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COEFFICIENTS OF DIFFERENTIALLY ALGEBRAIC SERIES

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Abstract

An Eisenstein-like criterion is proved for power series with algebraic coefficients satisfying algebraic differential equations of a certain general kind. The proof is elementary and the result extends earlier results of Hurwitz, Pólya and Popken

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1. Introduction

Algebraic coefficients of formal power series which satisfy certain kinds of equations possess a number of interesting arithmetic properties. We address here the problem of determining the shape of the denominators in such coefficients. A classical result of Eisenstein (see Pólya and Szegö [10, Chapter 3]) states that if a power series

$$(1) \qquad y = \sum_{n=0}^{\infty} a_n z^n$$

with rational coefficients represents an algebraic power series, that is, is a solution of an algebraic equation of the form

$$(2) \qquad P_K(z)y^K + P_{K-1}(z)y^{K-1} + \cdots + P_1(z)y + P_0(z) = 0$$

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where $P_i(z)$, $i = 0, \ldots, K$, are polynomials with rational coefficients, $P_K(z) \not\equiv 0$, then there exists a positive integer I such that $I^n a_n$ is integral for each $n \geq 1$. In [3], this theorem has been generalized to a wider class of power series, termed Eisensteinian power series. A formal power series of the form (1) with algebraic coefficients is said to be *Eisensteinian* if there exists a positive integer I such that $I^n a_n$ is an algebraic integer for all $n \geq 1$. The result in [3] says that if a power series (1) with algebraic coefficients satisfies an algebraic equation of the form (2) with all $P_i(z)$ being Eisensteinian series and $P_K(z) \not\equiv 0$, then y is also Eisensteinian. Passing from algebraic to linear differential equations, the situation is generally more complicated. It is convenient to introduce another definition. A power series (1) with algebraic coefficients is said to be an *H-series* if there exist a positive integer I and a nontrivial polynomial $Q(z)$, possibly dependent on I, with rational integral coefficients and with $Q(j) \neq 0$ for all natural j such that $Q(1)Q(2)\cdots Q(n)a_{I+n}$ is an algebraic integer for each $n \geq 1$. The simplest examples of H-series are of course polynomials with algebraic coefficients. A special case of what we proved in [4] asserts that if a power series (1) with algebraic coefficients satisfies a linear differential equation

$$P_K(z)y^{(K)} + \cdots + P_1(z)y' + P_0(z)y = 0,$$

where $P_i(z)$, $i = 0, \ldots, K$, are polynomials with algebraic coefficients, $P_K(z) \not\equiv 0$, then y is an H-series. Furthermore, if 0 is not a singularity of the differential equation, the result simplifies considerably. In this case, we have that $n!Q^n a_n$ is an algebraic integer for all $n \geq 1$, where Q is a fixed positive integer, and we say that y is an *H_0-series.*

It seems natural and more useful to consider not only solutions of linear differential equations, but also other more general differential equations. We treat here the case of (generalized) algebraic differential equations for which the shape of the denominators can also be determined. For brevity, following [5], we say that a power series which satisfies an algebraic differential equation is *differentially algebraic* (or DA for short). Hurwitz [1] proved that if a power series (1) with rational coefficients is DA, then there exist a positive integer I and a nontrivial polynomial $Q(z)$ with rational integral coefficients and with $Q(j) \neq 0$ for all natural j such that the prime factors in the denominators of the coefficients a_{I+n} divide $Q(I)Q(I+1)\cdots Q(I+n)$ for all $n \geq 1$. Kakeya [2] slightly improved this information by making explicit the degree of $Q(z)$. Pólya [9] refined Hurwitz's proof to deduce delicate estimates on the size of the coefficients extending a more restricted result of Pincherle [8]. Better and more general size estimates have been obtained by Popken [11], [12], Mahler [6], [7], Sibuya and Sperber [13]. In this paper, we pursue the original line of Hurwitz and prove a theorem analogous to that of Eisenstein for a class containing the DA series. The proof is algebraic in character and its basic ideas can be traced back to the works of Hurwitz, Pólya and Popken.

Let us begin with the following definition. A power series (1) with algebraic coefficients is said to be an *HPP-series* (or series of Hurwitz-Pólya-Popken type) if there exist a positive integer I and a nontrivial polynomial $Q(z)$, possibly depending on I, with rational integral coefficients and with $Q(j) \neq 0$ for all natural j such that

$$Q(1)^{[n/1]}Q(2)^{[n/2]}\cdots Q(n)^{[n/n]}a_{I+n}$$

is an algebraic integer for each $n \geq 1$, where $[x]$ denotes the integer part of real x. Throughout the paper, we make use of the standard abbreviation

$$y_i := d^i y/dz^i$$

for the derivatives of $y = y(z)$.

Our principal result is

THEOREM. *Let $y = \sum_{n=0}^{\infty} a_n z^n$ and for finitely many k, let $t_k = \sum_{n=0}^{\infty} t_{kn} z^n$ all be power series with algebraic coefficients. Assume y satisfies a (generalized) algebraic differential equation of the form*

$$P := \sum_{\mathbf{k}} t_k y^{k_0} y_1^{k_1} \cdots y_m^{k_m} = 0,$$

where P is a polynomial in $y, y_1, \dots, y_m$ with coefficients t_k. Assume also that the separant of this differential equation

$$\partial P/\partial y_m \not\equiv 0.$$

If all the t_k's are H-series, then y is an HPP*-series.*

2. Auxiliary lemmas

We first establish a few lemmas by following closely the proofs given in Popken [11]. The proof of the main theorem will be given in the next section. Apart from using algebraic identities from these lemmas, there are fundamental ideas from Hurwitz [1] and Pólya [9].

LEMMA 1. *Let m, k_μ $(\mu = 0, 1, \dots, m)$ be nonnegative rational integers. Let $t := t(z)$, $y := y(z)$ be functions differentiable arbitrarily often. Set*

$$T := t y^{k_0} y_1^{k_1} \cdots y_m^{k_m}.$$

(1) *For* $h \geq 0$, *we have*

$$\frac{d^h T}{dz^h} = \sum_{\chi} A_{h,\chi} t^{(\kappa)} y^{\chi_0} y_1^{\chi_1} \cdots y_{m+h}^{\chi_{m+h}}, \tag{3}$$

where $A_{h,\chi}$ *are rational integers, and the sum extends over the* $(m+h+2)$-*tuples* $\chi = (\kappa, \chi_0, \ldots, \chi_{m+h})$ *of nonnegative rational integers subject to*

$$\sum_{\tau=0}^{m+h} \chi_\tau = \sum_{\mu=0}^{m} k_\mu; \quad \kappa + \sum_{\tau=0}^{m+h} \tau\chi_\tau = h + \sum_{\mu=0}^{m} \mu k_\mu; \quad \sum_{\tau=1}^{h} \tau\chi_{m+\tau} \leq h; \quad \kappa \leq h. \tag{4}$$

(2) *Each summand on the right hand side of* (3) *contains at most one factor* y_τ *with* $\tau \geq m + \frac{1}{2}(h+1)$.

(3) *If* τ *is an integer, with* $\tau \geq m + \frac{1}{2}(h+1)$ *and if the summand of* (3) *divisible by* y_τ *is denoted by* $A_h(T) y_\tau$ ($A_h(T)$ *depends of course on* τ, z), *then*

$$A_h(T) = \sum_{\mu=0}^{m} \binom{h}{h-\tau+\mu} \frac{d^{h-\tau+\mu}}{dz^{h-\tau+\mu}} \frac{\partial T}{\partial y_\mu}. \tag{5}$$

Proof. (1) The assertion is clear for $h = 0$ and follows for $h > 0$ by induction.

(2) If the assertion were false, then on the right hand side of (3) there would exist a summand of at least two terms, say $y_{m+\tau_1}$ and $y_{m+\tau_2}$ with $\tau_1 \geq \frac{1}{2}(h+1)$ and $\tau_2 \geq \frac{1}{2}(h+1)$, and so with $\tau_1 + \tau_2 \geq h+1$. This contradicts the third condition in (4).

(3) Let τ be an integer, with $\tau \geq m + \frac{1}{2}(h+1)$, and let the term from (3) not divisible by y_τ be $B_h(T)$. Then

$$\frac{d^h T}{dz^h} = B_h(T) + A_h(T) y_\tau. \tag{6}$$

Let l, l_μ $(\mu = 0, \ldots, m)$ be nonnegative integers, and set

$$U := z^l y^{l_0} y_1^{l_1} \cdots y_m^{l_m}.$$

By Leibniz's formula, we have

$$\frac{d^h(TU)}{dz^h} = \sum_{g=0}^{h} \binom{h}{g} \frac{d^g T}{dz^g} \frac{d^{h-g} U}{dz^{h-g}}.$$

Substitute for the derivatives on the right from (6) and expand. By part (2) of the lemma, the coefficient of y_τ^2 is 0 and the coefficient of y_τ gives $A_h(TU)$.

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That is

$$A_h(TU) = \sum_{g=0}^{h} \binom{h}{g} (A_g(T)B_{h-g}(U) + A_{h-g}(U)B_g(T)),$$

$$0 = \sum_{g=0}^{h} \binom{h}{g} A_g(T)A_{h-g}(U).$$

Therefore

$$\begin{aligned}(7) \qquad A_h(TU) &= \sum_{g=0}^{h} \binom{h}{g} \{A_g(T)(B_{h-g}(U) + y_\tau A_{h-g}(U)) \\ &\qquad\qquad + A_{h-g}(U)(B_g(T) + y_\tau A_g(T))\} \\ &= \sum_{g=0}^{h} \binom{h}{g} A_{h-g}(T)\frac{d^g U}{dz^g} + A_{h-g}(U)\frac{d^g T}{dz^g}.\end{aligned}$$

Now assume the assertion holds for the expressions T and U and substitute for $A_{h-g}(T)$ and $A_{h-g}(U)$.

$$\begin{aligned}A_h(TU) &= \sum_{g=0}^{h} \binom{h}{g} \sum_{\mu=0}^{m} \binom{h-g}{h-g-\tau+\mu} \\ &\quad \times \left(\frac{d^g U}{dz^g} \frac{d^{h-g-\tau+\mu}}{dz^{h-g-\tau+\mu}} \frac{\partial T}{\partial y_\mu} + \frac{d^g T}{dz^g} \frac{d^{h-g-\tau+\mu}}{dz^{h-g-\tau+\mu}} \frac{\partial U}{\partial y_\mu} \right).\end{aligned}$$

We interchange the order of summation and note that $h-\tau+\mu \le h-\tau+m < h$ to get

$$\begin{aligned}A_h(TU) &= \sum_{\mu=0}^{m} \binom{h}{h-\tau+\mu} \sum_{g=0}^{h-\tau+\mu} \binom{h-\tau+\mu}{g} \\ &\quad \times \left\{ \frac{d^g U}{dz^g} \frac{d^{h-\tau+\mu-g}}{dz^{h-\tau+\mu-g}} \frac{\partial T}{\partial y_\mu} + \frac{d^g T}{dz^g} \frac{d^{h-\tau+\mu-g}}{dz^{h-\tau+\mu-g}} \frac{\partial U}{\partial y_\mu} \right\} \\ &= \sum_{\mu=0}^{m} \binom{h}{h-\tau+\mu} \left\{ \frac{d^{h-\tau+\mu}}{dz^{h-\tau+\mu}} \left(U \frac{\partial T}{\partial y_\mu} \right) + \frac{d^{h-\tau+\mu}}{dz^{h-\tau+\mu}} \left(T \frac{\partial U}{\partial y_\mu} \right) \right\} \\ &= \sum_{\mu=0}^{m} \binom{h}{h-\tau+\mu} \left\{ \frac{d^{h-\tau+\mu}}{dz^{h-\tau+\mu}} \frac{\partial (TU)}{\partial y_\mu} \right\}.\end{aligned}$$

We have thus proved that if the assertion holds for T and U, then it also holds for TU. We need only show now that the assertion holds for different factors $t = t(z), y, y_1, \ldots, y_m$ of T. If $T = t$, then both sides of (5) are equal to 0. If $T = y_i$ $(0 \le i \le m)$, then the left hand side of (5) is either 1

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or 0 according as $\tau = i + h$ or not, while the right hand side is clearly equal to $\binom{h}{h-\tau+i} d^{h-\tau+i} 1/dz^{h-\tau+i}$ and so is either 1 or 0 according as $\tau = i + h$ or not. Hence, the lemma follows.

LEMMA 2. *Let* m, k_μ $(\mu = 0, \dots, m)$, t, y *and* T *be as in Lemma* 1. *For each pair of nonnegative integers* s, h *with* $h \geq 2s + 1$, *we have*

$$\frac{d^h T}{dz^h} = \sum_{\sigma=0}^{s} y_{m+h-\sigma} \sum_{\mu=0}^{m} \binom{h}{\sigma - \mu} \frac{d^{\sigma-\mu}}{dz^{\sigma-\mu}} \frac{\partial T}{\partial y_{m-\mu}} + B_{m+h-s}, \tag{8}$$

where

$$B_{m+h-s} = \sum_{\chi} B_{m+h-s,\chi} t^{(\chi)} y^{\chi_0} y_1^{\chi_1} \cdots y_{m+h-s-1}^{\chi_{m+h-s-1}}$$

is a finite sum with rational integral coefficients $B_{m+h-s,\chi}$ *and the sum* $\sum_\chi$ *extends over* $(m+h-s+1)$*-tuples* $\chi = (\kappa, \chi_0, \chi_1, \dots, \chi_{m+h-s-1})$ *of nonnegative integers subject to*

$$\sum_{\tau=0}^{m+h-s-1} \chi_\tau = \sum_{\mu=0}^{m} k_\mu, \quad \kappa + \sum_{\tau=0}^{m+h-s-1} \tau\chi_\tau = h + \sum_{\mu=0}^{m} \mu k_\mu, \qquad \kappa \leq h. \tag{9}$$

PROOF. For each $\tau = m + h - \sigma$ $(\sigma = 0, \dots, s)$, because $h \geq 2s + 1$, the inequalities $\tau \geq m + h - s \geq m + \frac{1}{2}(h+1)$ are satisfied. By Lemma 1, there appears in each summand of (3) at most one factor $y_{m+h-\sigma}$ $(0 \leq \sigma \leq s)$ whose coefficient is

$$\begin{aligned} \sum_{\mu=0}^{m} \binom{h}{h-\tau+\mu} \frac{d^{h-\tau+\mu}}{dz^{h-\tau+\mu}} \frac{\partial T}{\partial y_\mu} &= \sum_{\mu=0}^{m} \binom{h}{\sigma+\mu-m} \frac{d^{\sigma+\mu-m}}{dz^{\sigma+\mu-m}} \frac{\partial T}{\partial y_\mu} \\ &= \sum_{\mu=0}^{m} \binom{h}{\sigma-\mu} \frac{d^{\sigma-\mu}}{dz^{\sigma-\mu}} \frac{\partial T}{\partial y_{m-\mu}}. \end{aligned}$$

Consequently, (8) is fulfilled with

$$B_{m+h-s} = \sum_{\chi'} A_{h,\chi'} t^{(\kappa)} y^{\chi_0} y_1^{\chi_1} \cdots y_{m+h-s-1}^{\chi_{m+h-s-1}},$$

where the sum $\sum_{\chi'}$ extends over all systems $\chi' = (\kappa, \chi_0, \chi_1, \dots, \chi_{m+h-s-1}, 0, \dots, 0)$ of nonnegative integers satisfying (9).

LEMMA 3. *Let* $y := y(z)$ *and for finitely many* k, $t_k := t_k(z)$ *be functions differentiable arbitrarily often. Assume* y *and all* t_k *together with all their derivatives at the origin take algebraic values. If* y *satisfies a (generalized) algebraic differential equation of the form*

$$P := \sum_{\mathbf{k}} t_k y^{k_0} y_1^{k_1} \cdots y_m^{k_m} = 0,$$

where P is a polynomial in $y, y_1, \dots, y_m$ with coefficients t_k, and if the separant $\partial P/\partial y_m \not\equiv 0$, then there exist a nonnegative integer s, and a nontrivial polynomial $p_s(z)$ with algebraic integral coefficients such that for all $h \geq 2s+1$, we have

$$p_s(h)y_{m+h-s}(0) = \sum_{\boldsymbol{\lambda}} D_{m+h-s,\boldsymbol{\lambda}} t_k^{(\lambda)}(0) y^{\lambda_0}(0) y_1^{\lambda_1}(0) \cdots y_{m+h-s-1}^{\lambda_{m+h-s-1}}(0), \tag{11}$$

where the $D_{m+h-s,\boldsymbol{\lambda}}$ are rational integers and the sum $\sum_{\boldsymbol{\lambda}}$ extends over $(m+h-s+1)$-tuples $\boldsymbol{\lambda} = (\lambda, \lambda_0, \dots, \lambda_{m+h-s-1})$ of nonnegative integers subject to

(12)

$$\sum_{\tau=0}^{m+h-s-1} \lambda_\tau \leq \max_{\mathbf{k}} \sum_{\mu=0}^{m} k_\mu, \quad \lambda + \sum_{\tau=0}^{m+h-s-1} \tau\lambda_\tau \leq h + \max_{\mathbf{k}} \sum_{\mu=0}^{m} \mu k_\mu, \qquad \lambda \leq h,$$

PROOF. Applying Lemma 2 to each expression $T = t_k y^{k_0} y_1^{k_1} \dots y_m^{k_m}$ of P with nonnegative integers s and h, $h \geq 2s+1$, we get

$$\frac{d^h P}{dz^h} = \sum_{\sigma=0}^{s} y_{m+h-\sigma} \sum_{\mu=0}^{m} \binom{h}{\sigma-\mu} \frac{d^{\sigma-\mu}}{dz^{\sigma-\mu}} \left(\frac{\partial P}{\partial y_{m-\mu}}\right) + C_{m+h-s},$$

where the finite sum

$$C_{m+h-s} = \sum_{\boldsymbol{\lambda}} C_{m+h-s,\boldsymbol{\lambda}} t_k^{(\lambda)} y^{\lambda_0} y_1^{\lambda_1} \cdots y_{m+h-s-1}^{\lambda_{m+h-s-1}}$$

has rational integral coefficients $C_{m+h-s,\boldsymbol{\lambda}}$, and the sum $\sum_{\boldsymbol{\lambda}}$ extends over $(m+h-s+1)$-tuples $\boldsymbol{\lambda}$ of nonnegative integers subject to (12). Making use of the differential equation and putting $z=0$, we get

$$\sum_{\sigma=0}^{s} y_{m+h-\sigma}(0) \sum_{\mu=0}^{m} \binom{h}{\sigma-\mu} \frac{d^{\sigma-\mu}}{dz^{\sigma-\mu}} \left(\frac{\partial P}{\partial y_{m-\mu}}\right)\Bigg|_{z=0} = D_{m+h-s}, \tag{13}$$

where now the finite sum

$$D_{m+h-s} = \sum_{\boldsymbol{\lambda}} D_{m+h-s,\boldsymbol{\lambda}} t_k^{(\lambda)}(0) y^{\lambda_0}(0) y_1^{\lambda_1}(0) \cdots y_{m+h-s-1}^{\lambda_{m+h-s-1}}(0)$$

has rational integral coefficients, and the sum extends over the same range of nonnegative integers as before. Since the separant is not identically zero, there is a nonnegative integer j such that

$$\left(\frac{d^j}{dz^j}\frac{\partial P}{\partial y_m}\right)_{z=0} \neq 0,$$

and so among the $j+1$ polynomials (in h)

$$p_\tau(h) = \sum_{\mu=0}^{m} \binom{h}{\tau-\mu} \left(\frac{d^{\tau-\mu}}{dz^{\tau-\mu}} \frac{\partial P}{\partial y_{m-\mu}}\right)_{z=0} \qquad (\tau = 0, 1, \dots, j) \tag{14}$$

at least $p_j(h)$ does not vanish identically in h. Let $p_s(h)$ be the first among the $(j+1)$ polynomials in (14) which does not vanish identically in h. Thus for $h \geq 2s+1$, (13) implies

$$p_s(h)y_{m+h-s}(0) = D_{m+h-s}. \tag{15}$$

Since y, t_k and all their derivatives take algebraic values at the origin, then (14) indicates that all coefficients of $p_s(h)$ are algebraic. Multiplying throughout (15) by a suitable positive integer and modifying $D_{m+h-s,\lambda}$ accordingly, we can be sure that all coefficients of $p_s(h)$ are algebraic integers.

3. Proof of the theorem

Since the t_k are H-series and there are only finitely many of them, we can find a nonnegative integer I and a nontrivial polynomial $Q(z)$ with rational integral coefficients, and with $Q(j) \neq 0$ for natural j, such that $Q(1)Q(2)\cdots Q(n)t_k^{(I+n)}(0)$ is an algebraic integer for each natural n, and for all k. By Lemma 3, we can find a nonnegative integer s, and a nontrivial polynomial $p_s(z)$ with algebraic integral coefficients such that for all $h \geq 2s+1$, (11) holds along with (12). Put $n = m+h-s$ so that $n \geq m+s+1$, and $p_s(h) = p(n)$. Then (11) reads

$$p(n)y_n(0) = \sum_{\lambda} D_{n,\lambda} t_k^{(\lambda)}(0) y^{\lambda_0}(0) y_1^{\lambda_1}(0) \cdots y_{n-1}^{\lambda_{n-1}}(0), \tag{16}$$

where the sum on the right hand side runs through nonnegative integers subject to

$$\sum_{\tau=0}^{n-1} \lambda_\tau \leq D, \quad \lambda + \sum_{\tau=0}^{n-1} \tau\lambda_\tau \leq n+S, \qquad \lambda \leq n-m+s, \tag{17}$$

where $D := \max_{\mathbf{k}} \sum_{\mu=0}^{m} k_\mu$, $S := -m+s+\max_{\mathbf{k}} \sum_{\mu=0}^{m} \mu k_\mu$. Let N be a positive integer such that $Ny(0)$, $Ny_1(0), \ldots, Ny_S(0)$ are algebraic integers. Multiplying (16) throughout by N^D and adjusting $D_{n,\lambda}$, $p(n)$ accordingly, we see that (16) and (17) still hold with $y_n(0)$ replaced by $Ny_n(0)$ for all n. We then assume for the rest of the proof that $y(0), y_1(0), \ldots, y_S(0)$ are algebraic integers. We may also assume with no loss of generality that $I > 2S$. Let R be a positive integer sufficiently large so that $R > I$ and $p(n) \neq 0$ for all $n > R$.

Now construct a sequence of algebraic integers (u_ν) by setting

$$u_1 = \text{denominator of } y_{S+1}(0),$$
$$u_2 = \text{denominator of } y_{S+2}(0), \dots,$$
$$u_{R-S} = \text{denominator of } y_R(0),$$
$$u_{R-S+1} = p(R+1), u_{R-S+2} = p(R+2), \dots, u_{R-S+\nu} = p(R+\nu).$$

Next construct another sequence of algebraic integers (U_ν) by setting

$$U_\nu = u_1^{[\nu/1]} u_2^{[\nu/2]} \cdots u_\nu^{[\nu/\nu]}.$$

Observe that for positive integers

(18) $1 \le \alpha < \nu, 1 \le \beta < \nu, \dots, 1 \le \psi < \nu$ and $\alpha + \beta + \cdots + \psi \le \nu$

we have that

(18)′ U_ν is divisible by the product $u_\nu U_\alpha U_\beta \cdots U_\psi$.

Let M be a positive integer such that $Mt_k^{(\lambda)}(0)$ are algebraic integers for $\lambda = 0, 1, \dots, I$ and for all k. Define

$$V(\nu) = MQ(1)Q(2)\cdots Q(I)Q(\nu),$$

so $V(\nu)$ is a polynomial in ν with rational integral coefficients. Now construct a final sequence of rational integers (W_ν) by $W_\nu = 1$ if $\nu \le 0$, and

$$W_\nu = V(1)^{[\nu/1]} V(2)^{[\nu/2]} \cdots V(\nu)^{[\nu/\nu]}.$$

Observe that $(18)'$ also holds for W_ν as well as U_ν.

We shall show by induction that $y_n(0) W_{n-S} U_{n-S}$ is an algebraic integer for all $n > S$. For $n = S+1, S+2, \dots, R$ from the definition of (u_ν), it is clear that $y_n(0) u_{n-S}$ is an algebraic integer, and so is $y_n(0) W_{n-S} U_{n-S}$. Now consider $n > R$, and assume $y_k(0) W_{k-S} U_{k-S}$ are algebraic integral for $S \le k \le n-1$. Rewrite (16) as

$$\begin{aligned} u_{n-S} y_n(0) &= p(n) y_n(0) \\ &= \sum_\lambda D_{n,\lambda} t_k^{(\lambda)}(0) y_a(0) y_b(0) \dots y_g(0) y_h(0) \cdots y_l(0), \end{aligned}$$

where $\lambda, a, b, \dots, g, h, \dots, l$ are nonnegative integers subject to $0 \le \lambda \le n-m+s$, $0 \le a \le n-1$, $0 \le b \le n-1, \dots, 0 \le l \le n-1$, $a \ge b \ge \cdots \ge l$, $\lambda + a + b + \cdots + l \le n + S$. We distinguish three separate cases.

CASE 1. $a, b, \dots, g, h, \dots, l \le S$. Then by the algebraic integrality of $y(0), y_1(0), \dots, y_S(0)$, we see that $y_a(0) y_b(0) \cdots y_l(0)$ is algebraic integral, and by $(18)'$, we have that $\frac{U_{n-S}}{u_{n-S}} D_{n,\lambda} y_a(0) \cdots y_l(0)$ is algebraic integral. Since

$W_{\lambda-I}t_k^{(\lambda)}(0)$ is algebraic integral for all k, and $\lambda \leq n-m+s < I+(n+S-I)$, that is, $\lambda - I < n - S$, then by $(18)'$ (for W_ν) we see that

$$D_{n,(\lambda)}W_{n-S}t_k^{(\lambda)}(0)\frac{U_{n-S}}{u_{n-S}}y_a(0)\cdots y_l(0) \tag{19}$$

is an algebraic integer.

CASE 2. $a > S$ and all other indices $b, \dots, l \leq S$. Since $a < n$, then by the induction hypothesis $W_{a-S}U_{a-S}y_a(0)y_b(0)\cdots y_l(0)$ is an algebraic integer. From $(18)'$ then $W_{a-S}(U_{n-S}/u_{n-S})y_a(0)y_b(0)\cdots y_l(0)$ is an algebraic integer. Now since $W_{\lambda-I}t_k^{(\lambda)}(0)$ is an algebraic integer, $a+\lambda-S-I \leq n-I$, and using $(18)'$ for W_ν, we have that statement (19) still holds in this case.

CASE 3. there are two or more indices, say, $a, b, \dots, g > S$ and the remaining indices $h, \dots, l \leq S$. Since $a, b, \dots, g < n$, then the induction hypothesis implies $W_{a-S}U_{a-S}y_a(0)W_{b-S}U_{b-S}y_b(0)\cdots W_{g-S}U_{g-S}y_g(0)y_h(0)\cdots y_l(0)$ is algebraic integral, and since $\lambda+(a-S)+(b-S)+\cdots+(g-S) \leq n-S$, $(18)'$ implies $W_{n-S-\lambda}(U_{n-S}/u_{n-S})y_a(0)y_b(0)\cdots y_l(0)$ is algebraic integral. Since $W_{\lambda-I}t_k^{(\lambda)}(0)$ is algebraic integral, we conclude also in this case that (19) holds.

Hence, $y_n(0)W_{n-S}U_{n-S}$ is algebraic integral for all $n > S$. Putting $q(n) = V(n)u_n$ $(n \geq 1)$, we get that $q(n)$ is a polynomial in n with algebraic integral coefficients such that

$$q^{[n/1]}(1)q^{[n/2]}(2)\cdots q^{[n/n]}(n)y_n(0) \tag{20}$$

is algebraic integral for all $n > S$. Let $r(n)$ be the product of all conjugate polynomials of $q(n)$. Then statement (20) is valid with $q(n)$ replaced by $r(n)$. Putting $F(n) = (r(1)\cdots r(S))^{S+1}r(S+n)$, we see that $F^{[n/1]}(1)F^{[n/2]}(2)\cdots F^{[n/n]}(n)y_{n+S}(0)$ is algebraic integral for all $n \geq 1$, that is, y is an HPP-series.

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