.$$

We also denote
$$\begin{aligned}\Pi_0 : (\mathbb{K}^{\mathcal{V}})^{k+1} &\longrightarrow \mathbb{K}^{\mathcal{V}}\\ \bar x &\longmapsto \bar x_0.\end{aligned}$$

For each $\pi \in \mathcal{P}$, we choose $\mu_\pi \in \mathcal{F}$ and $i_\pi \in \{1,\dots,k\}$ such that $\delta_{i_\pi}\mu_i = \pi$. Moreover, given $\bar a \in (\mathbb{K}^{\mathcal{F}})^{k+1}$, we define $\bar a' \in K^{\mathcal{V}}$ as the tuple with coordinates
$$a'_\gamma := \begin{cases} a(0,\gamma), & \text{if } \gamma \in \mathcal{F},\\ a(i_\gamma, \mu_\gamma), & \text{if } \gamma \in \mathcal{P}.\end{cases}$$

We define
$$W := \{\langle \bar a \in (\mathbb{K}^{\mathcal{F}})^{k+1}\rangle : \bar a' \in Z \ \wedge\ a(i,\gamma) = a(0,\delta_i\gamma) \text{ for } i = 1,\dots,k \text{ and } \gamma \in \mathcal{F}\}.$$

Notice that $\Pi_0(W)$ is equal to $\Pi_{\mathcal{F}}(Z)$, and therefore it is large. Thus, by (`nc-Wide`), there exists $\bar a \in \mathbb{K}^{\mathcal{F}}$ such that $\langle \bar a, \delta_1(\bar a), \dots, \delta_k(\bar a)\rangle \in W$. Finally, taking $a := a(0,\emptyset)$, we get $a^{\mathcal{V}} \in Z$. ⊣

LEMMA 4.7. *Let $(A, \bar\delta) \models T^{\bar\delta,nc}$. Let $Z \subseteq A^n \times (A^n)^k$ be L-definable with parameters in A, such that $\Pi_n(Z)$ is large. Then, there exists $\langle B, \bar\varepsilon\rangle \supseteq \langle A, \bar\delta\rangle$ and $\bar b \in B^n$ such that $B \succeq A$, $\langle B, \bar\varepsilon\rangle \models T^{\bar\delta,nc}$, and $\langle \bar b, \bar\varepsilon \bar b\rangle \in Z_B$.*

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PROOF. Same proof as for Lemma 3.14. $\dashv$

LEMMA 4.8. *Let $\langle B, \bar{\delta}\rangle \models T^{\bar{\delta},nc}$, $\langle C, \bar{\delta}\rangle \models T^{\bar{\delta},nc}_{deep}$, and $\langle A, \bar{\delta}\rangle$ be an $L(\bar{\delta})$-substructures of both models, such that B and C have the same $L(A)$-theory. Let $b \in B$ be such that $\langle B, \bar{\delta}\rangle \models \theta(b)$, where $\theta(x)$ is a quantifier free $L(\bar{\delta})$-formula with parameters in A. Then, there exists $c \in C$ such that $\langle C, \bar{\delta}\rangle \models \theta(c)$.*

PROOF. By Lemma 3.12 there exists U finite subset of Γ and an $L(A)$-formula $\psi(\bar{y})$ such that U is $\preceq$-initial and that $T^{\bar{\delta},nc} \models \theta(x) = \psi(x^U)$. Let $Y^B := \Psi(B)$ and $Y^C := \psi(C)$. Let

$$\mathcal{F} := \{\gamma \in U : \gamma b \notin \mathrm{acl}(A, b^{U<\gamma})\}, \text{ where we denote } b^{U<\gamma} := \langle \mu b : \mu < \gamma \wedge \mu \in U\rangle.$$

Define $\mathcal{B} := \Gamma \setminus \mathcal{F}$ and $\mathcal{P}$ be the set of $\preceq$-minimal elements of $\mathcal{B}$ (notice that $\mathcal{P}$ might be infinite). As usual, define $\mathcal{V} := \mathcal{F} \cup \mathcal{P}$.

For every $\gamma \in \Gamma$ there exists $q_\gamma \in A(x_{\mathcal{V}_{\leq\gamma}})$ such that $\gamma b = q_\gamma(b^{\mathcal{V}_{\leq\gamma}})$. Let β be the following $L(A)$-formula:

$$\beta(x_{\mathcal{V}}) := \psi(q_\gamma(x_{\mathcal{V}}) : \gamma \in U).$$

Notice that $\langle B, \bar{\delta}\rangle \models \beta(b^{\mathcal{V}})$. Let $\mathcal{V}_0 \subseteq \mathcal{V}$ be the set of indexes of the variables of β: w.l.o.g., we may assume that $\mathcal{V}_0$ is a $\preceq$-initial subset of Γ. Let $\mathcal{P}_0$ be the set of $\preceq$-maximal elements of $\mathcal{V}_0$. Define

$$Z := \{\bar{d} \in \mathbb{K}^{\mathcal{V}_0} : \langle B, \bar{\delta}\rangle \models \beta(\bar{d})\}.$$

Notice that $\Pi_{\mathcal{F}_0}(Z)$ contains $b^{\mathcal{F}_0}$, and therefore it is large. Thus, by (`nc-Deep`), there exists $c \in C$ such that $c^{\mathcal{V}_0} \in Z$, and therefore c^U satisfies ψ. $\dashv$

§5. Several commuting derivations. We now deal with the case when there are several *commuting* derivations $\delta_1, \ldots, \delta_k$. The technique used here for the treatment of the study of several derivations are a variant of [14]. In particular, we avoid as much as possible the algebraic approach in [25] based on autoreduced sets.

Let $\bar{\delta} := \langle \delta_1, \ldots, \delta_k\rangle$ and let $\eta_1, \ldots, \eta_k$ be commuting derivations on F. Let $T^{\bar{\delta}}$ be the $L^{\bar{\delta}}$-expansion of T saying that each δ_i is a derivation, that δ_i extends η_i for $i \leq k$, and that $\delta_i \circ \delta_j = \delta_j \circ \delta_i$, for $i, j \leq k$.

THEOREM 5.1. *Assume that T is model complete. Then, $T^{\bar{\delta}}$ has a model completion $T^{\bar{\delta}}_g$.*

5.1. Configurations. In order to give axioms for $T^{\bar{\delta}}_g$, we first need some more definitions and notations. We fix $\langle \mathbb{K}, \bar{\delta}\rangle \models T^{\bar{\delta}}$. We denote by K the field underlying $\mathbb{K}$.

Let Θ be the free commutative monoid generated by $\bar{\delta}$, with the canonical partial order $\preceq$ (notice that Θ is isomorphic to $\mathbb{N}^k$).

Notice that Θ is, canonically, a quotient of the free monoid Γ. For every $v \in \Gamma$ we denote by $[v] \in \Theta$ the equivalence class of v.

We fix the total order on Θ, given by

$$\theta \leq \theta' \text{ iff } |\theta| < |\theta'| \vee \big(|\theta| = |\theta'| \wedge \theta <_{lex} \theta'\big),$$

where $<_{lex}$ is the lexicographic order, and $|\langle \delta_1^{n_1}, \ldots, \delta_k^{n_k}\rangle| := n_1 + \cdots + n_k$.

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Given $a \in \mathbb{K}$ and $\theta \in \Theta$, we denote by $a^{<\theta} := \langle \mu a : \mu < \theta \rangle$, and similarly $a^{\leq \theta} := \langle \mu a : \mu \leq \theta \rangle$, and $a^{\Theta} := \langle \mu a : \mu \in \Theta \rangle$. Moreover, for each $\theta \in \Theta$ we have a variable x_θ, and we denote $x_{<\theta} := \langle x_\mu : \mu < \theta \rangle$. Moreover, given a set $A \subseteq \Theta$, we denote $x_A := \langle x_\theta : \theta \in A \rangle$, and $x_{A \leq \mu} := \langle x_\nu : \nu \in A \ \wedge \ \nu \leq \mu \rangle$. Given a rational function $q \in K(x_\Theta)$ and $\bar{a} \in K^\Theta$, we denote by

$$\frac{\partial q}{\partial \mu} := \frac{\partial q}{\partial x_\mu} \qquad \text{and} \qquad q(\bar{a}) =^\mu 0 \text{ iff } q(\bar{a}) = 0 \ \wedge \ \frac{\partial q}{\partial \mu}(\bar{a}) \neq 0.$$

Let K_0 be a differential subfield of K (i.e., such that $\bar{\delta}(K_0) \subseteq K_0$).

A *configuration* $\mathfrak{S}$ with parameters in K_0 is given by the following data.

(1) A $\preceq$-anti-chain $\mathcal{P} \subset \Theta$. Notice that, by Dickson's Lemma, $\mathcal{P}$ must be finite.

We distinguish two sets:

- $\mathcal{B} := \{\mu \in \Theta : \exists \pi \in \mathcal{P} \ \mu \succeq \pi\}$, the set of leaders.[2]
- $\mathcal{F} := \Theta \setminus \mathcal{B} = \{\mu \in \Theta : \forall \pi \in \mathcal{P} \ \mu \not\succeq \pi\}$, the set of free elements. Moreover, we define:
- $\mathcal{V} := \mathcal{F} \cup \mathcal{P}$.

Notice that $\mathcal{P}$ is the set of $\preceq$-minimal elements of $\mathcal{B}$. We assume that $\mathcal{F}$ is non-empty (equivalently, that $0 \in \mathcal{F}$).

(2) For every $\pi \in \mathcal{P}$ we are given a nonzero polynomial $p_\pi \in K_0[x_\pi, x_{\mathcal{F}<\pi}]$ which depends on x_π (i.e., its degree in x_π is nonzero).

Consider the quasi-affine variety (defined over K_0)

$$W_0 := \{x_\mathcal{V} \in K^\mathcal{V} : \bigwedge_{\pi \in \mathcal{P}} p_\pi(x_\mathcal{V}) =^\pi 0\}$$

(by quasi-affine variety we simply mean a subset of K^n which is locally closed in the Zariski topology. We don't consider its spectrum).

(3) Finally, we are given $W \subseteq W_0$, which is Zariski closed in W_0, such that W is defined (as a quasi-affine variety) over K_0 and such that $\Pi_\mathcal{F}(W)$ is large (where $\Pi_\mathcal{F} : K^\mathcal{V} \to K^\mathcal{F}$ is the canonical projection).

For the remainder of this section, we are given a configuration:

$$\mathfrak{S} := (W; p_\pi : \pi \in \mathcal{P}).$$

We want to impose some commutativity on $\mathfrak{S}$.

We define now some induced data.

For every $\alpha \in \Theta$ we define a rational function $f_\alpha \in K_0(x_\mathcal{V})$ and a tuple of rational functions F_α.

- If $\alpha \in \mathcal{V}$, then $f_\alpha := x_\alpha$ and $F_\alpha = \{f_\alpha\}$.
- For every $\pi \in \mathcal{P}$ and $v \in \Gamma$, we define $f_{v,\pi}$ by induction on v, and then

$$F_\alpha := \{f_{v,\pi} : v \in \Gamma, \pi \in \mathcal{P}, [v]\pi = \alpha\},$$

  and f_α is an arbitrary function in F_α.
  If $v = 0$, then $f_{v,\pi} := f_\pi = x_\pi$.
  If $v = \delta$ for some $\delta \in \bar{\delta}$, define

[2] We borrow the terminology from [38], who in turns borrows it from Ritt. Then $\mathcal{P}$ is the set of minimal leaders.

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$$f_{\delta,\pi} :=- \frac{p_\pi^\delta + \sum_{\mu\in\mathcal{F}} \frac{\partial p_\pi}{\partial\mu} \cdot f_{\delta\mu}}{\frac{\partial p_\pi}{\partial\pi}}. \tag{5}$$

If $v = \delta w$ with $0 \neq w \in \Gamma$, define

$$f_{\delta w,\pi} := f_{w,\pi}^\delta + \sum_{\mu\in\mathcal{V}} \frac{\partial f_{w,\pi}}{\partial\mu} \cdot f_{\delta,\mu},$$

where $f_{\delta,\mu} := f_{\delta\mu}$ when $\mu \in \mathcal{F}$.

We also define g_α and $g_{w,\pi}$ as the restriction to W of f_α and $f_{w,\pi}$, respectively, and G_α the family $\{g_{w,\pi} : w \in \Gamma, \pi \in \mathcal{P}, [w]\pi = \alpha\}$.

Let $\theta \in \Theta$ be the $\preceq$-l.u.b. of $\mathcal{P}$ and d be the dimension of W. We say that $\mathfrak{S}$ *commutes* at $\alpha \in \Theta$ if, for every $g, g' \in G_\alpha$, g and g' coincide outside a subset of W of dimension less than d (i.e., they coincide "almost everywhere" on W). We say that $\mathfrak{S}$ commutes locally if it commutes at every $\alpha \leq \theta$, and it commutes globally if it commutes at every $\alpha \in \Theta$.

The main result that makes the machinery work is the following.

THEOREM 5.2. *$\mathfrak{S}$ commutes locally iff it commutes globally.*

We will say then that $\mathfrak{S}$ commutes if it commutes locally (equivalently, globally).

In order to prove the above theorem, we need some preliminary definitions and results. It suffices to prove the theorem for every K_0-irreducible component of W of dimension d. Thus, w.l.o.g. we may *assume* for the remainder of this subsection that W is K_0-irreducible. Then, the condition that W commutes at a certain α becomes that G_α is a singleton.

We denote by $K_0[W]$ the ring of regular functions on W (that is, the restriction to W of polynomial maps $K_0^{\mathcal{V}} \to K_0$). Since we are assuming that W is K_0-irreducible, $K_0[W]$ is an integral domain. Therefore, we can consider its fraction field $K_0(W)$. An element of $K_0(W)$ is the restriction to W of a rational function in $K_0(x_{\mathcal{V}})$; in particular, the functions g_α and $g_{w,\pi}$ are in $K_0(W)$.

Given $\delta \in \bar{\delta}$, we define the following function:

$$\begin{aligned} R_0^\delta : K_0(x_{\mathcal{V}}) &\longrightarrow K_0(x_{\mathcal{V}}) \\ h &\longmapsto h^\delta + \sum_{\mu\in\mathcal{V}} \frac{\partial h}{\partial\mu} \cdot f_{\delta,\mu}. \end{aligned}$$

Notice that R_0^δ is the unique derivation extending δ such that $R_0^\delta(x_\mu) = f_{\delta,\mu}$, for every $\mu \in \mathcal{V}$. Given $w = w_1 \dots w_\ell \in \Gamma$, we can define $R_0^w : K_0(x_{\mathcal{V}}) \to K_0(x_{\mathcal{V}})$ as the composition $R_0^w := R_0^{w_1} \circ \cdots \circ R_0^{w_\ell}$.

We have, for every $v, w \in \Gamma$ and every $\pi \in \mathcal{P}$,

$$R_0^w(f_{v,\pi}) = f_{wv,\pi}. \tag{6}$$

If we restrict R_0^δ to $K_0(x_{\mathcal{F}})$ and compose with the restriction to W, we obtain a derivation $R_1^\delta : K_0(x_{\mathcal{F}}) \to K_0(W)$. An equivalent definition is that R_1^δ is the unique derivation extending δ such that $R_1^\delta(x_\mu) = g_{\delta\mu}$ for every $\mu \in \mathcal{F}$. Finally, observe that $K_0(W)$ is an algebraic extension of $K_0(x_{\mathcal{F}})$. So, R_1^δ extends uniquely to a derivation R^δ from $K_0(W)$ to the algebraic closure of $K_0(W)$.

We will consider the objects x_μ both as variables and as functions. Observe that, for every $\mu \in \mathcal{V}$, g_μ is the restriction of x_μ to W.

REMARK 5.3. Let $f \in K_0(x_\mathcal{V})$ and $h \in K_0(W)$ be the restriction of f to W. Then, we have $h = f(g_\mathcal{V})$ (we are seeing f as a rational function). Therefore,

$$R^\delta(h) = f^\delta(g_\mathcal{V}) + \sum_{\mu \in \mathcal{V}} \frac{\partial f}{\partial \mu} \restriction_W \cdot R^\delta(g_\mu) = f^\delta \restriction_W + \sum_{\mu \in \mathcal{V}} \frac{\partial f}{\partial \mu} \restriction_W \cdot g_{\delta,\mu}.$$

LEMMA 5.4. *Let $\mu \in \mathcal{V}$. Then,*

$$R^\delta(g_\mu) = g_{\delta,\mu}. \tag{7}$$

PROOF. If $\mu \in \mathcal{F}$, the conclusion follows by definition of R^δ_1.
If $\mu = \pi \in \mathcal{P}$, observe that g_π satisfies the algebraic condition

$$p_\pi(g_\pi, g_{\mathcal{F}<\pi}) =^\pi 0.$$

Therefore,

$$R^\delta(g_\pi) = -\frac{p_\pi^\delta + \sum_{\beta \in \mathcal{F}<\pi} \frac{\partial p_\pi}{\partial \beta} \cdot R^\delta(g_\beta)}{\frac{\partial p_\pi}{\partial \pi}}(g_\pi, g_{\mathcal{F}<\pi}) = f_{\delta,\pi} \restriction_W = g_{\delta,\pi}. \qquad \dashv$$

REMARK 5.5. The image of R^δ is already in $K_0(W)$ (no need to take the algebraic closure). Indeed, as a K_0-algebra, $K_0(W)$ is generated by $(g_\mu : \mu \in \mathcal{V})$. Thus, it suffices to show that $R^\delta(g_\mu) \in K_0(W)$, which follows from Lemma 5.4.

LEMMA 5.6. *For all $w \in \Gamma$, $R^w(g_{v,\pi}) = g_{wv,\pi}$.*

PROOF. Apply (7) and (6). $\dashv$

LEMMA 5.7. *Let $\delta, \varepsilon \in \bar{\delta}$, $f \in K_0(x_\mathcal{V})$, and $h \in K_0(W)$ be the restriction of f to W. Then,*

$$[R^\varepsilon, R^\delta]h = \sum_{\mu \in \mathcal{V}} \frac{\partial f}{\partial \mu} \restriction_W \cdot (R^\varepsilon g_{\delta,\mu} - R^\delta g_{\varepsilon,\mu}). \tag{8}$$

PROOF. (FIRST PROOF) Let $\lambda = [\varepsilon, \delta]$ be the Lie bracket of ε and δ, and $R^\lambda = [R^\varepsilon, R^\delta]$ be corresponding Lie bracket. We can write $h = f(g_\mathcal{V})$, where we see f as a rational function: therefore, since R^λ is a derivation, we have

$$R^\lambda h = h^\lambda + \sum_{\mu \in \mathcal{V}} \frac{\partial f}{\partial \mu}(g_\mathcal{V}) \cdot R^\lambda(g_\mu) = \sum_{\mu \in \mathcal{V}} \frac{\partial f}{\partial \mu} \restriction_W \cdot (R^\varepsilon g_{\delta,\mu} - R^\delta g_{\varepsilon,\mu}).$$

(SECOND PROOF) Since both the LHS and the RHS of (8) define a derivation on $K_0(W)$, it suffices to show that they are equal when $f = x_\mu$ for some $\mu \in \mathcal{F}$.
Then, RHS is equal to $R^\varepsilon g_{\delta,\mu} - R^\delta g_{\varepsilon,\mu}$, which is equal to $R^\varepsilon R^\delta g_\mu - R^\delta R^\varepsilon g_\mu$, i.e., the LHS. $\dashv$

We can finally prove the theorem.

PROOF OF THEOREM 5.2. Assume that $\mathfrak{S}$ commutes locally (and that W is K_0-irreducible). Let $\alpha \in \mathcal{B}$: we show, by induction on α, that $\mathfrak{S}$ commutes at α. Let

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$\pi, \pi' \in \mathcal{P}$ and $w, w' \in \Gamma$ be such that $[w]\pi = [w']\pi' = \alpha$: we need to show that $g_{w,\pi} = g_{w',\pi'}$.

If $\pi = \pi'$, we can reduce to the case when $w = v\delta\varepsilon u$ and $w' = v\varepsilon\delta u$ for some $\delta, \varepsilon \in \bar{\delta}$, $v, u \in \Gamma$.

If $u \neq 0$, we have, by inductive hypothesis, that $g_{v\delta\varepsilon,\pi} = g_{v\varepsilon\delta,\pi}$ and therefore

$$g_{w,\pi} = R^u g_{v\delta\varepsilon,\pi} = R^u g_{v\varepsilon\delta,\pi} = g_{w',\pi},$$

which is the thesis.

If instead $u = 0$, we have

$$g_{w,\pi} - g_{w',\pi} = [R^\varepsilon, R^\delta] g_{v,\pi} = \sum_{\mu \in \mathcal{V}} \frac{\partial f_{v,\pi}}{\partial \mu} \restriction_W \cdot (R^\varepsilon g_{\delta,\mu} - R^\delta g_{\varepsilon,\mu}).$$

Fix some μ in the above sum. Notice that $\mu < [v]\delta$, and therefore $\delta\varepsilon\mu = \varepsilon\delta\mu < \alpha$. Therefore, by inductive hypothesis, $R^\varepsilon g_{\delta,\mu} = g_{\varepsilon\delta\mu} = R^\delta g_{\varepsilon,\mu}$. Thus, all summands are 0 and $g_{w,\pi} - g_{w',\pi} = 0$, and we are done.

If $\pi \neq \pi'$, let $\beta := \pi \vee \pi'$; by definition, $\beta \leq \theta$, and therefore, by assumption, $G_\beta = \{g_\beta\}$. Let $u, u', w \in \Gamma$ be such that

$$\beta = [u]\pi = [u']\pi', \quad \alpha = [w]\beta.$$

By the previous case, we have

$$g_{v,\pi} = g_{wu,\pi} = R^w g_{u,\pi} = R^w g_\beta = R^w g_{u',\pi'} = g_{wu',\pi'} = g_{v'\pi'}. \qquad \dashv$$

REMARK 5.8. If $\mathfrak{S}$ commutes globally, then the derivations $R^{\delta_1}, \dots, R^{\delta_k}$ commute with each other.

The following remark motivates the definition of the functions g_μ.

REMARK 5.9. Let $b \in K$ be such that $b^{\mathcal{V}} \in W$. Then, for every $w \in \Gamma$ and $\pi \in \mathcal{P}$,

$$b^{[w]\pi} = g_{w,\pi}(b).$$

Therefore, for every $\mu \in \Theta$,

$$b^\mu = g_\mu(b).$$

PROOF. By induction on w. $\dashv$

COROLLARY 5.10. *Let $b \in K$ be such that $b^{\mathcal{V}} \in W$ and $b^{\mathcal{F}}$ is algebraically independent over K_0. Then, $\mathfrak{S}$ commutes.*

PROOF. Remember that we are assuming that W is K_0-irreducible. By Remark 5.9, for every $\mu \in \Theta$ and $g, g' \in G_\mu$, $g(b^{\mathcal{V}}) = g'(b^{\mathcal{V}})$. Since W is irreducible and $b^{\mathcal{V}}$ is generic in W (over K_0), we have that $g = g'$. $\dashv$

5.2. The axioms.

DEFINITION 5.11. The axioms of $T_g^{\bar{\delta}}$ are the axioms of $T^{\bar{\delta}}$ plus the following axiom scheme:

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($\boldsymbol{k}$-Deep) Let $\mathfrak{S} = (W; p_\pi : \pi \in \mathcal{P})$ be a commutative configuration.[3] Let $X \subseteq \mathbb{K}^{\mathcal{V}}$ be $L(\mathbb{K})$-definable, such that $\Pi_{\mathcal{F}}(X)$ is large. Then, there exists $a \in \mathbb{K}$ such that $a^{\mathcal{V}} \in X$.

Notice that the above is the analogue of the axiom scheme (Deep). We don't have an analogue for the axiom scheme (Wide). Notice also that the axiom scheme (k-Deep) is first-order expressible thanks to Theorem 5.2.

THEOREM 5.12.

(1) *$T_g^{\bar\delta}$ is a consistent and complete extension of $T^{\bar\delta}$.*

(2) *If T is model-complete, then $T_g^{\bar\delta}$ is an axiomatization for the model completion of $T^{\bar\delta}$.*

(3) *If T eliminates quantifiers, then $T_g^{\bar\delta}$ eliminates quantifiers.*

(4) *For every $L^{\bar\delta}$-formula $\alpha(\bar x)$ there exists an L-formula $\beta(\bar x)$ such that*
$$T_g^{\bar\delta} \models \forall \bar x\big(\alpha(\bar x) \leftrightarrow \beta(\bar x^{\Theta})\big).$$

(5) *For every $\langle \mathbb{K}, \bar\delta\rangle \models T_g^{\bar\delta}$, for every $\bar a$ tuple in $\mathbb{K}$ and $B \subseteq \mathbb{K}$, the $L^{\bar\delta}$-type of $\bar a$ over B is determined by the L-type of $\bar a^{\Theta}$ over B^{Θ}.*

We assume that T eliminates quantifiers. We use the criterion in Proposition 3.2(3) to show that $T_g^{\bar\delta}$ is the model completion of $T^{\bar\delta}$ and it eliminates quantifiers. We will do it in two lemmas.

LEMMA 5.13. *Let $\langle A, \bar\delta\rangle \models T^{\bar\delta}$. Let $\mathfrak{S} = (W; p_\pi : \pi \in \mathcal{P})$ be a commutative configuration with parameters in A, and $X \subseteq \mathbb{K}^{\mathcal{V}}$ be an $L(A)$-definable set, such that $\Pi_{\mathcal{F}}(X)$ is large.*

Then, there exists $\langle B, \bar\delta\rangle \supseteq \langle A, \bar\delta\rangle$ and $b \in B$ such that $B \succeq A$, $\langle B, \bar\delta\rangle \models T^{\bar\delta}$, and $b^{\mathcal{V}} \in X$.

PROOF. Let $B \succeq A$ be $|A|^+$-saturated. Let $\bar b = (b_v : v \in \mathcal{V}) \in X_B$ be such that $b_{\mathcal{F}} := \Pi_{\mathcal{F}}(\bar b)$ is algebraically independent over A. Let $I \subset B$ be such that I is disjoint from $b_{\mathcal{F}}$ and $D := b_{\mathcal{F}} \cup I$ is a transcendence basis of B over A. We extend each derivation $\delta \in \bar\delta$ to B in the following way. It suffices to specify the value of δ on each $c \in D$.

If $c \in I$, we define $\delta c := 0$.

If $c = b_\mu$ for some $\mu \in \mathcal{F}$, we define $\delta b_\mu := g_{\delta\mu}(\bar b)$.

By definition, it is clear that $b_0^{\mathcal{V}} = \bar b \in X$. Thus, it suffices to show that the extensions of $\bar\delta$ commute on all B. Again, it suffices to show that, for every $c \in D$ and every $\delta, \varepsilon \in \bar\delta$,
$$\delta\varepsilon c = \varepsilon\delta c.$$
If $c \in I$, then both sides are equal to 0, and we are done.

Suppose now that $c = b_\mu$ for some $\mu \in \mathcal{F}$, we have to show that
$$\delta\varepsilon b_\mu = \varepsilon\delta b_\mu. \tag{9}$$

[3] We are no longer assuming that W is irreducible.

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Since the argument is delicate we prefer to give two different proofs with two different approaches.

(First proof) For this first proof, we replace W with its A-irreducible component containing $\bar{b}$. Thus, we may assume that W is A-irreducible. Since $\bar{b}$ is generic in W, the map $A(W) \to A(\bar{b})$, $h \mapsto h(\bar{b})$ is an isomorphism of A-algebras. Via the above isomorphism, the derivation R^δ on $A(W)$ corresponds to the derivation δ on $A(\bar{b})$. The assumptions plus Theorem 5.2 and Remark 5.8 imply that R^δ and R^ε commute: therefore, also δ and ε commute on $A(\bar{b})$.

(Second proof) Since $\bar{b} \in W$, we have that $\delta b_\pi = f_{\delta,\pi}(\bar{b})$ for every $\pi \in \mathcal{P}$. Thus, by definition, for every $v \in \mathcal{V}$,

$$\delta b_v = f_{\delta,v}(\bar{b}).$$

Denote $f := f_{\varepsilon\mu}$. By definition, the LHS of (9) is equal to

$$\begin{aligned}\delta g_{\varepsilon\mu}(\bar{b}) = \delta f(\bar{b}) = f^\delta(\bar{b}) + \sum_{v \in \mathcal{V}} \frac{\partial f}{\partial v}(\bar{b}) \cdot \delta b_v = \\ = f^\delta(\bar{b}) + \sum_{v \in \mathcal{V}} \frac{\partial f}{\partial v}(\bar{b}) \cdot f_{\delta,v}(\bar{b}) = f_{\delta,\varepsilon\mu}(\bar{b}) = g_{\delta,\varepsilon\mu}(\bar{b}).\end{aligned} \tag{10}$$

Similarly, the RHS of (9) is equal to $g_{\varepsilon,\delta\mu}(\bar{b})$. Finally, since $\mathfrak{S}$ commutes, $g_{\delta,\varepsilon\mu} = g_{\varepsilon,\delta\mu}$, and we are done. ⊣

Lemma 5.14. *Let $\langle B, \bar{\delta}\rangle \models T^{\bar{\delta}}$, $\langle C, \bar{\delta}\rangle \models T_g^{\bar{\delta}}$, and $\langle A, \bar{\delta}\rangle$ be a common substructure, such that B and C have the same $L(A)$-theory. Let $\gamma(x)$ be a quantifier-free $L^{\bar{\delta}}$-formula with parameters in A. Let $b \in B$ be such that $\langle B, \bar{\delta}\rangle \models \gamma(b)$. Then, there exists $c \in C$ such that $\langle C, \bar{\delta}\rangle \models \gamma(c)$.*

Proof. W.l.o.g., we may assume that the only constants in the language L are the elements of F, the only function symbols are $+$ and $\cdot$, thus, an $L(\bar{\delta})$-substructure is a differential subring containing F. Let A' (resp., A'') be the relative algebraic closure (as fields, or equivalently as L-structures) of A inside B (resp., C). Since A has the same L-type in B and C, there exists isomorphism ϕ of L-structures between of A' and A'' extending the identity on A. Moreover, any derivations on A extends uniquely to A' and A'': thus, ϕ is also an isomorphism of $L^{\bar{\delta}}$-structures. Thus, w.l.o.g. we may assume that A is relatively algebraically closed in B and in C.

Let

$$\mathcal{B} := \{\mu \in \Theta : \mu b \in \operatorname{acl}(A b^{<\mu})\}$$

and $\mathcal{F} := \Theta \setminus \mathcal{B}$. If $\mathcal{F}$ is empty, then $b \in \operatorname{acl}(A) = A$, and we are done.

Otherwise, let $\mathcal{P}$ be the set of $\preceq$-minimal elements of $\mathcal{B}$. For every $\pi \in \mathcal{P}$, let $p_\pi \in A[x_\pi, x_{\mathcal{F}<\pi}]$ be such that $p_\pi(\pi b, b^{\mathcal{F}<\pi}) =^\pi 0$. Let W be the A-irreducible component of

$$W_0 := \{d_{\mathcal{V}} \in B^{\mathcal{V}} : \bigwedge_{\pi \in \mathcal{P}} p_\pi(d_{\mathcal{V}}) =^\pi 0\}$$

containing $b^{\mathcal{V}}$.

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CLAIM 2. *$\mathfrak{S} := (W; p_\pi : \pi \in \mathcal{P})$ is a commutative configuration (over A).*

Again, we prefer to give two different proofs.

(FIRST PROOF) Since $b^{\mathcal{V}}$ is generic (over A) in the A-irreducible variety W, the map $\Phi : A(W) \mapsto A(b^{\mathcal{V}})$, $h \mapsto h(b^{\mathcal{V}})$ is an isomorphism of A-algebrae. Via the above isomorphism, R^δ corresponds to the derivation δ on $A(b^{\mathcal{V}}) = A(b^\Theta)$: thus, the maps R^δ commute with each other. Therefore, for every $\pi, \pi' \in \mathcal{P}$ and $w, w' \in \Gamma$ with $[w]\pi = [w']\pi'$,

$$g_{w,\pi} = R^w g_\pi = \Phi((b^\pi)^{[w]}) = \Phi((b^{\pi'})^{[w']}) = g_{w',\pi'}.$$

(SECOND PROOF) By Corollary 5.10.
Let $\beta(z_\Theta)$ be a quantifier-free $L(A)$-formula such that

$$T^{\bar{\delta}} \cup \mathrm{Diag}_L(A) \vdash \gamma(x) \leftrightarrow \beta(x^\Theta).$$

Let

$$X := \{x_{\mathcal{V}} \in W : \bar{d} \in W : B \models \beta\big((g_\mu(\bar{d}))_{\mu\in\Theta}\big)\}.$$

Since $b^{\mathcal{V}} \in X$, and X is $L(A)$-definable, we have that $\Pi_{\mathcal{F}}(X)$ is large. Thus, there exists $c \in C$ such that $c^{\mathcal{V}} \in X$. Since $c^{\mathcal{V}} \in W$, by Remark 5.9 for every $\mu \in \Theta$ we have $c^\mu = g_\mu(c^{\mathcal{V}})$. Therefore, $\langle C, \bar{\delta}\rangle \models \beta(c^\Theta)$. ⊣

5.2.1. Addendum. Call a configuration $\mathfrak{S} = (W; p_\pi : \pi \in \mathcal{P})$ "K_0-irreducible" if W is K_0-irreducible. In the definition of a configuration we did not impose that it is irreducible. However, by the proof of the Amalgamation Lemma 5.14, it seems that it would suffice to impose in Axiom (k-`Deep`) that only irreducible configurations need to be satisfied. The reason why we did not restrict ourselves to irreducible configurations is that we don't know if we can impose irreducibility in a first-order way.

QUESTION 5.15. *Let $(W_i : i \in I)$ be an L-definable family of varieties in K^n. Is the set*

$$\{i \in I : W_i \text{ is } K\text{-irreducible}\}$$

definable?

§6. Stability and NIP. In this section we see that some of the model theoretic properties of T are inherited by $T_g^{\bar{\delta},?}$. In a following paper we will consider other properties. We assume basic knowledge about stable and NIP theories: see [40, 51].

THEOREM 6.1.
(1) *If T is stable, then $T_g^{\bar{\delta},?}$ is stable.*
(2) *If T is NIP, then $T_g^{\bar{\delta},?}$ is NIP.*

The above theorem follows immediately from the following one.

THEOREM 6.2. *Let U be an L-theory. Let $\bar{\delta}$ be a set of new unary function symbols. Let U' be an $L^{\bar{\delta}}$-theory expanding U. Assume that, for every $L^{\bar{\delta}}$-formula $\alpha(\bar{x})$ there exists and L-formula $\beta(\bar{y})$ such that*

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$$U' \models \forall \bar{x}\ \alpha(\bar{x}) \leftrightarrow \beta(\bar{x}^\Gamma),$$

where $\bar{x}^\Gamma$ is the set of $\bar{\delta}$-terms in the variables $\bar{x}$.

Then, for every $(M, \bar{\delta}) \models U'$ and every $\bar{a}$ tuple in M and B subset of M, the $L^{\bar{\delta}}$-type of $\bar{a}$ over B is uniquely determined by the L-type of $\bar{a}^\Gamma$ over B^Γ.

Moreover:

(1) *If U is stable, then U' is stable.*
(2) *If U NIP, then U' is NIP.*

Proof. The results follow easily by applying the following criteria.

(1) [50, Theorem II. 2.13] A theory U is stable iff, for every subset A of a model M of U, and for every sequence $(\bar{a}_n)_{n\in\mathbb{N}}$ of tuples in M, if $(\bar{a}_n)_{n\in\mathbb{N}}$ is an indiscernible sequence, then it is totally indiscernible.

(2) [51, Proposition 2.8] A theory U is NIP iff, for every formula $\phi(\bar{x}; \bar{y})$ and for any indiscernible sequence $(\bar{a}_i : i \in I)$ and tuple $\bar{b}$, there is some end segment $I_0 \subseteq I$ such that $\phi(a_i; b)$ is "constant" on I_0: that is, either for every $i \in I_0$ $\phi(\bar{a}_i; \bar{b})$ holds, or for every $i \in I_0$ $\neg\phi(\bar{a}_i; \bar{b})$ holds. ⊣

§7. Pierce–Pillay axioms. We give now an extra axiomatization for T^δ_g, in the "geometric" style of Pierce and Pillay [39]. We won't use this axiomatization, but it may be of interest.

Let $\langle \mathbb{K}, \delta \rangle \models T^\delta$.

Let $W \subseteq \mathbb{K}^n$ be an algebraic variety defined over $\mathbb{K}$. We define the twisted tangent bundle τW of W w.r.t. δ in the same way as in [39] (see also [35], where it is called "prolongation").

Let $\bar{x} := \langle x_1, \dots, x_n \rangle$. Let $\mathbb{K}^* \succeq \mathbb{K}$ and $\bar{a} := \langle a_1, \dots, a_n \rangle \in (\mathbb{K}^*)^n$. We define:

$$I(\bar{a}/\mathbb{K}) := \{p \in \mathbb{K}[\bar{x}] : p(\bar{a}) = 0\},$$
$$I(W/\mathbb{K}) := \{p \in \mathbb{K}[\bar{x}] : \forall \bar{c} \in W\ p(\bar{c}) = 0\}.$$

Let $\bar{p} = (p_1, \dots, p_\ell) \in \mathbb{K}[\bar{x}]^\ell$. Define

$$V_{\mathbb{K}}(\bar{p}) := \{\bar{c} \in \mathbb{K}^n : p_i(\bar{c}) = 0, i = 1, \dots, \ell\},$$

and $(\bar{p})_{\mathbb{K}}$ to be the ideal of $\mathbb{K}[\bar{x}]$ generated by $p_1, \dots, p_\ell$.

Definition 7.1. Assume that $I(W/\mathbb{K}) = (p_1, \dots, p_\ell)_{\mathbb{K}}$. The twisted tangent bundle of W (w.r.t. δ) is the algebraic variety $\tau^\delta W \subseteq \mathbb{K}^n \times \mathbb{K}^n$

$$\tau^\delta W := \{\langle \bar{x}, \bar{y} \rangle \in \mathbb{K}^n \times \mathbb{K}^n : p_i^{[\delta]}(\bar{x}, \bar{y}) = 0, i = 1, \dots, n\} \subseteq \mathbb{K}^n \times \mathbb{K}^n,$$

where $p^{[\delta]}$ was introduced in Definition 3.6.

Notice that the definition of $\tau^\delta W$ does not depend on the choice of polynomials $\bar{p}$ such that $I(W/\mathbb{K}) = (\bar{p})_{\mathbb{K}}$. Notice also that, when $\delta = 0$, the twisted tangent bundle $\tau^0 W$ coincides with the tangent bundle.

The importance in this context of the twisted tangent bundle is due to the following two facts:

Remark 7.2. If $\bar{a} \in W$, then $\langle \bar{a}, \delta\bar{a} \rangle \in \tau^\delta W$.

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FACT 7.3. *Let $L \supset \mathbb{K}$ be a field. Let $\bar{b} \in W$ be (as interpreted in L) such that $\bar{b}$ is generic in W over $\mathbb{K}$ (that is,* $\mathrm{rk}(\bar{b}/\mathbb{K}) = \dim(W)$*). Let $\bar{c} \in L^n$ be such that $\langle \bar{b}, \bar{c} \rangle \in \tau^\delta W$.*

Then, there exists a derivation ε on L extending δ and such that $\varepsilon \bar{b} = \bar{c}$.

PROOF. It is a known result: see [57, Chapter II, Section 17, Theorem 39]. See also [26, Theorem VIII.5.1], [39], and [19, Lemma 1.1]. $\dashv$

We want to write an axiom scheme generalizing Pierce–Pillay to T_g^δ.

An idea would be to use the following:

(PP-wrong) Let $W \subseteq \mathbb{K}^n$ be an algebraic variety which is defined over $\mathbb{K}$ and $\mathbb{K}$-irreducible. Let $U \subseteq \tau^\delta W$ be an $L(\mathbb{K})$-definable set, such that the projection of U over W is large in W (i.e., of the same dimension as W). Then, there exists $\bar{a} \in W$ such that $\langle \bar{a}, \delta \bar{a} \rangle \in U$.

However, there is an issue with the above axiom scheme: we don't know how to express it in a first-order way! The reason is the following: given a definable family of tuples of polynomials $(\bar{p}_i : i \in I)$, while each $\tau^\delta(V_\mathbb{K}(\bar{p}_i))$ is definable, we do not know whether the family $\big(\tau^\delta(V_\mathbb{K}(\bar{p}_i)) : i \in I\big)$ is definable. We leave it as an open problem, and we will use a different axiom scheme.

QUESTION 7.4. *Let $\big(\bar{p}_i : i \in I\big)$ be a definable family of tuples of polynomials. Is there a definable family $\big(\bar{q}_i : i \in I\big)$ of tuples of polynomials, such that $I(V_\mathbb{K}(\bar{p}_i)/\mathbb{K}) = (\bar{q}_i)_\mathbb{K}$ for every $i \in I$?*

The above question is related to Question 5.15: notice that "W is $\mathbb{K}$-irreducible" is equivalent to "$I(W/\mathbb{K})$ is prime", and the latter, by [11] (see also [49]), is a definable property of the parameters of the formula defining $I(W/\mathbb{K})$.

We need some additional definitions and results before introducing the true axiom scheme. Fix $\bar{p} \in \mathbb{K}[\bar{x}]^\ell$, and let $W := V_\mathbb{K}(\bar{p})$. Given $\bar{a} \in W$, the twisted tangent space of $\bar{p}$ at $\bar{a}$ is

$$\tau^\delta_{\bar{a}}(\bar{p}) := \{\bar{y} \in \mathbb{K}^n : p_i^{[\delta]}(\bar{a}, \bar{y}) = 0 : i = 1, \dots, \ell\}.$$

Moreover, $\tau^0(\bar{p})$ is the usual tangent space at $\bar{a}$ of $V_\mathbb{K}(\bar{p})$.

REMARK 7.5. Let $\bar{a} \in (K^*)^n$, we define $J := I(\bar{a}/\mathbb{K})$ and $F := \mathbb{K}(\bar{a}) \subseteq \mathbb{K}^*$. Let J' be the ideal of $F[\bar{x}]$ generated by J and let $S := \tau^0_{\bar{a}} J'$ be the tangent space at $\bar{a}$ (as an F-vector space). Then, the dimension of S as an F-vector space is equal to $\mathrm{rk}(\bar{a}/\mathbb{K})$. Indeed, let S' be the set of derivations on F which are 0 on $\mathbb{K}$: then, by Fact 7.3, S and S' are isomorphic as F-vector spaces. By [57, Chapter II, Section 17, Theorem 41], the dimension of S' as F-vector space is equal to $\mathrm{rk}(\bar{a}/\mathbb{K})$.

Let $W := V_\mathbb{K}(\bar{p})$; the following lemma shows that, under suitable conditions, we can replace $I(W/\mathbb{K})$ with $I(\bar{p}/\mathbb{K})$. We need to introduce some notations:

Let $d_0 := \dim(W)$ and for every $d \in \mathbb{N}$, we define

$$\mathrm{Reg}^d(\bar{p}) := \{\bar{c} \in W : \dim(\tau^0_{\bar{a}}(\bar{p})) = d\}$$

and $\mathrm{Reg}(\bar{p}) := \mathrm{Reg}^{d_0}(\bar{p})$.

LEMMA 7.6. *Let $\bar{a} \in (\mathbb{K}^*)^n$ be such that $\bar{a} \in \mathrm{Reg}(\bar{p})$ and $rk(\bar{a}/\mathbb{K}) \geq d_0$. Let $J := I(\bar{a}/\mathbb{K})$ and $W := V_\mathbb{K}(\bar{p})$.*

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Then,

(1) $\dim(\mathrm{Reg}(\bar{p})) = \mathrm{rk}(\bar{a}/\mathbb{K}) = d_0$;
(2) $\tau^0_{\bar{a}}(\bar{p}) = \tau^0_{\bar{a}}(J)$;
(3) $\tau^\delta_{\bar{a}}(\bar{p}) = \tau^\delta_{\bar{a}}(J)$;
(4) *for every* $\bar{b} \in (\mathbb{K}^*)^n$ *such that* $\langle \bar{a}, \bar{b}\rangle \in \tau^\delta(\bar{p})$ *there exists* δ' *derivation on* $\mathbb{K}^*$ *extending* δ *and such that* $\delta'(\bar{a}) = \bar{b}$.

Proof. (1) It is clear:

$$d_0 \leq \mathrm{rk}(\bar{a}/\mathbb{K}) \leq \dim(\mathrm{Reg}(\bar{p})) \leq \dim(W) = d_0.$$

(2) Since $\bar{p}(\bar{a}) = 0$, we have $(\bar{p})_{\mathbb{K}} \subseteq J$, and therefore

$$V_{\mathbb{K}}(J) \subseteq V_{\mathbb{K}}(\bar{p}) = W.$$

Thus, $\dim(V_{\mathbb{K}}(J)) \leq \dim W = d_0$. Therefore, $\bar{a}$ is also a generic point of $V_{\mathbb{K}}(J)$. So, by Remark 7.5,

$$dim(\tau^0_{\bar{a}}(J)) = \mathrm{rk}(\bar{a}/\mathbb{K}) = d_0.$$

Therefore, the vector space $\tau^0_{\bar{a}}(\bar{p})$ contains $\tau^0_{\bar{a}}(J)$ and has the same dimension d_0: thus, they are equal.

(3) Notice that $\tau^0_{\bar{a}}(\bar{p})$ and $\tau^\delta_{\bar{a}}(\bar{p})$ are vector spaces of the same dimension, and the same happens for $\tau^\delta_{\bar{a}}(J)$. Moreover, $\tau^\delta_{\bar{a}}(\bar{p})$ contains $\tau^\delta_{\bar{a}}(J)$ and has the same dimension, and therefore $\tau^\delta_{\bar{a}}(\bar{p}) = \tau^\delta_{\bar{a}}(J)$.

(4) It follows from Fact 7.3. ⊣

Given $m, n, d \in \mathbb{N}$, let

$$(\bar{p}_{m,n,d}(\bar{x}, \bar{a}) : \bar{a} \in \mathbb{K}^\ell)$$

be a parameterization (definable in the language of rings) of all m-tuples of polynomials in $\mathbb{K}[x_1, \dots, x_n]$ of degree at most d. We will write $\bar{p}$ instead of $\bar{p}_{m,n,d}$.

For each $\bar{a} \in \mathbb{K}^\ell$, let $W_{\bar{a}} := V_{\mathbb{K}}(\bar{p}(\bar{x}, \bar{a}))$ and $U_{\bar{a}} := \mathrm{Reg}(\bar{p}(\bar{x}, \bar{a})) \subseteq W_{\bar{a}}$. Let $\Pi_n : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n$ be the canonical projection onto the first n coordinates.

We can finally write the axiom scheme.

(PP) Let $m, n, d \in \mathbb{N}$ and $\bar{p} := \bar{p}_{m,n,d}(\bar{x}, \bar{y})$. Let $\bar{a} \in \mathbb{K}^\ell$ and $X \subseteq \tau^\delta(\bar{p}(\bar{x}, \bar{a}))$ be $L(\mathbb{K})$-definable. Assume that $\Pi_n(X) \subseteq U_{\bar{a}}$ and $\dim(\Pi_n(X)) = \dim(W_{\bar{a}})$. Then, there exists $\bar{c} \in \mathbb{K}^n$ such that $\langle \bar{c}, \delta\bar{c}\rangle \in X$.

Theorem 7.7. $T^\delta_{\mathrm{PP}} := T^\delta \cup$ (PP) *is an axiomatization of* T^δ_g.

Proof. Since we can take $\bar{p}$ to be the empty tuple, and therefore $W = \mathbb{K}^n$, it is clear that (PP) implies (`Wide`).

We have to prove the opposite. Since T^δ_g is complete, it suffices to show that T^δ_{PP} is consistent. W.l.o.g., we may assume that T has elimination of quantifiers. To show that T^δ_{PP} is consistent, it suffices to prove the following:

Claim 3. *Let* $m, n, d, \bar{p}, \bar{a}, X$ *be as in* (PP). *Then there exists* $\mathbb{K}^* \succeq \mathbb{K}$ *and* $\bar{b} \in (\mathbb{K}^*)^n$ *such that* $\langle \bar{b}, \delta\bar{b}\rangle \in X$.

Let $\mathbb{K}^* \succ \mathbb{K}$ be sufficiently saturated and $\bar{b} \in \mathbb{K}^{*n}$ such that $\bar{b} \in \Pi_n(X)$ and $\mathrm{rk}(\bar{b}/K) = \dim(\Pi_n(X)) = \dim(W_{\bar{a}})$. Let $\bar{c} \in \mathbb{K}^{*n}$ be such that $\langle \bar{b}, \bar{c}\rangle \in X$. By Lemma 7.6 there exists δ' derivation on $\mathbb{K}^*$ extending δ and such that $\delta(\bar{b}) = \bar{c}$. ⊣

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Giving the analogue axiomatization for $T_g^{\bar{\delta},nc}$ is not difficult and the reader can provide the details.

On the other hand, we won't try to give a similar axiomatization for $T_g^{\bar{\delta}}$, since already when $T = ACF$ it is an arduous task: see [27, 37, 38].

§8. Conjectures and open problems. We conclude the paper with a list of open problems, remarks and some ideas.

8.1. Elimination of imaginaries.

Conjecture 8.1. $T_g^{\bar{\delta},?}$ has elimination of imaginaries modulo T^{eq}.

A few particular cases are known, when $T_g^{\bar{\delta},?}$ is one of the following:

- $\mathrm{DCF}_{0,\mathrm{m}}$: see [32].
- RCF or certain theories of Henselian valued fields, endowed with m commuting generic derivations: see [8, 14] for a proof based on M. Tressl's idea; see also [5, 34, 41] for different proofs.
- $\mathrm{DCF}_{0,\mathrm{m,nc}}$ (see [36]).

We have also established the validity of the above conjecture for certain topological structures. Drawing upon established techniques, it is probable that the conjecture can be proven for T simple (as proved in [33, 36]). However, for the general case, we believe that novel approaches are required (although some progress has been made in [5]).

8.2. Definable types. Let $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta},?}$. Given a type $p \in S^n_{L^\delta}(\mathbb{K})$, let $\bar{a}$ be a realization of p; we define $\tilde{p} \in S_L^{n\times\Gamma}(\mathbb{K})$ as the L-type of $\bar{a}^\Gamma$ over $\mathbb{K}$.

Open problem 8.2. *Is it true that p is definable iff $\tilde{p}$ is definable? We conjecture that it is true when $T_g^{\bar{\delta},?} = T_g^{\bar{\delta}}$.*

8.3. Zariski closure. Given $X \subseteq \mathbb{K}^n$, denote by X^{Zar} be the Zariski closure of X.

Open problem 8.3 (See [16]). (1) *Let $\big(X_i : i \in I\big)$ be an L-definable family of subsets of $\mathbb{K}^n$. Is $\big(X_i^{Zar} : i \in I\big)$ also L-definable?*

(2) *Assume that* (1) *holds for $\mathbb{K}$. Let $\langle \mathbb{K}, \bar{\delta} \rangle \models T_g^{\bar{\delta},?}$. Let $\big(X_i : i \in I\big)$ be an L^δ-definable family of subsets of $\mathbb{K}^n$. Is $\big(X_i^{Zar} : i \in I\big)$ also L^δ-definable?*

8.4. Monoid actions. Let Λ be a monoid generated by a k-tuple $\bar{\delta}$: we consider Λ as a quotient of the free monoid Γ. We can consider actions of Λ on models of T such that each δ_i is a derivation: we have a corresponding theory T^Λ whose language is L^δ and with axioms given by T, the conditions that each δ_i is a derivation, and, for every $\gamma, \gamma' \in \Gamma$ which induce the same element of Λ, the axiom $\forall x\, \gamma x = \gamma' x$.

Open problem 8.4. *Under which conditions on Λ the theory T^Λ has a model completion?*

Conjecture 8.5. Let Γ_ℓ be the free monoid in ℓ generators, and Θ_k be the free commutative monoid in k generators. Then, for Λ equal either to $\Gamma_\ell \times \Theta_k$ or to

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$\Gamma_\ell * \theta_k$, T^Λ has a model completion (where $\times$ is the Cartesian product, and $*$ is the free product). More generally, for Γ equal to a combination of free and Cartesian products of finitely many copies of $\mathbb{N}$, T^Λ has a model completion.

As a consequence of [28], when $T = ACF_0$ and $\Lambda = \Gamma_\ell \times \Theta_k$, T^Λ has a model companion.

Maybe the following conditions on Λ suffice for T^Λ to have a model completion:

Let $\preceq$ be the canonical quasi ordering on Λ given by $\alpha \preceq \beta\alpha$ for every $\alpha, \beta \in \Lambda$; we assume that:

- $\preceq$ is a well-founded partial ordering;
- for every $\lambda \in \Lambda$, the set $\{\alpha \in \Lambda : \alpha \preceq \lambda\}$ is finite;
- for every $\alpha, \beta \in \Lambda$, if they have an upper bound, then they have a least upper bound;
- let $X \subset \Lambda$ be finite; assume that X is $\preceq$-initial in Λ; then, $\Lambda \setminus X$ has finitely many $\preceq$-minimal elements;
- if $\alpha_1\delta_1 = \alpha_2\delta_2$ for some $\alpha_i \in \Lambda$ and $\delta_i \in \bar{\delta}$, then δ_1 and δ_2 commute with each other; moreover, there exists $\beta \in \Lambda$ such that $\alpha_1 = \delta_2\beta$ and $\alpha_2 = \delta_1\beta$.

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DIPARTIMENTO DI MATEMATICA E INFORMATICA “ULISSE DINI”
UNIVERSITÀ DI FIRENZE
E-mail: antongiulio.fornasiero@gmail.com
URL: https://sites.google.com/site/antongiuliofornasiero/

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI “RENATO CACCIOPPOLI”
UNIVERSITÀ DEGLI STUDI DI NAPOLI “FEDERICO II”
E-mail: giuseppina.terzo@unina.it