WEAK FAMILIES OF MAPS

J. C. Taylor

(received March 12, 1965)

1. Introduction. Let Ω be an index set and for each $\alpha \in \Omega$ let $f_\alpha : X \to X_\alpha$ be a function where X and X_α are sets. Assume that, for each α, a topology $\underline{O}_\alpha$ is given for X_α. Then, as is well-known, the functions f_α and the topologies $\underline{O}_\alpha$ determine a topology for X. This is the so-called weak or initial topology, which is generated by $\bigcup_\alpha \{ f_\alpha^{-1} O \mid O \in \underline{O}_\alpha \}$.

Bourbaki [1] shows that the weak topology is the unique topology $\underline{O}$ for X satisfying the following condition: a function $f : Y \to X$ is $(\underline{T}, \underline{O})$-continuous if and only if, for each α, $f_\alpha \circ f$ is $(\underline{T}, \underline{O}_\alpha)$-continuous. This suggests that the concept of a weak topology could be defined using the language of category theory.

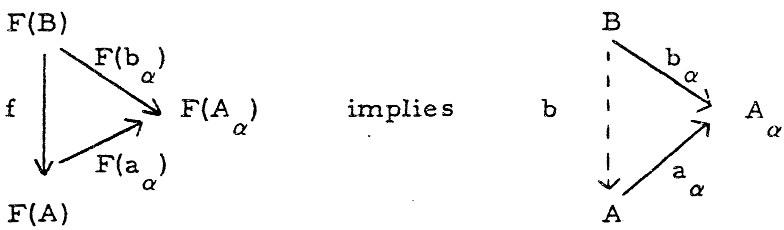
Let $\underline{A}$ denote the category of topological spaces, and let $\underline{E}$ denote the category of sets. Denote by $a : A \to B, b, \ldots$ the morphisms of $\underline{A}$ and by $f : X \to Y, g, \ldots$ those of $\underline{E}$. Let $F : \underline{A} \to \underline{E}$ denote the forgetful functor.

Denote by A the space $(X, \underline{O})$, by A_α the space $(X_\alpha, \underline{O}_\alpha)$, and let (f_α) be a family of functions $f_\alpha : X \to X_\alpha$. The topology of the space A is the weak topology determined by (f_α) and (A_α) if and only if the following assertion holds. A function $f : Y \to X$ is of the form $F(b)$, for $b : B \to A$, if and only if, for each α, there exists $b_\alpha : B \to A_\alpha$ with $f_\alpha \circ f = F(b_\alpha)$.

https://doi.org/10.4153/CMB-1965-057-7 Published online by Cambridge University Press

Assume that the topology of A is the weak topology determined by (f_α) and (A_α). Since the identity map 1_X is $F(1_A)$, each f_α is of the form $F(a_\alpha)$. Hence, as is well known, the family (f_α) determines a family (a_α) of morphisms of $\underline{A}$.

The family (a_α) of morphisms a_α of $\underline{A}$, with common domain A, has the following property: if a function $f : F(B) \to F(A)$ is such that there exists a family (b_α) of morphisms $b_\alpha : B \to A_\alpha$ with, for each α, $F(a_\alpha) \circ f = F(b_\alpha)$, then there exists a unique $b : B \to A$ with $F(b) = f$ and, for each α, $a_\alpha \circ b = b_\alpha$. In terms of diagrams, (a_α) is such that the commutativity, for each α, of



with $F(b) = f$.

Let $\underline{A}$ and $\underline{E}$ be arbitrary categories, and let $F : \underline{A} \to \underline{E}$ be a covariant functor. The purpose of this expository note is to provide some examples and to discuss some elementary properties of families (a_α) of morphisms a_α of $\underline{A}$ which have the above property. When $\underline{E}$ is the trivial category with a unique morphism such families define the direct products that exist in $\underline{A}$.

While the theory outlined here is essentially a translation of Bourbaki's theory of initial structures [2] into the language of categories, it differs in several respects. The emphasis here is on families of morphisms of $\underline{A}$, rather than on the determination of an object of $\underline{A}$ by families of morphisms of $\underline{E}$ and families of objects of $\underline{A}$. Further, the theory of initial structures restricts $\underline{A}$ to be the category determined by a type

https://doi.org/10.4153/CMB-1965-057-7 Published online by Cambridge University Press

of structure, $\underline{E}$ to be the category of sets, and F to be the forgetful functor.

In the case where the index set is a singleton this theory is to be found, in its dual form, in a recent paper of Ehresmann [3].

I would like to thank I. Connell for some interesting conversations on rings during the preparation of this note.

2. Weak families. Let $\underline{A}$ and $\underline{E}$ be two categories and denote by $F : \underline{A} \to \underline{E}$ a covariant functor.

A family (a_α) of morphisms $a_\alpha : A \to A_\alpha$ of $\underline{A}$ will be called an F-weak family or a weak family if it has the following property: if a morphism $f : F(B) \to F(A)$ is such that there exists a family (b_α) of morphisms $b_\alpha : B \to A_\alpha$ with, for each α, $F(a_\alpha) \circ f = F(b_\alpha)$, then there exists a unique $b : B \to A$ with $F(b) = f$ and, for each α, $a_\alpha \circ b = b_\alpha$. A morphism $a : A \to C$ of $\underline{A}$ is called weak if it is a weak family when viewed as a family indexed by a one-point set. In the terminology of Ehresmann [3], a weak morphism is an $(\underline{E}, F)$ injection.

If (a_α) is a family of morphisms $a_\alpha : A \to A_\alpha$ the family (A_α) will be called the range of the family and the object A will be referred to as its domain. Two families (a_α) and (a'_α), with the same range and with domains A and A', will be called isomorphic if there is an isomorphism a of $\underline{A}$ with, for each α, $a_\alpha \circ a = a'_\alpha$.

Examples.

1. A family (a_α) of morphisms of $\underline{A}$ will be said to have the left cancellation property (LCP) if $b=c$ whenever, for each α, $a_\alpha \circ b = a_\alpha \circ c$. When $\underline{A} = \underline{E}$ and F is the identity functor it follows that (a_α) is weak if and only if (a_α) has the LCP. In particular, a morphism is weak if and only if

it is a monomorphism. Consequently, a weak family can be thought of as a generalized monomorphism.

2. Let $\underline{A}$ be the category of uniform spaces, $\underline{E}$ be the category of sets, and let F be the forgetful functor. Then, a family (a_α) of uniformly continuous functions is weak if and only if the uniformity on $F(A)$, A being the domain, is the weak uniformity defined by the functions $F(a_\alpha)$ and the uniformities $\underline{U}_\alpha$ on the sets $F(A_\alpha)$.

3. Let $\underline{E}$ be the category with a unique morphism. There is a unique functor $F : \underline{A} \to \underline{E}$. A family (a_α) of morphisms a_α is weak if and only if $(A, (a_\alpha))$ is a direct product of the family (A_α). Hence, weak families might well be called relative direct products.

4. Let $\underline{A}$ be the category of groups. Take F to be the forgetful functor from $\underline{A}$ to the category of sets. A family (a_α) of group homomorphisms $a_\alpha : A \to A_\alpha$ is F-weak if and only if $\bigcap_\alpha \ker(a_\alpha)$ is the trivial subgroup of A. In particular a group homomorphism is weak if and only if it is a monomorphism.

5. Let $\underline{A}$ denote the category of vector spaces over the field of rationals. For $\underline{E}$ take the category of abelian groups, and let $F(A)$ denote the underlying abelian group of A. Then, every family of linear transformations $a_\alpha : A \to A_\alpha$ is weak.

6. Let $\underline{A}$ denote a ring R, with unit, viewed as a category with one object 1 and morphisms the elements of the ring, the law of composition being ring multiplication. A left R-module defines a covariant functor $M : \underline{A} \to \underline{E}$ where $\underline{E}$ is the category of abelian groups.

A ring element r is M-weak if, for a group homomorphism f, $r \cdot f(x) = s \cdot x$ for all $x \in M(1)$ implies that there exists a unique $t \in R$ with $f(x) = t \cdot x$ for all $x \in M(1)$. In order that M-weak elements exist, it is necessary that M have zero annihilator. Clearly, an element of the ring with a left inverse is M-weak for all such modules M.

https://doi.org/10.4153/CMB-1965-057-7 Published online by Cambridge University Press

Conversely, if r is M-weak for all modules M with zero annihilator then r is left-invertible. Consider the left R-module $M = R \times R/(r)$, where (r) is the principal left ideal determined by r. Define $f : M \to M$ by $f(x, y) = (o, y)$. Then $r \cdot f(x, y) = 0 \cdot (x, y) = 0$. The group homomorphism f is of the form $f(x, y) = t \cdot (x, y)$ if and only if $R/(r) = \{0\}$. This is equivalent to r being left-invertible.

Every ring R is a left R-module with zero annihilator. The element 0 is R-weak if and only if every endomorphism f of the additive group of R is given by left multiplication with some element of the ring. For example, 0 is a $\mathbb{Z}$-weak element of $\mathbb{Z}$.

7. Let $\underline{A}$ again denote a ring R viewed as a category, and let $\underline{E}$ now be the category of sets. Let F be the composition of the functor corresponding to R as a left R-module with the forgetful functor from the category of groups to $\underline{E}$.

A ring element r is F-weak if, for a function f, $r \cdot f(x) = s \cdot x$ for all $x \in F(1)$ implies that there exists $t \in R$ with $f(x) = t \cdot x$ for all $x \in F(1)$. When $0 \neq 1$ this is equivalent to r not being a left divisor of zero.

Assume $r \in R$ is not a left divisor of zero. Let $f : R \to R$ be a function for which there exists s with $r \cdot f(x) = s \cdot x$ for all $x \in R$. Let $t = f(1)$. Then, $r \cdot f(x) = (r \cdot t) \cdot x$, for all $x \in R$. Since r is not a left divisor of zero, $f(x) = t \cdot x$ for all $x \in r$. In other words, r is F-weak.

Assume that r is F-weak. Then $r \neq 0$. Let r be a left divisor of zero and let $p \in R$ be such that $r \cdot p = 0$ and $p \neq 0$. Define $f : R \to R$ by $f(x) = 0$ if $x \neq p$ and $f(p) = p$. Then, $r \cdot f(x) = 0 \cdot x$ for all $x \in R$. Hence, there exists $t \in R$ with $f(x) = t \cdot x$ for all $x \in R$. Since $1 \neq p$, $0 = f(1) = t$. This is a contradiction.

3. Elementary properties of weak families. As might be expected, a weak family (a_α) is determined up to isomorphism by its range and $(F(a_\alpha))$.

https://doi.org/10.4153/CMB-1965-057-7 Published online by Cambridge University Press

PROPOSITION 1. Let (a_α) and (a_α') be two weak families with the same range. They are isomorphic if, for each α, $F(a_\alpha) = F(a_\alpha')$.

Proof: If A and A' are the respective domains of (a_α) and (a_α'), then $F(A) = F(A') = X$. Therefore, $F(a_\alpha) \circ 1_X = F(a_\alpha') = F(a_\alpha') \circ 1_X = F(a_\alpha)$. It follows that there are unique morphisms $a : A' \to A$ and $a' : A \to A'$ such that, for each α, $a_\alpha \circ a = a_\alpha'$ and $a_\alpha' \circ a' = a_\alpha$, and $F(a) = F(a')$ $= 1_X$. The uniqueness condition in the definition of a weak family implies that a and a' are inverse to one another.

A family (a_α) of morphisms will be said to have the left cancellation property (L C P) if, for each α, $a_\alpha \circ b = a_\alpha \circ c$ implies $b = c$. If the family (a_α) defines a direct product in $\underline{A}$ of the family (A_α) of objects A_α of $\underline{A}$, then (a_α) has the L C P.

In general, if (A_α) has the L C P the family $(F(a_\alpha))$ need not have this property. However, when F has a left adjoint the family $(F(a_\alpha))$ inherits the L C P from (a_α).

Since a weak family can be thought of as a generalized or relative direct product, the question arises as to whether a family (a_α) that defines a direct product in $\underline{A}$ is weak.

PROPOSITION 2. Let (a_α) define a direct product in $\underline{A}$. The following are equivalent:

(1) (a_α) is weak;

(2) for each family (b_α) of morphisms of $\underline{A}$, with domain B and range (A_α), there is a unique morphism $f : F(B) \to F(A)$ with, for each α, $F(a_\alpha) \circ f = F(b_\alpha)$.

https://doi.org/10.4153/CMB-1965-057-7 Published online by Cambridge University Press

In particular, (a_α) is weak if F has a left adjoint or, more generally, if $(F(a_\alpha))$ has the L C P.

Proof: Since (a_α) defines a direct product in $\underline{A}$, for each family (b_α) of morphisms of $\underline{A}$, with domain B and range (A_α), there is a unique $b : B \to A$ with, for each α, $a_\alpha \circ b = b_\alpha$. Consequently, there is at most one $f : F(B) \to F(A)$ of the form F(b) where b satisfies, for each α, $a_\alpha \circ b = b_\alpha$. From this observation, it follows immediately that (1) and (2) are equivalent.

Examples.

8. Let both $\underline{A}$ and $\underline{E}$ be the category of topological spaces and let F be the functor defined by the generalized Stone-Čech compactification. It is well known that F does not preserve direct products [4]. However, every family (a_α) of continuous functions that defines a direct product is F-weak.

9. Let $\underline{A} = \underline{E}$ be the category of abelian groups and let F be the functor obtained by associating with each group A the tensor product $A \otimes \mathbb{Q}$. Denote by A a direct product of the modules $\mathbb{Z}_i$, where $i = 1, 2, 3, \ldots$ and by (a_i) the family of projections $a_i : A \to \mathbb{Z}_i$. The family (a_i) is not F-weak. Clearly, for each i, $F(\mathbb{Z}_i) = \mathbb{Z}_i \otimes \mathbb{Q}$ is the zero group and $F(A) = A \otimes \mathbb{Q}$ is not the zero group. Hence, there are at least two morphisms $f_1, f_2 : F(A) \to F(A)$ with, for each i, $F(a_i) \circ f_j = F(a_i) = 0$.

The following proposition is a converse to proposition 2.

PROPOSITION 3. If (a_α) is a weak family in $\underline{A}$ for which the family $(F(a_\alpha))$ defines a direct product of the family $(F(A_\alpha))$, then (a_α) defines a direct product of the family (A_α).

https://doi.org/10.4153/CMB-1965-057-7 Published online by Cambridge University Press

Proof: Let (b_α) be a family of morphisms $b_\alpha : B \to A_\alpha$ of $\underline{A}$. There is a unique map $f : F(B) \to F(A) = X$ with, for each α, $f_\alpha \circ f = F(b_\alpha)$. Since (a_α) is weak and $F(a_\alpha) = f_\alpha$, there is a unique map $b : B \to A$ with, for each α, $a_\alpha \circ b = b_\alpha$. Hence, (a_α) defines a direct product of (A_α).

Let $\underline{B}$ be a third category and let $F : \underline{A} \to \underline{E}$ be equal to HG, where $G : \underline{A} \to \underline{B}$ and $H : \underline{B} \to \underline{E}$.

PROPOSITION 4. When H is faithful, a family (a_α) of morphisms of $\underline{A}$ is G-weak if it is F-weak. If the family of morphisms (a_α) is G-weak and the family $(G(a_\alpha))$ is H-weak, then (a_α) is F-weak.

Proof: Assume that (b_α) is a family of morphisms $b_\alpha : B \to A_\alpha$ and that $g : G(B) \to G(A)$ is such that, for each α, $G(a_\alpha) \circ g = G(b_\alpha)$. Then, for each α, $F(a_\alpha) \circ H(g) = F(b_\alpha)$. Consequently, there is a unique $b : B \to A$ with $F(b) = H(g)$ and, for each α, $a_\alpha \circ b = b_\alpha$.

Since $F(b) = HG(b) = H(g)$, the faithfulness of H implies that $g = G(b)$. Clearly, there is at most one b with $G(b) = g$ and satisfying the condition $a_\alpha \circ b = b_\alpha$ for each α.

Let (b_α) be a family of morphisms $b_\alpha : B \to A_\alpha$ and let $f : F(B) \to F(A)$ be such that, for each α, $F(a_\alpha) \circ f = F(b_\alpha)$. Since $(G(a_\alpha))$ is H-weak, there is a unique $g : G(B) \to G(A)$ with, for each α, $G(a_\alpha) \circ g = G(b_\alpha)$ and $H(g) = f$. The G-weakness of (a_α) implies that there exists a unique $b : B \to A$ with, for each α, $a_\alpha \circ b = b_\alpha$ and $G(b) = g$.

Clearly, $F(b) = f$. It remains to show the uniqueness of b. Let $b' : B \to A$ be such that, for each α, $a_\alpha \circ b' = b_\alpha$ and

https://doi.org/10.4153/CMB-1965-057-7 Published online by Cambridge University Press

$F(b') = f$. Then, $G(b') = g$ since, for each α, $G(a_\alpha) \circ G(b') = G(b_\alpha)$ and $H\, G(b') = F(b') = f$. It then follows from the G-weakness of (a_α) that $b' = b$.

Examples.

10. Let $\underline{A}$ be a ring R viewed as a category and let $\underline{B}$ denote the category of abelian groups. Let $G : \underline{A} \to \underline{B}$ be the functor corresponding to R as a left R-module and let $H : \underline{B} \to \underline{E}$, where $\underline{E}$ is the category of sets, be the forgetful functor. Then, $F = HG$ is the functor of example 7. The faithfulness of H and proposition 4 imply that every $r \in R$ which is not a divisor of zero is G-weak.

11. Let $\underline{A} = \underline{B}$ be the category of abelian groups and let $\underline{E}$ be the trivial category with one morphism. Denote by G the functor of example 9 and by H the unique functor from $\underline{B}$ to $\underline{E}$. The family (a_i) of example 9 is then $F = HG$-weak, but it is not G-weak.

4. Weak families and direct products. Let Ω be an index set, and for each $\alpha \in \Omega$ let $I(\alpha)$ be an index set. For each $\beta \in I(\alpha)$ let $a'_{\alpha\beta} : A_\alpha \to A_{\alpha\beta}$ be a morphism of $\underline{A}$, and for each $\alpha \in \Omega$ let $a_\alpha : A \to A_\alpha$. Define $a_{\alpha\beta}$ to be $a'_{\alpha\beta} \circ a_\alpha$.

PROPOSITION 5. If (a_α) and, for each α, $(a'_{\alpha\beta})$ are weak families, the family $(a_{\alpha\beta})$ is weak. Conversely, if $(a_{\alpha\beta})$ is weak the family (a_α) is weak.

Proof: Let $(b_{\alpha\beta})$ be a family of morphisms $b_{\alpha\beta} : B \to A_{\alpha\beta}$. Assume $f : F(B) \to F(A)$ is such that, for each α and β, $F(a_{\alpha\beta}) \circ f = F(b_{\alpha\beta})$.

Let $f_\alpha : F(B) \to F(A_\alpha)$ be the morphism $F(a_\alpha) \circ f$. Since, for each α, $(a'_{\alpha\beta})$ is weak, there is, for each α, a unique morphism $b_\alpha : B \to A_\alpha$ with $F(b_\alpha) = f_\alpha$ and, for each β, $a'_{\alpha\beta} \circ b_\alpha = b_{\alpha\beta}$.

https://doi.org/10.4153/CMB-1965-057-7 Published online by Cambridge University Press

Since $f_\alpha = F(a_\alpha) \circ f = F(b_\alpha)$, there is a unique $b : B \to A$ with $F(b) = f$ and, for each α, $a_\alpha \circ b = b_\alpha$. It remains to show that b is the unique morphism with $F(b) = f$ and, for each α and β, $a_{\alpha\beta} \circ b = b_{\alpha\beta}$.

Assume $F(b') = f$ and that, for each α and β, $a_{\alpha\beta} \circ b' = b_{\alpha\beta}$. Let $b'_\alpha = a_\alpha \circ b'$. Then, $F(b'_\alpha) = F(a_\alpha) \circ f = f_\alpha$ and, for each β, $a'_{\alpha\beta} \circ b'_\alpha = b_{\alpha\beta}$. Therefore, $b'_\alpha = b_\alpha$. From this it follows immediately that $b = b'$.

The proof of the converse is similar.

A morphism e of $\underline{A}$ will be called an <u>embedding</u> if e is weak and $F(e)$ is a monomorphism. An object A of $\underline{A}$ will be called a <u>subobject</u> of B if there is an embedding $e : A \to B$.

PROPOSITION 6. Let (A_α) be a family of objects A_α of $\underline{A}$. Let $(\Pi A_\alpha, pr_\alpha)$ be a direct product of the family (A_α). Assume that (pr_α) is weak and that $(F(pr_\alpha))$ has the L C P. The following statements are equivalent:

(1) A is a subobject of ΠA_α;

(2) there is a weak family (a_α) with domain A and range (A_α) for which $(F(a_\alpha))$ has the L C P.

<u>Proof</u>: There is a 1-1 correspondence between families (a_α) with domain A and range (A_α) and morphisms $e : A \to \Pi A_\alpha$. To the morphism e corresponds the family (a_α) where, for each α, $a_\alpha = pr_\alpha \circ e$. Since (pr_α) is weak, proposition 5 shows that e is weak if and only if the corresponding family (a_α) is weak.

If $(F(a_\alpha))$ has the L C P it is clear that $F(e)$ is a monomorphism. Conversely, since $(F(pr_\alpha))$ has the L C P,

https://doi.org/10.4153/CMB-1965-057-7 Published online by Cambridge University Press

the family $(F(a_\alpha))$ has the L C P whenever $F(e)$ is a monomorphism.

Example.

12. Let $\underline{A}$ be the category of topological spaces, let $\underline{E}$ be the category of sets, and let F be the forgetful functor. A family (a_α) with domain A is such that $(F(a_\alpha))$ has the L C P if and only if the functions a_α separate the points of $F(A)$. Hence, the embedding lemma in [5] is a particular case of the result in proposition 6. This proposition also shows that a similar embedding lemma holds for uniform spaces.

REFERENCES

1. N. Bourbaki, Topologie Générale, Actualités Sci. Ind. 1142 3éme ed., 1961.

2. ________, Théorie des Ensembles, Actualités Sci. Ind. 1258, 1957.

3. D. Ehresmann, Structures Quotient, Comm. Math. Helvetici, vol. 38 (1964), 219-242.

4. L. Gillman and M. Jerison, Rings of Continuous Functions, New York, 1960.

5. J. L. Kelley, General Topology, New York, 1955.

McGill University

https://doi.org/10.4153/CMB-1965-057-7 Published online by Cambridge University Press