Nagoya Math. J., **229** (2018), 169–214
DOI 10.1017/nmj.2016.68

ON GALOIS EQUIVARIANCE OF HOMOMORPHISMS BETWEEN TORSION CRYSTALLINE REPRESENTATIONS

YOSHIYASU OZEKI

Abstract. Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field. Let $(\pi_n)_{n\geqslant 0}$ be a system of p-power roots of a uniformizer $\pi=\pi_0$ of K with $\pi_{n+1}^p=\pi_n$, and define G_s (resp. G_∞) the absolute Galois group of $K(\pi_s)$ (resp. $K_\infty:=\bigcup_{n\geqslant 0}K(\pi_n)$). In this paper, we study G_s-equivariantness properties of G_∞-equivariant homomorphisms between torsion crystalline representations.

CONTENTS

Received May 22, 2015. Revised November 15, 2016. Accepted November 24, 2016.
2010 Mathematics subject classification. 11F85, 11S20.
Supported by the JSPS Fellowships for Young Scientists.

§1. Introduction

Let p be a prime number and $r \geqslant 0$ an integer. Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field and absolute ramification index e. Let $\pi = \pi_0$ be a uniformizer of K and π_n a p^nth root of π such that $\pi_{n+1}^p = \pi_n$ for all $n \geqslant 0$. For any integer $s \geqslant 0$, we put $K_{(s)} = K(\pi_s)$. We also put $K_\infty = \bigcup_{n\geqslant 0} K_{(n)}$. We denote by G_K, G_s and G_∞ absolute Galois groups of K, $K_{(s)}$ and K_∞, respectively. By definition we have the following decreasing sequence of Galois groups:

$$G_K = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_\infty.$$

Since K_∞ is a strict APF extension of K, the theory of fields of norm implies that G_∞ is isomorphic to the absolute Galois group of some field of characteristic p. Therefore, representations of G_∞ have easy interpretations via Fontaine's étale φ-modules. Hence it seems natural to pose the following question:

QUESTION 1.1. Let T be a $\mathbb{Z}_p$- or $\mathbb{Q}_p$-representation of G_K. How small can we choose $s \geqslant 0$ to recover "enough" information of $T|_{G_s}$ from that of $T|_{G_\infty}$?

Nowadays there is an interesting insight of Breuil for this question; he showed that representations of G_K arising from finite flat group schemes or p-divisible groups over the integer ring of K are "determined" by their restriction to G_∞. Furthermore, for $\mathbb{Q}_p$-representations, Kisin proved the following theorem in [Kis] (which was a conjecture of Breuil): the restriction

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

functor from the category of crystalline $\mathbb{Q}_p$-representations of G_K into the category of $\mathbb{Q}_p$-representations of G_∞ is fully faithful.

In this paper, we give some partial answers to Question 1.1 for *torsion crystalline representations*. A torsion $\mathbb{Z}_p$-representation T of G_K is torsion crystalline with Hodge–Tate weights in $[0, r]$ if it can be written as the quotient of lattices in some crystalline $\mathbb{Q}_p$-representation of G_K with Hodge–Tate weights in $[0, r]$. Let $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ be the category of them. In the case $r = 1$, such representations are equivalent to finite flat representations. (Here, a torsion $\mathbb{Z}_p$-representation of G_K is finite flat if it arises from the generic fiber of some p-power order finite flat commutative group scheme over the integer ring of K.) We denote by $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ the category of torsion $\mathbb{Z}_p$-representations of G_∞. The first main result in this paper is as follows.

THEOREM 1.2. (Full Faithfulness Theorem) *Suppose* $e(r - 1) < p - 1$. *Then the restriction functor* $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K) \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ *is fully faithful.*

Before this work, some results were already known. First, the full faithfulness theorem was proved by Breuil for $e = 1$ and $r < p - 1$ via the Fontaine–Laffaille theory [Br2, the proof of Théorèm 5.2]. He also proved the theorem under the assumptions $p > 2$ and $r \leqslant 1$ as a consequence of a study of the category of finite flat group schemes [Br3, Theorem 3.4.3]. Later, his result was extended to the case $p = 2$ in [Kim], [La], [Li4] (proved independently). In particular, the case $p = 2$ of the full faithfulness theorem is a consequence of their works. On the other hand, Abrashkin proved the full faithfulness in the case where $p > 2, r < p$ and K is a finite unramified extension of $\mathbb{Q}_p$ [Ab2, Section 8.3.3]. His proof is based on calculations of ramification bounds for torsion crystalline representations. In [Oz2], a proof of Theorem 1.2 under the assumption $er < p - 1$ is given via $(\varphi, \hat{G})$-modules (which was introduced by Tong Liu [Li2] to classify lattices in semistable representations). We should remark that Abrashkin's approach implies that *calculations of ramification bounds induce full faithfulness results on restriction functors* such as our theorems. However, known results on ramification bounds for torsion crystalline representations are not sufficient to obtain our results. Conversely, our results possibly help us to study ramification bounds for them.

Our proof of Theorem 1.2 is similar to the proof for the main result of [Oz2], but we need more careful considerations for $(\varphi, \hat{G})$-modules. In fact, we prove a full faithfulness theorem for torsion representations arising from certain classes of $(\varphi, \hat{G})$-modules (cf. Theorem 4.9), which immediately gives our main theorem. In addition, our study gives a result as below which is the second main result of this paper (here, we define $\log_p(x) := -\infty$ for any real number $x \leqslant 0$).

THEOREM 1.3. *Suppose that p is odd and $s > n-1+\log_p(r-(p-1)/e)$. Let T and T' be objects of $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ which are killed by p^n. Then any G_∞-equivariant homomorphism $T \to T'$ is G_s-equivariant.*

For torsion semistable representations, a similar result was shown in [CL2, Theorem 3], which was a consequence of a study of ramification bounds. The bound appearing in their theorem was $n-1+\log_p(nr)$. By applying our arguments given in this paper, we can obtain a generalization of their result; our refined condition is $n-1+\log_p r$ (see Theorem 4.17). Some other consequences of our study are described in Section 4.7. Motivated by the full faithfulness theorem (= Theorem 1.2) and Theorem 1.3, we raise the following question.

QUESTION 1.4. Does there exist a constant c depending on e, r and p so that any G_∞-equivariant homomorphism in the category $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ is G_s-equivariant for $s > c$? Moreover, can we choose c to be $\log_p(r-(p-1)/e)$?

On the other hand, there exist counter examples of the full faithfulness theorem when we ignore the condition $e(r-1) < p-1$. Let $\mathrm{Rep}_{\mathrm{tor}}(G_1)$ be the category of torsion $\mathbb{Z}_p$-representations of G_1.

THEOREM 1.5. (= Special case of Corollary 5.15) *Suppose that K is a finite extension of $\mathbb{Q}_p$, and also suppose $e \mid (p-1)$ or $(p-1) \mid e$. If $e(r-1) \geqslant p-1$, the restriction functor $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K) \to \mathrm{Rep}_{\mathrm{tor}}(G_1)$ is not full (in particular, the restriction functor $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K) \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ is not full).*

In particular, if $p=2$, then the full faithfulness never hold for any finite extension K of $\mathbb{Q}_2$ and any $r \geqslant 2$. Theorem 1.5 implies that the condition "$e(r-1) < p-1$" in Theorem 1.2 is optimal for many finite extensions K of $\mathbb{Q}_p$.

Now we describe the organization of this paper. In Section 2, we set up notations and summarize facts we need later. In Section 3, we define variant notions of $(\varphi, \hat{G})$-modules and give some basic properties. They are needed to study certain classes of potentially crystalline representations and restrictions of semistable representations. In Section 4, we study technical torsion $(\varphi, \hat{G})$-modules which are related with torsion (potentially) crystalline representations. The key result in this section is the full faithfulness result Theorem 4.9 on them, which allows us to prove our main results immediately. Finally, in Section 5, we calculate the smallest integer r for a given torsion representation T such that T admits a crystalline lift with Hodge–Tate weights in $[0, r]$. We mainly study the rank two case. We use our full faithfulness theorem to assure the nonexistence of crystalline lifts with small Hodge–Tate weights. Theorem 1.5 is a consequence of studies of this section.

Notation and convention: Throughout this paper, we fix a prime number p. Except in Section 5, we always assume that *p is odd*.

For any topological group H, we denote by $\mathrm{Rep}_{\mathrm{tor}}(H)$ (resp. $\mathrm{Rep}_{\mathbb{Z}_p}(H)$, resp. $\mathrm{Rep}_{\mathbb{Q}_p}(H)$) the category of torsion $\mathbb{Z}_p$-representations of H (resp. the category of free $\mathbb{Z}_p$-representations of H, resp. the category of $\mathbb{Q}_p$-representations of H). All $\mathbb{Z}_p$-representations (resp. $\mathbb{Q}_p$-representations) in this paper are always assumed to be finitely generated over $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) and continuous.

For any field F, we denote by G_F the absolute Galois group of F (for a fixed separable closure of F).

§2. Preliminaries

In this section, we recall definitions and basic properties for Kisin modules and $(\varphi, \hat{G})$-modules. Throughout Sections 2–4, we always assume that p is an odd prime.

2.1 Basic notations

Let k be a perfect field of characteristic p, $W(k)$ the ring of Witt vectors with coefficients in k, $K_0 = W(k)[1/p]$, K a finite totally ramified extension of K_0 of degree e, $\overline{K}$ a fixed algebraic closure of K. Throughout this paper, we fix a uniformizer π of K. Let $E(u)$ be the minimal polynomial of π over K_0. Then $E(u)$ is an Eisenstein polynomial. For any integer $n \geqslant 0$, we fix a system $(\pi_n)_{n\geqslant 0}$ of p^nth roots of π in $\overline{K}$ such that $\pi_{n+1}^p = \pi_n$. Let $R = \varprojlim \mathcal{O}_{\overline{K}}/p$, where $\mathcal{O}_{\overline{K}}$ is the integer ring of $\overline{K}$ and the transition maps are given by the pth power map. For any integer $s \geqslant 0$, we write $\underline{\pi_s} := (\pi_{s+n})_{n\geqslant 0} \in R$ and $\underline{\pi} := \underline{\pi_0} \in R$. Note that we have ${\underline{\pi_s}}^{p^s} = \underline{\pi}$.

Let L be the completion of an unramified algebraic extension of K with residue field k_L. Then π_s is a uniformizer of $L_{(s)} := L(\pi_s)$ and $L_{(s)}$ is a totally ramified degree ep^s extension of $L_0 := W(k_L)[1/p]$. We set

$L_\infty := \bigcup_{n\geqslant 0} L_{(n)}$. We put $G_{L,s} := G_{L_{(s)}} = \mathrm{Gal}(\overline{L}/L_{(s)})$ and $G_{L,\infty} := G_{L_\infty} = \mathrm{Gal}(\overline{L}/L_\infty)$. By definitions, we have $L = L_{(0)}$ and $G_{L,0} = G_L$. Put $\mathfrak{S}_{L,s} = W(k_L)[\![u_s]\!]$ (resp. $\mathfrak{S}_L = W(k_L)[\![u]\!]$) with an indeterminate u_s (resp. u). We equip a Frobenius endomorphism φ of $\mathfrak{S}_{L,s}$ (resp. $\mathfrak{S}_L$) by $u_s \mapsto u_s^p$ (resp. $u \mapsto u^p$) and the Frobenius on $W(k_L)$. We embed the $W(k_L)$-algebra $W(k_L)[u_s]$ (resp. $W(k_L)[u]$) into $W(R)$ via the map $u_s \mapsto [\underline{\pi_s}]$ (resp. $u \mapsto [\underline{\pi}]$), where $[*]$ stands for the Teichmüller representative. This embedding extends to an embedding $\mathfrak{S}_{L,s} \hookrightarrow W(R)$ (resp. $\mathfrak{S}_L \hookrightarrow W(R)$). By identifying u with $u_s^{p^s}$, we regard $\mathfrak{S}_L$ as a subalgebra of $\mathfrak{S}_{L,s}$. It is readily seen that the embedding $\mathfrak{S}_L \hookrightarrow \mathfrak{S}_{L,s} \hookrightarrow W(R)$ is compatible with the Frobenius endomorphisms. If we denote by $E_s(u_s)$ the minimal polynomial of π_s over K_0, with indeterminate u_s, then we have $E_s(u_s) = E(u_s^{p^s})$. Therefore, we have $E_s(u_s) = E(u)$ in $\mathfrak{S}_{L,s}$. We note that the minimal polynomial of π_s over L_0 is $E_s(u_s)$.

Let $S^{\mathrm{int}}_{L_0,s}$ (resp. $S^{\mathrm{int}}_{L_0}$) be the p-adic completion of the divided power envelope of $W(k_L)[u_s]$ (resp. $W(k_L)[u]$) with respect to the ideal generated by $E_s(u_s)$ (resp. $E(u)$). There exist a unique Frobenius map $\varphi: S^{\mathrm{int}}_{L_0,s} \to S^{\mathrm{int}}_{L_0,s}$ (resp. $\varphi: S^{\mathrm{int}}_{L_0} \to S^{\mathrm{int}}_{L_0}$) and monodromy $N: S^{\mathrm{int}}_{L_0,s} \to S^{\mathrm{int}}_{L_0,s}$ defined by $\varphi(u_s) = u_s^p$ (resp. $\varphi(u) = u^p$) and $N(u_s) = -u_s$ (resp. $N(u) = -u$). Put $S_{L_0,s} = S^{\mathrm{int}}_{L_0,s}[1/p] = L_0 \otimes_{W(k_L)} S^{\mathrm{int}}_{L_0,s}$ (resp. $S_{L_0} = S^{\mathrm{int}}_{L_0}[1/p] = L_0 \otimes_{W(k_L)} S^{\mathrm{int}}_{L_0}$). We equip $S^{\mathrm{int}}_{L_0,s}$ and $S_{L_0,s}$ (resp. $S^{\mathrm{int}}_{L_0}$ and S_{L_0}) with decreasing filtrations $\mathrm{Fil}^i S^{\mathrm{int}}_{L_0,s}$ and $\mathrm{Fil}^i S_{L_0,s}$ (resp. $\mathrm{Fil}^i S^{\mathrm{int}}_{L_0,s}$ and $\mathrm{Fil}^i S_{L_0,s}$) by the p-adic completion of the ideal generated by $E_s^j(u_s)/j!$ (resp. $E^j(u)/j!$) for all $j \geqslant 0$. The inclusion $W(k_L)[u_s] \hookrightarrow W(R)$ (resp. $W(k_L)[u] \hookrightarrow W(R)$) via the map $u_s \mapsto [\underline{\pi_s}]$ (resp. $u \mapsto [\underline{\pi}]$) induces φ-compatible inclusions $\mathfrak{S}_{L,s} \hookrightarrow S^{\mathrm{int}}_{L_0,s} \hookrightarrow A_{\mathrm{cris}}$ and $S_{L_0,s} \hookrightarrow B^+_{\mathrm{cris}}$ (resp. $\mathfrak{S}_L \hookrightarrow S^{\mathrm{int}}_{L_0} \hookrightarrow A_{\mathrm{cris}}$ and $S_{L_0} \hookrightarrow B^+_{\mathrm{cris}}$). By these inclusions, we often regard these rings as subrings of B^+_{cris}. By identifying u with $u_s^{p^s}$ as before, we regard $S^{\mathrm{int}}_{L_0}$ (resp. S_{L_0}) as a φ-stable (but not N-stable) subalgebra of $S^{\mathrm{int}}_{L_0,s}$ (resp. $S_{L_0,s}$). By definitions, we have $\mathfrak{S}_L = \mathfrak{S}_{L,0}$, $S^{\mathrm{int}}_{L_0,0} = S^{\mathrm{int}}_{L_0}$ and $S_{L_0,0} = S_{L_0}$ (cf. Figure 1).

Convention: For simplicity, if $L = K$, then we often omit the subscript "L" from various notations (e.g., $G_{K_s} = G_s$, $G_{K_\infty} = G_\infty$, $\mathfrak{S}_K = \mathfrak{S}$, $\mathfrak{S}_{K,s} = \mathfrak{S}_s$).

2.2 Kisin modules

Let $r, s \geqslant 0$ be integers. A *φ-module over* $\mathfrak{S}_{L,s}$ is an $\mathfrak{S}_{L,s}$-module $\mathfrak{M}$ equipped with a φ-semilinear map $\varphi: \mathfrak{M} \to \mathfrak{M}$. A morphism between two φ-modules $(\mathfrak{M}_1, \varphi_1)$ and $(\mathfrak{M}_2, \varphi_2)$ over $\mathfrak{S}_{L,s}$ is an $\mathfrak{S}_{L,s}$-linear map $\mathfrak{M}_1 \to \mathfrak{M}_2$

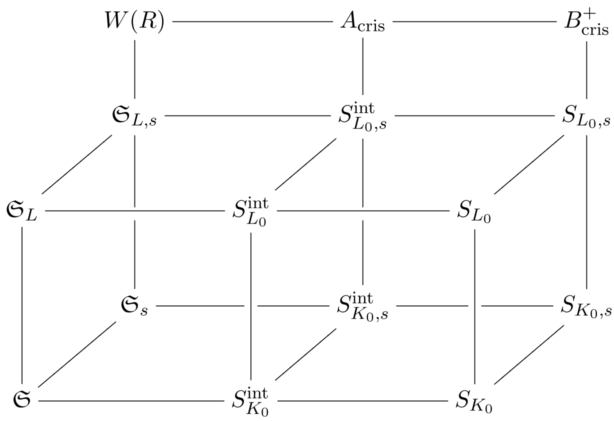

Figure 1.
Ring extensions.

compatible with φ_1 and φ_2. Denote by $'\mathrm{Mod}^r_{/\mathfrak{S}_{L,s}}$ the category of φ-modules $(\mathfrak{M}, \varphi)$ over $\mathfrak{S}_{L,s}$ *of height* $\leqslant r$ in the sense that $\mathfrak{M}$ is of finite type over $\mathfrak{S}_{L,s}$ and the cokernel of $1 \otimes \varphi : \mathfrak{S}_{L,s} \otimes_{\varphi, \mathfrak{S}_{L,s}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E_s(u_s)^r$.

Let $\mathrm{Mod}^r_{/\mathfrak{S}_{L,s}}$ be the full subcategory of $'\mathrm{Mod}^r_{/\mathfrak{S}_{L,s}}$ consisting of finite free $\mathfrak{S}_{L,s}$-modules. We call an object of $\mathrm{Mod}^r_{/\mathfrak{S}_{L,s}}$ a *free Kisin module of height* $\leqslant r$ *(over* $\mathfrak{S}_{L,s}$*)*.

Let $\mathrm{Mod}^r_{/\mathfrak{S}_{L,s,\infty}}$ be the full subcategory of $'\mathrm{Mod}^r_{/\mathfrak{S}_{L,s}}$ consisting of finite $\mathfrak{S}_{L,s}$-modules which are killed by some power of p and have projective dimension 1 in the sense that $\mathfrak{M}$ has a two term resolution by finite free $\mathfrak{S}_{L,s}$-modules. We call an object of $\mathrm{Mod}^r_{/\mathfrak{S}_{L,s,\infty}}$ a *torsion Kisin module of height* $\leqslant r$ *(over* $\mathfrak{S}_{L,s}$*)*.

For any free or torsion Kisin module $\mathfrak{M}$ over $\mathfrak{S}_{L,s}$, we define a $\mathbb{Z}_p[G_{L,\infty}]$-module $T_{\mathfrak{S}_{L,s}}(\mathfrak{M})$ by

$$T_{\mathfrak{S}_{L,s}}(\mathfrak{M}) := \begin{cases} \mathrm{Hom}_{\mathfrak{S}_{L,s},\varphi}(\mathfrak{M}, W(R)) & \text{if } \mathfrak{M} \text{ is free,} \\ \mathrm{Hom}_{\mathfrak{S}_{L,s},\varphi}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)) & \text{if } \mathfrak{M} \text{ is torsion.} \end{cases}$$

Here a $G_{L,\infty}$-action on $T_{\mathfrak{S}_{L,s}}(\mathfrak{M})$ is given by $(\sigma.g)(x) = \sigma(g(x))$ for $\sigma \in G_{L,\infty}$, $g \in T_{\mathfrak{S}}(\mathfrak{M})$, $x \in \mathfrak{M}$.

Convention: For simplicity, if $L = K$, then we often omit the subscript "L" from various notations (e.g., $\mathrm{Mod}^r_{/\mathfrak{S}_{K,s,\infty}} = \mathrm{Mod}^r_{/\mathfrak{S}_{s,\infty}}$, $T_{\mathfrak{S}_{K,s}} = T_{\mathfrak{S}_s}$). Also, if $s = 0$, we often omit the subscript "s" from various notations (e.g., $\mathrm{Mod}^r_{/\mathfrak{S}_{L,0,\infty}} = \mathrm{Mod}^r_{/\mathfrak{S}_{L,\infty}}$, $T_{\mathfrak{S}_{L,0}} = T_{\mathfrak{S}_L}$, $\mathrm{Mod}^r_{/\mathfrak{S}_{K,0,\infty}} = \mathrm{Mod}^r_{/\mathfrak{S}_{\infty}}$, $T_{\mathfrak{S}_{K,0}} = T_{\mathfrak{S}}$).

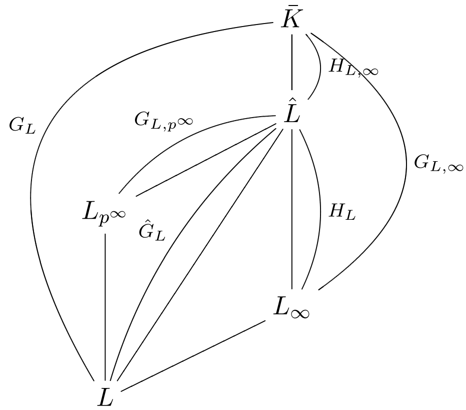

Figure 2.
Galois groups of field extensions.

Proposition 2.1. (1) [Kis, Corollary 2.1.4 and Proposition 2.1.12] *The functor* $T_{\mathfrak{S}_{L,s}}: \mathrm{Mod}^r_{/\mathfrak{S}_{L,s}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ *is exact and fully faithful.*

(2) [CL1, Corollaries 2.1.6, 3.3.10 and 3.3.15] *The functor* $T_{\mathfrak{S}_{L,s}}$: $\mathrm{Mod}^r_{/\mathfrak{S}_{L,s,\infty}} \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ *is exact and faithful. Furthermore, it is full if* $er < p-1$.

2.3 $(\varphi, \hat{G})$-modules

The notion of $(\varphi, \hat{G})$-modules is introduced by Liu in [Li2] to classify lattices in semistable representations. We recall definitions and properties of them. We continue to use same notations as above.

Let L_{p^∞} be the field obtained by adjoining all p-power roots of unity to L. We denote by $\hat{L}$ the composite field of L_∞ and L_{p^∞}. We define $H_L := \mathrm{Gal}(\hat{L}/L_\infty)$, $H_{L,\infty} := \mathrm{Gal}(\overline{K}/\hat{L})$ $G_{L,p^\infty} := \mathrm{Gal}(\hat{L}/L_{p^\infty})$ and $\hat{G}_L := \mathrm{Gal}(\hat{L}/L)$ (cf. Figure 2). Furthermore, putting $L_{(s),p^\infty} = L_{(s)}L_{p^\infty}$, we define $\hat{G}_{L,s} = \mathrm{Gal}(\hat{L}/L_{(s)})$ and $G_{L,s,p^\infty} := \mathrm{Gal}(\hat{L}/L_{(s),p^\infty})$.

Since $p > 2$, it is known that $L_{(s),p^\infty} \cap L_\infty = L_{(s)}$ and thus $\hat{G}_{L,s} = G_{L,s,p^\infty} \rtimes H_{L,s}$ (cf. [Li1, Lemma 5.1.2]). Furthermore, G_{L,s,p^∞} is topologically isomorphic to $\mathbb{Z}_p$.

Lemma 2.2. *The natural map* $G_{L,s,p^\infty} \to G_{K,s,p^\infty}$ *defined by* $g \mapsto g|_{\hat{K}}$ *is bijective.*

Proof. By replacing L_s with L, we may assume $s = 0$. It suffices to prove $\hat{K} \cap L_{p^\infty} = K_{p^\infty}$. Since G_{K,p^∞} is isomorphic to $\mathbb{Z}_p$, we know that any finite subextension of $\hat{K}/K_{p^\infty}$ is of the form $K_{(s),p^\infty}$ for some $s \geqslant 0$. Assume that

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

we have $\hat{K}\cap L_{p^\infty}\neq K_{p^\infty}$. Then we have $K_{(1)}\subset \hat{K}\cap L_{p^\infty}\subset L_{p^\infty}$. Thus π_1 is contained in $L_{p^\infty}\cap L_\infty = L$. However, since L is unramified over K, this contradicts the fact that π is a uniformizer of L. ∎

We fix a topological generator τ of G_{K,p^∞}. We also denote by τ the preimage of $\tau\in G_{K,p^\infty}$ under the bijection $G_{L,p^\infty}\simeq G_{K,p^\infty}$ of the above lemma. Note that τ^{p^s} is a topological generator of G_{L,s,p^∞}.

For any $g\in G_K$, we put $\underline{\varepsilon}(g)=g(\underline{\pi})/\underline{\pi}\in R$, and define $\underline{\varepsilon}:=\underline{\varepsilon}(\tilde{\tau})$. Here, $\tilde{\tau}\in G_K$ is any lift of $\tau\in \hat{G}_K$ and then $\underline{\varepsilon}(\tilde{\tau})$ is independent of the choice of the lift of τ. With these notations, we also note that we have $g(u)=[\underline{\varepsilon}(g)]u$ (recall that $\mathfrak{S}$ is embedded in $W(R)$). An easy computation shows that $\tau(\underline{\pi})/\underline{\pi}=\tau^{p^s}(\underline{\pi_s})/\underline{\pi_s}=\underline{\varepsilon}$. Therefore, we have $\tau(u)/u=\tau^{p^s}(u_s)/u_s=[\underline{\varepsilon}]$.

We put $t=-\log([\underline{\varepsilon}])\in A_{\mathrm{cris}}$. Denote by $\nu: W(R)\to W(\overline{k})$ the unique lift of the projection $R\to\overline{k}$, which extends to a map $\nu: B^+_{\mathrm{cris}}\to W(\overline{k})[1/p]$. For any subring $A\subset B^+_{\mathrm{cris}}$, we put $I_+A=\mathrm{Ker}(\nu \text{ on } B^+_{\mathrm{cris}})\cap A$. For any integer $n\geqslant 0$, let $t^{\{n\}}:=t^{r(n)}\gamma_{\tilde{q}(n)}(\frac{t^{p-1}}{p})$ where $n=(p-1)\tilde{q}(n)+r(n)$ with $\tilde{q}(n)\geqslant 0$, $0\leqslant r(n)<p-1$ and $\gamma_i(x)=\frac{x^i}{i!}$ is the standard divided power. We define a subring $\mathcal{R}_{L_0,s}$ (resp. $\mathcal{R}_{L_0}$) of B^+_{cris} as below:

$$\mathcal{R}_{L_0,s}:=\Big\{\sum_{i=0}^{\infty} f_i t^{\{i\}}\mid f_i\in S_{L_0,s} \text{ and } f_i\to 0 \text{ as } i\to\infty\Big\}$$

$$\Big(\text{resp.}\quad \mathcal{R}_{L_0}:=\Big\{\sum_{i=0}^{\infty} f_i t^{\{i\}}\mid f_i\in S_{L_0} \text{ and } f_i\to 0 \text{ as } i\to\infty\Big\}\Big).$$

Put $\widehat{\mathcal{R}}_{L,s}=\mathcal{R}_{L_0,s}\cap W(R)$ (resp. $\widehat{\mathcal{R}}_L=\mathcal{R}_{L_0}\cap W(R)$) and $I_{+,L,s}=I_+\widehat{\mathcal{R}}_{L,s}$ (resp. $I_{+,L}=I_+\widehat{\mathcal{R}}_L$). By definitions, we have $\mathcal{R}_{L_0,0}=\mathcal{R}_{L_0}$, $\widehat{\mathcal{R}}_{L,0}=\widehat{\mathcal{R}}_L$ and $I_{+,L,0}=I_{+,L}$. Lemma 2.2.1 in [Li2] shows that $\widehat{\mathcal{R}}_{L,s}$ (resp. $\mathcal{R}_{L_0,s}$) is a φ-stable $\mathfrak{S}_{L,s}$-subalgebra of $W(R)$ (resp. B^+_{cris}), and ν induces $\mathcal{R}_{L_0,s}/I_+\mathcal{R}_{L_0,s}\simeq L_0$ and $\widehat{\mathcal{R}}_{L,s}/I_{+,L,s}\simeq S^{\mathrm{int}}_{L_0,s}/I_+S^{\mathrm{int}}_{L_0,s}\simeq \mathfrak{S}_{L,s}/I_+\mathfrak{S}_{L,s}\simeq W(k_L)$. Furthermore, $\widehat{\mathcal{R}}_{L,s}, I_{+,L,s}, \mathcal{R}_{L_0,s}$ and $I_+\mathcal{R}_{L_0,s}$ are $G_{L,s}$-stable, and $G_{L,s}$-actions on them factors through $\hat{G}_{L,s}$. For any torsion Kisin module $\mathfrak{M}$ over $\mathfrak{S}_{L,s}$, we equip $\widehat{\mathcal{R}}_{L,s}\otimes_{\varphi,\mathfrak{S}_{L,s}}\mathfrak{M}$ with a Frobenius by $\varphi_{\widehat{\mathcal{R}}_{L,s}}\otimes\varphi_{\mathfrak{M}}$. It is known that the natural map $\mathfrak{M}\to\widehat{\mathcal{R}}_{L,s}\otimes_{\varphi,\mathfrak{S}_{L,s}}\mathfrak{M}$ given by $x\mapsto 1\otimes x$ is an injection (cf. [Oz1, Corollary 2.12]). By this injection, we regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S}_{L,s})$-stable submodule of $\widehat{\mathcal{R}}_{L,s}\otimes_{\varphi,\mathfrak{S}_{L,s}}\mathfrak{M}$.

DEFINITION 2.3. A *free* (resp. *torsion*) *$(\varphi,\hat{G}_{L,s})$-module of height $\leqslant r$* over $\mathfrak{S}_{L,s}$ is a triple $\hat{\mathfrak{M}}=(\mathfrak{M},\varphi_{\mathfrak{M}},\hat{G}_{L,s})$ where

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a free (resp. torsion) Kisin module of height $\leqslant r$ over $\mathfrak{S}_{L,s}$;
(2) $\hat{G}_{L,s}$ is an $\widehat{\mathcal{R}}_{L,s}$-semilinear $\hat{G}_{L,s}$-action on $\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}$ which induces a continuous $G_{L,s}$-action on $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}$;
(3) the $\hat{G}_{L,s}$-action commutes with $\varphi_{\widehat{\mathcal{R}}_{L,s}} \otimes \varphi_{\mathfrak{M}}$;
(4) $\mathfrak{M} \subset (\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M})^{H_L}$;
(5) $\hat{G}_{L,s}$ acts on the $W(k_L)$-module $(\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M})/I_{+,L,s}(\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M})$ trivially.

A morphism between two $(\varphi, \hat{G}_{L,s})$-modules $\hat{\mathfrak{M}}_1 = (\mathfrak{M}_1, \varphi_1, \hat{G})$ and $\hat{\mathfrak{M}}_2 = (\mathfrak{M}_2, \varphi_2, \hat{G})$ is a morphism $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ of φ-modules over $\mathfrak{S}_{L,s}$ such that $\widehat{\mathcal{R}}_{L,s} \otimes f : \widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}_1 \to \widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}_2$ is $\hat{G}_{L,s}$-equivariant. We denote by $\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s}}$ (resp. $\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s,\infty}}$) the category of free (resp. torsion) $(\varphi, \hat{G}_{L,s})$-modules of height $\leqslant r$ over $\mathfrak{S}_{L,s}$. We often regard $\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}$ as a $G_{L,s}$-module via the projection $G_{L,s} \twoheadrightarrow \hat{G}_{L,s}$.

For any free or torsion $(\varphi, \hat{G}_{L,s})$-module $\hat{\mathfrak{M}}$ over $\mathfrak{S}_{L,s}$, we define a $\mathbb{Z}_p[G_{L,s}]$-module $\hat{T}_{L,s}(\hat{\mathfrak{M}})$ by

$$\hat{T}_{L,s}(\hat{\mathfrak{M}}) = \begin{cases} \mathrm{Hom}_{\widehat{\mathcal{R}}_{L,s},\varphi}(\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}, W(R)) & \text{if } \mathfrak{M} \text{ is free,} \\ \mathrm{Hom}_{\widehat{\mathcal{R}}_{L,s},\varphi}(\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)) & \text{if } \mathfrak{M} \text{ is torsion.} \end{cases}$$

Here, $G_{L,s}$ acts on $\hat{T}_{L,s}(\hat{\mathfrak{M}})$ by $(\sigma.f)(x) = \sigma(f(\sigma^{-1}(x)))$ for $\sigma \in G_{L,s}$, $f \in \hat{T}_{L,s}(\hat{\mathfrak{M}})$, $x \in \widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}$. Then, there exists a natural $G_{L,\infty}$-equivariant map

$$\theta_{L,s} : T_{\mathfrak{S}_{L,s}}(\mathfrak{M}) \to \hat{T}_{L,s}(\hat{\mathfrak{M}})$$

defined by $\theta(f)(a \otimes x) = a\varphi(f(x))$ for $f \in T_{\mathfrak{S}_{L,s}}(\mathfrak{M})$, $a \in \widehat{\mathcal{R}}_{L,s}, x \in \mathfrak{M}$. We have the following theorem.

THEOREM 2.4. [Li2, Theorem 2.3.1(1)] and [CL2, Theorem 3.1.3(1)] *The map $\theta_{L,s}$ is an isomorphism of $\mathbb{Z}_p[G_{L,\infty}]$-modules.*

Convention: For simplicity, if $L = K$, then we often omit the subscript "L" from various notations (e.g., "a $(\varphi, \hat{G}_{K,s})$-module" = "a $(\varphi, \hat{G}_s)$-module", $\mathrm{Mod}^{r,\hat{G}_{K,s}}_{/\mathfrak{S}_{K,s}} = \mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s}$, $\mathrm{Mod}^{r,\hat{G}_{K,s}}_{/\mathfrak{S}_{K,s,\infty}} = \mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_{s,\infty}}$, $\hat{T}_{K,s} = \hat{T}_s$, $\theta_{K,s} = \theta_s$). Furthermore, if $s = 0$, we often omit the subscript "s" from various notations (e.g., $\mathrm{Mod}^{r,\hat{G}_{L,0}}_{/\mathfrak{S}_{L,0}} = \mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L}$, $\mathrm{Mod}^{r,\hat{G}_{L,0}}_{/\mathfrak{S}_{L,0,\infty}} = \mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_{L,\infty}}$, $\hat{T}_{L,0} = \hat{T}_L$, $\mathrm{Mod}^{r,\hat{G}_{K,0}}_{/\mathfrak{S}_{K,0}} = \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$, "a $(\varphi, \hat{G}_{K,0})$-module" = "a $(\varphi, \hat{G})$-module", $\hat{T}_{K,0} = \hat{T}$, $\theta_{K,0} = \theta$).

We denote by $\mathrm{Rep}^{r,\mathrm{st}}_{\mathbb{Q}_p}(G_{L,s})$ (resp. $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathbb{Q}_p}(G_{L,s})$, resp. $\mathrm{Rep}^{r,\mathrm{st}}_{\mathbb{Z}_p}(G_{L,s})$, resp. $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathbb{Z}_p}(G_{L,s})$) the category of semistable $\mathbb{Q}_p$-representations of $G_{L,s}$ with Hodge–Tate weights in $[0,r]$ (resp. the category of crystalline $\mathbb{Q}_p$-representations of $G_{L,s}$ with Hodge–Tate weights in $[0,r]$, resp. the category of lattices in semistable $\mathbb{Q}_p$-representations of $G_{L,s}$ with Hodge–Tate weights in $[0,r]$, resp. the category of lattices in crystalline $\mathbb{Q}_p$-representations of $G_{L,s}$ with Hodge–Tate weights in $[0,r]$).

There exists $\mathfrak{t} \in W(R) \smallsetminus pW(R)$ such that $\varphi(\mathfrak{t}) = pE(0)^{-1}E(u)\mathfrak{t}$. Such $\mathfrak{t}$ is unique up to units of $\mathbb{Z}_p$ (cf. [Li2, Example 2.3.5]). Now we define the full subcategory $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}}$ (resp. $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$) of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$ (resp. $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$) consisting of objects $\hat{\mathfrak{M}}$ which satisfy the following condition; $\tau(x) - x \in u^p\varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ for any $x \in \mathfrak{M}$.

The following results are important properties for the functor $\hat{T}_{L,s}$.

THEOREM 2.5. (1) [Li2, Theorem 2.3.1(2)] *The functor $\hat{T}$ induces an anti-equivalence of categories between* $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$ *and* $\mathrm{Rep}^{r,\mathrm{st}}_{\mathbb{Z}_p}(G_K)$.

(2) [GLS, Proposition 5.9] and [Oz2, Theorem 19] *The functor $\hat{T}$ induces an anti-equivalence of categories between* $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}}$ *and* $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathbb{Z}_p}(G_K)$.

(3) [Oz1, Corollaries 2.8 and 5.34] *The functor* $\hat{T}_{L,s} : \mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s,\infty}} \to \mathrm{Rep}_{\mathrm{tor}}(G_{L,s})$ *is exact and faithful. Furthermore, it is full if $er < p-1$.*

2.4 $(\varphi,\hat{G})$-modules, Breuil modules and filtered (φ,N)-modules

We recall some relations between Breuil modules and $(\varphi,\hat{G})$-modules. Here we give a rough sketch only. For more precise information, see [Br1, Section 6], [Li1, Section 5] and the proof of [Li2, Theorem 2.3.1(2)].

Let $\hat{\mathfrak{M}}$ be a free $(\varphi,\hat{G}_{L,s})$-module over $\mathfrak{S}_{L,s}$. If we put $\mathcal{D} := S_{L_0,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}$, then $\mathcal{D}$ has a structure of a Breuil module over $S_{L_0,s}$ which corresponds to the semistable representation $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}_{L,s}(\hat{\mathfrak{M}})$ of $G_{L,s}$ (for the definition and properties of Breuil modules, see [Br1]). Thus $\mathcal{D}$ is equipped with a Frobenius $\varphi_{\mathcal{D}}(= \varphi_{S_{L_0,s}} \otimes \varphi_{\mathfrak{M}})$, a decreasing filtration $(\mathrm{Fil}^i\mathcal{D})_{i\geqslant 0}$ of $S_{L_0,s}$-submodules of $\mathcal{D}$ and a L_0-linear monodromy operator $N : \mathcal{D} \to \mathcal{D}$ which satisfy certain properties (for example, Griffiths transversality).

Putting $D = \mathcal{D}/I_+S_{L_0,s}\mathcal{D}$, we can associate a filtered (φ,N)-module over $L_{(s)}$ as following: $\varphi_D := \varphi_{\mathcal{D}} \bmod I_+S_{L_0,s}\mathcal{D}$, $N_D := N_{\mathcal{D}} \bmod I_+S_{L_0,s}\mathcal{D}$ and $\mathrm{Fil}^i D_{L_{(s)}} := f_{\pi_s}(Fil^i(\mathcal{D}))$. Here, $f_{\pi_s} : \mathcal{D} \to D_{L_{(s)}}$ is the projection defined by $\mathcal{D} \twoheadrightarrow \mathcal{D}/\mathrm{Fil}^1 S_{L_0,s}\mathcal{D} \simeq D_{L_{(s)}}$. Proposition 6.2.1.1 of [Br1] implies that there exists a unique φ-compatible section $s : D \hookrightarrow \mathcal{D}$ of $\mathcal{D} \twoheadrightarrow D$. By this

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

embedding, we regard D as a submodule of $\mathcal{D}$. Then we have $N_{\mathcal{D}}|_D = N_D$ and $N_{\mathcal{D}} = N_{S_{L_0,s}} \otimes \mathrm{Id}_D + \mathrm{Id}_{S_{L_0,s}} \otimes N_D$ under the identification $\mathcal{D} = S_{L_0,s} \otimes_{L_{(s)}} D$.

The $G_{L,s}$-action on $\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}$ extends to $B^+_{\mathrm{cris}} \otimes_{\widehat{\mathcal{R}}_{L,s}} (\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}) \simeq B^+_{\mathrm{cris}} \otimes_{S_{L_0,s}} \mathcal{D}$. This action is in fact explicitly written as follows:

$$g(a \otimes x) = \sum_{i=0}^{\infty} g(a)\gamma_i\Big(-\log\Big(\frac{g[\underline{\pi_s}]}{[\underline{\pi_s}]}\Big)\Big) \otimes N^i_{\mathcal{D}}(x)$$
$$\text{for } g \in G_{L,s},\ a \in B^+_{\mathrm{cris}},\ x \in \mathcal{D}. \tag{1}$$

By this explicit formula, we can obtain an easy relation between $N_{\mathcal{D}}$ and τ^{p^s}-action on $\hat{\mathfrak{M}}$ as follows: first we recall that $t = -\log(\tau([\underline{\pi}])/[\underline{\pi}]) = -\log(\tau^{p^s}([\underline{\pi_s}])/[\underline{\pi_s}])$. By the formula, for any $n \geqslant 0$ and $x \in \mathcal{D}$, an induction on n shows that we have

$$(\tau^{p^s} - 1)^n(x) = \sum_{m=n}^{\infty}\Big(\sum_{\substack{i_1+\cdots i_n=m \\ i_j \geqslant 0}} \frac{m!}{i_1! \cdots i_n!}\Big)\gamma_m(t) \otimes N^m_{\mathcal{D}}(x) \in B^+_{\mathrm{cris}} \otimes_{S_{L_0,s}} \mathcal{D}$$

and in particular we see $\frac{(\tau^{p^s}-1)^n}{n}(x) \to 0$ p-adically as $n \to \infty$. Hence we can define

$$\log(\tau^{p^s})(x) := \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(\tau^{p^s} - 1)^n}{n}(x) \in B^+_{\mathrm{cris}} \otimes_{S_{L_0,s}} \mathcal{D}.$$

It is not difficult to check the equation

$$\log(\tau^{p^s})(x) = t \otimes N_{\mathcal{D}}(x). \tag{2}$$

2.5 Base changes for Kisin modules

Let $\mathfrak{M}$ be a free or torsion Kisin module of height $\leqslant r$ over $\mathfrak{S}_L$ (resp. over $\mathfrak{S}$). We put $\mathfrak{M}_{L,s} = \mathfrak{S}_{L,s} \otimes_{\mathfrak{S}_L} \mathfrak{M}$ (resp. $\mathfrak{S}_L = \mathfrak{S}_L \otimes_{\mathfrak{S}} \mathfrak{M}$) and equip $\mathfrak{M}_{L,s}$ (resp. $\mathfrak{M}_L$) with a Frobenius by $\varphi = \varphi_{\mathfrak{S}_{L,s}} \otimes \varphi_{\mathfrak{M}}$ (resp. $\varphi = \varphi_{\mathfrak{S}_L} \otimes \varphi_{\mathfrak{M}}$). Then it is not difficult to check that $\mathfrak{M}_{L,s}$ (resp. $\mathfrak{M}_L$) is a free or torsion Kisin module of height $\leqslant r$ over $\mathfrak{S}_{L,s}$ (resp. over $\mathfrak{S}_L$) (here we recall that $E_s(u_s) = E(u_s^{p^s}) = E(u)$). Hence we obtained natural functors

$$\mathrm{Mod}^r_{/\mathfrak{S}_L} \to \mathrm{Mod}^r_{/\mathfrak{S}_{L,s}} \quad \text{and} \quad \mathrm{Mod}^r_{/\mathfrak{S}_{L,\infty}} \to \mathrm{Mod}^r_{/\mathfrak{S}_{L,s,\infty}}$$
$$\Big(\text{resp. } \mathrm{Mod}^r_{/\mathfrak{S}} \to \mathrm{Mod}^r_{/\mathfrak{S}_L} \text{ and } \mathrm{Mod}^r_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_{L,\infty}}\Big).$$

By definition, we immediately see that we have $T_{\mathfrak{S}_L}(\mathfrak{M}) \simeq T_{\mathfrak{S}_{L,s}}(\mathfrak{M}_{L,s})$ (resp. $T_{\mathfrak{S}}(\mathfrak{M})|_{G_{L_\infty}} \simeq T_{\mathfrak{S}_L}(\mathfrak{M}_L)$). In particular, it follows from Proposition 2.1(1) that the following holds:

PROPOSITION 2.6. *The functor* $\mathrm{Mod}^r_{/\mathfrak{S}_L} \to \mathrm{Mod}^r_{/\mathfrak{S}_{L,s}}$ *is fully faithful.*

2.6 Base changes for $(\varphi, \hat{G})$-modules

Let $\hat{\mathfrak{M}}$ be a free or torsion $(\varphi, \hat{G}_L)$-module (resp. $(\varphi, \hat{G})$-module) of height $\leqslant r$ over $\mathfrak{S}_L$ (resp. over $\mathfrak{S}$). The $G_{L,s}$ action on $\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$ (resp. the G_L action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$) extends to $\widehat{\mathcal{R}}_{L,s} \otimes_{\widehat{\mathcal{R}}_L} (\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}) \simeq \widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_{L,s}} \mathfrak{M}_{L,s}$ (resp. $\widehat{\mathcal{R}}_L \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \simeq \widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}_L$), which factors through $\hat{G}_{L,s}$ (resp. $\hat{G}_L$). Then it is not difficult to check that $\mathfrak{M}_{L,s}$ (resp. $\mathfrak{M}_L$) has a structure of a $(\varphi, \hat{G}_{L,s})$-module (resp. $(\varphi, \hat{G}_L)$-module). Hence we obtained natural functors

$$\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L} \to \mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s}} \qquad \text{and} \qquad \mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_{L,\infty}} \to \mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s,\infty}}$$
$$\left(\text{resp. } \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}} \to \mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L} \text{ and } \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_{L,\infty}}\right).$$

By definition, we immediately see that we have $\hat{T}_L(\hat{\mathfrak{M}})|_{G_{L,s}} \simeq \hat{T}_{L,s}(\hat{\mathfrak{M}}_{L,s})$ (resp. $\hat{T}(\hat{\mathfrak{M}})|_{G_L} \simeq \hat{T}_L(\hat{\mathfrak{M}}_L)$). Similar to Proposition 2.6, we can prove the following.

PROPOSITION 2.7. *The functor* $\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L} \to \mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s}}$ *is fully faithful.*

The proposition immediately follows from the full faithfulness property of Theorem 2.5(1) and the lemma below.

LEMMA 2.8. *Let K' be a finite totally ramified extension of K. Then the restriction functor from the category of semistable $\mathbb{Q}_p$-representations of G_K into the category of semistable $\mathbb{Q}_p$-representations of $G_{K'}$ is fully faithful.*

Proof. Let V and V' be semistable $\mathbb{Q}_p$-representations of G_K and let $f: V \to V'$ be a $G_{K'}$-equivariant homomorphism. Considering the morphism of filtered (φ, N)-modules over K' corresponding to f, we can check without difficulty that f is in fact a morphism of filtered (φ, N)-modules over K. This is because K' is totally ramified over K_0 as same as K. This gives the desired result. $\square$

§3. Variants of free $(\varphi, \hat{G})$-modules

In this section, we define some variant notions of $(\varphi, \hat{G})$-modules. We continue to use same notation as in the previous section. In particular, p is odd.

3.1 Definitions

We start with some definitions which are our main concern in this and the next section.

DEFINITION 3.1. We define the category $\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_L}$ (resp. $\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_L}$) as follows. An object is a triple $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G}_{L,s})$ where

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a free Kisin module of height $\leqslant r$ over $\mathfrak{S}_L$;
(2) $\hat{G}_{L,s}$ is an $\widehat{\mathcal{R}}_L$-semilinear $\hat{G}_{L,s}$-action on $\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$ (resp. an $\widehat{\mathcal{R}}_{L,s}$-semilinear $\hat{G}_{L,s}$-action on $\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$) which induces a continuous $G_{L,s}$-action on $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$;
(3) the $\hat{G}_{L,s}$-action commutes with $\varphi_{\widehat{\mathcal{R}}_L} \otimes \varphi_{\mathfrak{M}}$ (resp. $\varphi_{\widehat{\mathcal{R}}_{L,s}} \otimes \varphi_{\mathfrak{M}}$);
(4) $\mathfrak{M} \subset (\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M})^{H_L}$ (resp. $\mathfrak{M} \subset (\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M})^{H_L}$);
(5) $\hat{G}_{L,s}$ acts on the $W(k_L)$-module $(\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M})/I_{+,L}(\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M})$ (resp. $(\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M})/I_{+,L,s}(\widehat{\mathcal{R}}_{L,s} \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M})$) trivially.

Morphisms are defined by the obvious way. By replacing "free" of (1) with "torsion", we define the category $\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,\infty}}$ (resp. $\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,\infty}}$).

REMARK 3.2. The category $\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_L}$ is very similar to the category $\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s}}$ from Definition 2.3, and so it may give the reader a little confusion. The differences between these categories are as follows.

	$\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s}}$	$\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_L}$
the base ring	$\mathfrak{S}_{L,s}$	$\mathfrak{S}_L$
coefficients of $G_{L,s}$-actions	$\widehat{\mathcal{R}}_{L,s}$	$\widehat{\mathcal{R}}_L$

For any object $\hat{\mathfrak{M}}$ of $\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_L}$ or $\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,\infty}}$, we define a $\mathbb{Z}_p[G_{L,s}]$-module $\hat{T}_{L,s}(\hat{\mathfrak{M}})$ by

$$\hat{T}_{L,s}(\hat{\mathfrak{M}}) = \begin{cases} \mathrm{Hom}_{\widehat{\mathcal{R}}_L,\varphi}(\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}, W(R)) & \text{if } \mathfrak{M} \text{ is free,} \\ \mathrm{Hom}_{\widehat{\mathcal{R}}_L,\varphi}(\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)) & \text{if } \mathfrak{M} \text{ is torsion.} \end{cases}$$

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

Here, $G_{L,s}$ acts on $\hat{T}_{L,s}(\hat{\mathfrak{M}})$ by $(\sigma.f)(x)=\sigma(f(\sigma^{-1}(x)))$ for $\sigma\in G_{L,s}$, $f\in\hat{T}_{L,s}(\hat{\mathfrak{M}})$, $x\in\widehat{\mathcal{R}}_L\otimes_{\varphi,\mathfrak{S}_L}\mathfrak{M}$. Similar to the above, for any object $\hat{\mathfrak{M}}$ of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_L}$ or $\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,\infty}}$, we define a $\mathbb{Z}_p[G_{L,s}]$-module $\hat{T}_{L,s}(\hat{\mathfrak{M}})$ by

$$\hat{T}_{L,s}(\hat{\mathfrak{M}})=\begin{cases}\mathrm{Hom}_{\widehat{\mathcal{R}}_{L,s},\varphi}(\widehat{\mathcal{R}}_{L,s}\otimes_{\varphi,\mathfrak{S}_L}\mathfrak{M},W(R)) & \text{if } \mathfrak{M} \text{ is free,}\\ \mathrm{Hom}_{\widehat{\mathcal{R}}_{L,s},\varphi}(\widehat{\mathcal{R}}_{L,s}\otimes_{\varphi,\mathfrak{S}_L}\mathfrak{M},\mathbb{Q}_p/\mathbb{Z}_p\otimes_{\mathbb{Z}_p}W(R)) & \text{if } \mathfrak{M} \text{ is torsion.}\end{cases}$$

On the other hand, we have natural functors $\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L}\to\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_L}\to\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_L}\to\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s}}$ and

$$\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_{L,\infty}}\to\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,\infty}}\to\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,\infty}}\to\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s,\infty}}$$

and it is readily seen that these functors are compatible with $\hat{T}_L$ and $\hat{T}_{L,s}$. In particular, the functors $\hat{T}_{L,s}$ on $\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_L}$ and $\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_L}$ take their values in $\mathrm{Rep}^{r,\mathrm{st}}_{\mathbb{Z}_p}(G_{L,s})$ since we have an equivalence of categories $\hat{T}_{L,s}:\mathrm{Mod}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,s}}\overset{\sim}{\to}\mathrm{Rep}^{r,\mathrm{st}}_{\mathbb{Z}_p}(G_{L,s})$ by Theorem 2.5.

In the rest of this section, we study free cases. We leave studies for torsion cases to the next section.

Convention: For simplicity, if $L=K$, then we often omit the subscript "L" from various notations (e.g., $\mathrm{Mod}^{r,\hat{G}_{K,s}}_{/\mathfrak{S}_K}=\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}}$, $\widetilde{\mathrm{Mod}}^{r,\hat{G}_{K,s}}_{/\mathfrak{S}_K}=\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}$). Furthermore, if $s=0$, we often omit the subscript "s" from various notations (e.g., $\mathrm{Mod}^{r,\hat{G}_{L,0}}_{/\mathfrak{S}_L}=\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L}$, $\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,0}}_{/\mathfrak{S}_{L,0}}=\widetilde{\mathrm{Mod}}^{r,\hat{G}_L}_{/\mathfrak{S}_L}$).

3.2 The functors $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}\to\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}}\to\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}\to\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s}$

Now we consider the functors $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}\to\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}}\to\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}\to\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s}$. At first, by Proposition 2.6, we see that the functor $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}\to\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s}$ is fully faithful. It follows from this fact and Theorem 2.5(1) that the functor $\hat{T}_s:\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}\to\mathrm{Rep}^{r,\mathrm{st}}_{\mathbb{Z}_p}(G_s)$ is fully faithful. It is clear that the functor $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}}\to\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}$ is fully faithful and thus so is $\hat{T}_s:\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}}\to\mathrm{Rep}^{r,\mathrm{st}}_{\mathbb{Z}_p}(G_s)$. Combining this with Theorem 2.5(1) and Lemma 2.8, we obtain that the functor $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}\to\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}}$ is also fully faithful. Furthermore, we prove the following.

Proposition 3.3. *The functor* $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}} \to \widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}$ *is an equivalence of categories.*

Summary, we obtained the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}} \arrow[r, hook] \arrow[d, "\hat{T}", "\wr"'] & \mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}} \arrow[r, "\sim"] \arrow[drr, hook, "\hat{T}_s"] & \widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}} \arrow[r, hook] \arrow[dr, hook, "\hat{T}_s"] & \mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s} \arrow[d, "\hat{T}_s", "\wr"'] \\
\mathrm{Rep}^{r,\mathrm{st}}_{\mathbb{Z}_p}(G_K) \arrow[rrr, hook, "\text{restriction}"] & & & \mathrm{Rep}^{r,\mathrm{st}}_{\mathbb{Z}_p}(G_s).
\end{tikzcd}$$

Remark 3.4. The functor $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}} \hookrightarrow \mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s}$ may not be possibly essentially surjective. In fact, under some conditions, there exists a representation of G_K which is crystalline over K_s but not of finite height. For more precise information, see [Li2, Example 4.2.3].

Before starting the proof of Proposition 3.3, we give an explicit formula such as (1) for an object of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}$. The argument below follows the method of [Li2]. Let $\hat{\mathfrak{M}}$ be an object of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}$. Let $\hat{\mathfrak{M}}_s$ be the image of $\hat{\mathfrak{M}}$ for the functor $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}} \to \mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s}$. Put $\mathcal{D} = S_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and also put $\mathcal{D}_s = S_{K_0,s} \otimes_{\varphi,\mathfrak{S}_s} \mathfrak{M}_s = S_{K_0,s} \otimes_{S_{K_0}} \mathcal{D}$. Then $\mathcal{D}_s$ has a structure of a Breuil module and also $D = \mathcal{D}_s / I_+ S_{K_0,s} \mathcal{D}_s$ has a structure of a filtered (φ, N)-module corresponding to $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}_s(\hat{\mathfrak{M}}_s)$ (see Section 2.4), which is isomorphic to $\mathcal{D}/I_+ S_{K_0} \mathcal{D}$ as a φ-module over K_0. By [Li1, Lemma 7.3.1], we have a unique φ-compatible section $D \hookrightarrow \mathcal{D}$ and we regard D as a submodule of $\mathcal{D} \subset \mathcal{D}_s$ by this section. Then we have $\mathcal{D} = S_{K_0} \otimes_{K_0} D$ and $\mathcal{D}_s = S_{K_0,s} \otimes_{K_0} D$. By the explicit formula (1) for $\hat{\mathfrak{M}}_s$, we know that

$$\hat{G}_s(D) \subset (K_0[\![t]\!] \cap \mathcal{R}_{K_0,s}) \otimes_{K_0} D.$$

(Note that $\mathcal{R}_{K_0,s}$ can be regarded as a subring of $K_0[\![t, u_s]\!]$ via [Li1, Lemma 7.1.2].) Hence, taking any K_0-basis $e_1, \ldots, e_d$ of D, there exist $A_s(t) \in M_{d\times d}(K_0[\![t]\!])$ such that $\tau^{p^s}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A_s(t)$. Since $A_s(0) = \mathrm{I}_d$, we see that $\log(A_s(t)) \in M_{d\times d}(K_0[\![t]\!])$ is well defined. On the other hand, choose $g_0 \in G_s$ such that $\chi_p(g_0) \neq 1$, where χ_p is the p-adic cyclotomic character. Since $g_0\tau^{p^s} = (\tau^{p^s})^{\chi_p(g_0)} g_0$, we have $A_s(\chi_p(g_0)t) = A_s(t)^{\chi_p(g_0)}$ and thus we also have $\log(A_s(\chi_p(g_0)t)) = \chi_p(g_0)\log(A_s(t))$. Since $\log(A_s(0)) = \log(I_d) = 0$, we can write $\log(A_s(t))$ as $tB(t)$ for some $B(t) \in M_{d\times d}(K_0[\![t]\!])$. Then we have $\chi_p(g_0)tB(\chi_p(g_0)t) = \chi_p(g_0)tB(t)$, that is, $B(\chi_p(g_0)t) = B(t)$. Hence the assumption $\chi_p(g_0) \neq 1$ implies that $B(t)$ is a constant. Putting $N_s = B(t) \in M_{d\times d}(K_0)$, we obtain

$$\tau^{p^s}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)\Big(\sum_{i=0}^{\infty} N_s^i \gamma_i(t)\Big).$$

Now we define $N_D : D \to D$ by $N(e_1, \ldots, e_d) = (e_1, \ldots, e_d)p^{-s}N_s$ and also define $N_{\mathcal{D}} := N_{S_{K_0}} \otimes \mathrm{Id}_D + \mathrm{Id}_{S_{K_0}} \otimes N_D$. (Note that we have $N_D\varphi_D = p\varphi_D N_D$ and thus N_D is nilpotent.) It is a routine work to check the following:

(3)
$$g(a \otimes x) = \sum_{i=0}^{\infty} g(a)\gamma_i(-\log([\underline{\varepsilon}(g)])) \otimes N_D^i(x) \quad \text{for } g \in G_s, a \in B^+_{\mathrm{cris}}, x \in D.$$

Since we have

$$g(f) = \sum_{i=0}^{\infty} \gamma_i(-\log([\underline{\varepsilon}(g)]))N^i_{S_{K_0}}(f) \tag{4}$$

for any $g \in G_K$ and $f \in S_{K_0}$, we obtain the following explicit formula:

(5)
$$g(a \otimes x) = \sum_{i=0}^{\infty} g(a)\gamma_i(-\log([\underline{\varepsilon}(g)])) \otimes N_{\mathcal{D}}^i(x) \quad \text{for } g \in G_s, a \in B^+_{\mathrm{cris}}, x \in \mathcal{D}.$$

In particular, as in Section 2.4, we can show that

$$\log(\tau^{p^s})(x) = p^s t \otimes N_{\mathcal{D}}(x) \tag{6}$$

for any $x \in \mathcal{D}$.

Proof of Proposition 3.3. We continue to use the above notation. It suffices to prove that the G_s-action on $\widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ preserves $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Note that we have $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = (\mathcal{R}_{K_0} \otimes_{K_0} D) \cap (W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$, $G_s(\mathfrak{M}) \subset \widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \subset W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and $G_s(\mathcal{R}_{K_0}) \subset \mathcal{R}_{K_0}$. Thus it is enough to show $G_s(D) \subset \mathcal{R}_{K_0} \otimes_{K_0} D$. This quickly follows from (3). In fact, we have

$$g(x) = \sum_{i=0}^{\infty} \gamma_i(-\log([\underline{\varepsilon}(g)])) \otimes N_D^i(x) \in \mathcal{R}_{K_0} \otimes_{K_0} D \quad \text{for } x \in D,\ g \in G_s. \qquad \square$$

3.3 Relations with crystalline representations

We know that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}_s(\hat{\mathfrak{M}})$ is semistable over K_s for any object $\hat{\mathfrak{M}}$ of $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}}$ or $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}$. This subsection is devoted to prove a criterion, for $\hat{\mathfrak{M}}$, that describes when $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}_s(\hat{\mathfrak{M}})$ becomes crystalline.

Following [Fo2, Section 5] we set $I^{[m]}B^+_{\mathrm{cris}} := \{x \in B^+_{\mathrm{cris}} \mid \varphi^n(x) \in \mathrm{Fil}^m B^+_{\mathrm{cris}}$ for all $n \geqslant 0\}$. For any subring $A \subset B^+_{\mathrm{cris}}$, we put $I^{[m]}A = A \cap I^{[m]}B^+_{\mathrm{cris}}$. Furthermore, we also put $I^{[m+]}A = I^{[m]}A.I_+A$ (here, I_+A is defined in Section 2.3). By [Fo2, Proposition 5.1.3] and the proof of [Li2, Lemma 3.2.2], we know that $I^{[m]}W(R)$ is a principal ideal which is generated by $\varphi(\mathfrak{t})^m$.

Now we recall Theorem 2.5(2): if $\mathfrak{M}$ is an object of $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s}$, then $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}_s(\hat{\mathfrak{M}})$ is crystalline if and only if $\tau^{p^s}(x) - x \in u_s^p(I^{[1]}W(R) \otimes_{\varphi,\mathfrak{S}_s} \mathfrak{M})$ for any $x \in \mathfrak{M}$. However, if such $\mathfrak{M}$ descends to a Kisin module over $\mathfrak{S}$, then we can show the following.

THEOREM 3.5. *Let $\hat{\mathfrak{M}}$ be an object of $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}}$ or $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}$. Then the following is equivalent:*

(1) *$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}_s(\hat{\mathfrak{M}})$ is crystalline;*
(2) *$\tau^{p^s}(x) - x \in u^p(I^{[1]}W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ for any $x \in \mathfrak{M}$;*
(3) *$\tau^{p^s}(x) - x \in I^{[1+]}W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ for any $x \in \mathfrak{M}$.*

Proof. (1) $\Rightarrow$ (2): The proof here mainly follows that of [GLS, Proposition 4.7]. We may suppose $\hat{\mathfrak{M}}$ is an object of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}$. Put $\mathcal{D} = S_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and $D = \mathcal{D}/I_+S_{K_0}\mathcal{D}$ as in the previous subsection. We fix a $\varphi(\mathfrak{S})$-basis $(\hat{e}_1, \ldots, \hat{e}_d)$ of $\mathfrak{M} \subset \mathcal{D}$ and denote by $(e_1, \ldots, e_d)$ the image of $(\hat{e}_1, \ldots, \hat{e}_d)$ for the projection $\mathcal{D} \to D$. Then $(e_1, \ldots, e_d)$ is a K_0-basis of D. As described before the proof of Proposition 3.3, we regard D as a φ-stable submodule of $\mathcal{D}$, and we have $N_D : D \to D$ and $N_{\mathcal{D}} : D_{\mathcal{D}} \to D_{\mathcal{D}}$.

Now we consider a matrix $X \in GL_{d\times d}(S_{K_0})$ such that $(\hat{e}_1, \ldots, \hat{e}_d) = (e_1, \ldots, e_d)X$. We define $\tilde{S} = W(k)[\![u^p, u^{ep}/p]\!]$ as in [GLS, Section 4], which is a sub $W(k)$-algebra of $S^{\mathrm{int}}_{K_0}$ with the property $N_{S_{K_0}}(\tilde{S}) \subset u^p\tilde{S}$. By an easy computation we have $U = X^{-1}BX + X^{-1}N_{S_{K_0}}(X)$. Here, $B \in M_{d\times d}(K_0)$ and $U \in M_{d\times d}(S_{K_0})$ are defined by $N_D(e_1, \ldots, e_d) = (e_1, \ldots, e_d)B$ and $N_{\mathcal{D}}(\hat{e}_1, \ldots, \hat{e}_d) = (\hat{e}_1, \ldots, \hat{e}_d)U$. By the same proof as in the former half part of the proof of [GLS, Proposition 4.7], we obtain $X, X^{-1} \in M_{d\times d}(\tilde{S}[1/p])$.

On the other hand, let $\hat{\mathfrak{M}}_s$ be the image of $\hat{\mathfrak{M}}$ for the functor $\widetilde{\mathrm{Mod}}_{/\mathfrak{S}}^{r,\hat{G}_s} \to \mathrm{Mod}_{\mathfrak{S}_s}^{r,\hat{G}_s}$. Now we recall that $\mathcal{D}_s = S_{K_0,s} \otimes_{\varphi,\mathfrak{S}_s} \mathfrak{M}_s$ has a structure of the Breuil module corresponding to $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}_s(\hat{\mathfrak{M}}_s)$ Denote by $N_{\mathcal{D}_s}$ its monodromy operator. By the formula (2) for $\hat{\mathfrak{M}}_s$ and the formula (6) for $\hat{\mathfrak{M}}$, we see that $p^s N_{\mathcal{D}} = N_{\mathcal{D}_s}$ on $\mathcal{D}$. Therefore, $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}_s(\hat{\mathfrak{M}})$ is crystalline if and only if $N_{\mathcal{D}_s}$ mod $I_+ S_{K_0,s}\mathcal{D}_s$ is zero, which is equivalent to say that $N_D = (N_{\mathcal{D}}$ mod $I_+ S_{K_0}\mathcal{D})$ is zero, that is, $B = 0$. Therefore, the latter half part of the proof [GLS, Proposition 4.7] gives the assertion (2).

(2) $\Rightarrow$ (3): This is clear.

(3) $\Rightarrow$ (1): Suppose that (3) holds. We denote by $\hat{\mathfrak{M}}_s$ the image of $\hat{\mathfrak{M}}$ for the functor $\widetilde{\mathrm{Mod}}_{/\mathfrak{S}}^{r,\hat{G}_s} \to \mathrm{Mod}_{\mathfrak{S}_s}^{r,\hat{G}_s}$ as above. We claim that, for any $x \in \mathfrak{M}_s$, we have $\tau^{p^s}(x) - x \in I^{[1+]}W(R) \otimes_{\varphi_s} \mathfrak{M}_s$. Let $x = a \otimes y \in \mathfrak{M}_s = \mathfrak{S}_s \otimes_{\mathfrak{S}} \mathfrak{M}$ where $a \in \mathfrak{S}_s$ and $y \in \mathfrak{M}$. Then

$$\tau^{p^s}(x) - x = \tau^{p^s}(\varphi(a))(\tau^{p^s}(y) - y) + (\tau^{p^s}(\varphi(a)) - \varphi(a))y$$

and thus it suffices to show $\tau^{p^s}(\varphi(a)) - \varphi(a) \in I^{[1+]}W(R)$. This follows from the lemma below and thus we obtained the claim. By the claim and [Oz2, Theorem 21], we know that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}_s(\hat{\mathfrak{M}}_s) \simeq \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{T}_s(\hat{\mathfrak{M}})$ is crystalline. ∎

LEMMA 3.6. (1) *We have* $I^{[1]}W(R) \cap u^\ell B^+_{\mathrm{cris}} = u^\ell I^{[1]}W(R)$ *for* $\ell \geqslant 0$.
(2) *We have* $g(a) - a \in uI^{[1]}W(R)$ *for* $g \in G$ *and* $a \in \mathfrak{S}$.

Proof. This is due to [GLS, the proof of Proposition 7] but we write a proof here.

(1) Take $x = u^\ell y \in I^{[1]}W(R)$ with $y \in B^+_{\mathrm{cris}}$. By [Li4, Lemma 3.2.2] we have $y \in W(R)$. Now we remark that $uz \in \mathrm{Fil}^n W(R)$ with $z \in W(R)$ implies $z \in \mathrm{Fil}^n W(R)$ since u is a unit of B^+_{dR}. Hence $u^\ell y \in I^{[1]}W(R)$ implies $y \in I^{[1]}W(R)$.

(2) By the relation (4), we see that $g(a) - a \in I^{[1]}W(R)$. On the other hand, if $i > 0$, we can write $N^i_{S_{K_0}}(a) = ub_i$ for some $b_i \in \mathfrak{S}$. Thus by the relation (4) again we obtain $g(a) - a \in uB^+_{\mathrm{cris}}$. Then the result follows from (1). ∎

§4. Variants of torsion $(\varphi, \hat{G})$-modules

In this section, we mainly study full subcategories of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}$ defined below and also study representations associated with them. As a consequence, we prove theorems in Introduction. We use same notation as in Sections 2 and 3. In particular, p is odd. In below, let v_R be the valuation of R normalized such that $v_R(\underline{\pi}) = 1/e$ and, for any real number $x \geqslant 0$, we denote by $\mathfrak{m}_R^{\geqslant x}$ the ideal of R consisting of elements a with $v_R(a) \geqslant x$.

Let J be an ideal of $W(R)$ which satisfies the following conditions:

- $J \not\subset pW(R)$;
- J is a principal ideal;
- J is φ-stable and G_s-stable in $W(R)$.

By the above first and second assumptions for J, the image of J under the projection $W(R) \twoheadrightarrow R$ is of the form $\mathfrak{m}_R^{\geqslant c_J}$ for some real number $c_J \geqslant 0$.

Definition 4.1. We denote by $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$ the full subcategory of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}$ consisting of objects $\hat{\mathfrak{M}}$ which satisfy the following condition:

$$\tau^{p^s}(x) - x \in JW(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \quad \text{for any } x \in \mathfrak{M}.$$

We denote by $\widetilde{\mathrm{Rep}}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s)$ the essential image of the functor $\hat{T}_s : \widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G_s)$ restricted to $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$.

By definition, we have relations

$$\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty} \subset \widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J'}_{/\mathfrak{S}_\infty} \qquad \text{and} \qquad \widetilde{\mathrm{Rep}}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s) \subset \widetilde{\mathrm{Rep}}^{r,\hat{G}_s,J'}_{\mathrm{tor}}(G_s)$$

for $J \subset J'$.

4.1 Full faithfulness for $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$

For the beginning of a study of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$, we prove the following full faithfulness result.

Proposition 4.2. *Let r and r' be nonnegative integers with $c_J > pr/(p-1)$. Let $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{N}}$ be objects of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$ and $\widetilde{\mathrm{Mod}}^{r',\hat{G}_s,J}_{/\mathfrak{S}_\infty}$, respectively. Then we have $\mathrm{Hom}(\hat{\mathfrak{M}}, \hat{\mathfrak{N}}) = \mathrm{Hom}(\mathfrak{M}, \mathfrak{N})$. (Here, two "Hom"s are defined by obvious manners.)*

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

In particular, if $c_J > pr/(p-1)$, then the forgetful functor $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ is fully faithful.

Proof. A very similar proof of [Oz2, Lemma 7] proceeds, and hence we only give a sketch here. Let $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{N}}$ be objects of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$ and $\widetilde{\mathrm{Mod}}^{r',\hat{G}_s,J}_{/\mathfrak{S}_\infty}$, respectively. Let $f:\mathfrak{M}\to\mathfrak{N}$ be a morphism of Kisin modules over $\mathfrak{S}$. Put $\hat{f} = W(R)\otimes f : W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}\to W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$. Choose any lift of $\tau\in\hat{G}$ to G_K; we denote it also by τ. Since the $\hat{G}_s$-action for $\hat{\mathfrak{M}}$ is continuous, it suffices to prove that $\Delta(1\otimes x)=0$ for any $x\in\mathfrak{M}$ where $\Delta := \tau^{p^s}\circ\hat{f} - \hat{f}\circ\tau^{p^s}$. We use induction on n such that $p^n\mathfrak{N}=0$.

Suppose $n=1$. Since $\Delta = (\tau^{p^s}-1)\circ\hat{f} - \hat{f}\circ(\tau^{p^s}-1)$, we obtain the following:

$$(0):\quad \text{For any } x\in\mathfrak{M},\quad \Delta(1\otimes x)\in\mathfrak{m}_R^{\geqslant c(0)}(R\otimes_{\varphi,\mathfrak{S}}\mathfrak{N})$$

where $c(0)=c_J$. Since $\mathfrak{M}$ is of height $\leqslant r$, we further obtain the following for any $i\geqslant 1$ inductively:

$$(i):\quad \text{For any } x\in\mathfrak{M},\quad \Delta(1\otimes x)\in\mathfrak{m}_R^{\geqslant c(i)}(R\otimes_{\varphi,\mathfrak{S}}\mathfrak{N})$$

where $c(i) = pc(i-1) - pr = (c_J - pr/(p-1))p^i + pr/(p-1)$. The condition $c_J > pr/(p-1)$ implies that $c(i)\to\infty$ as $i\to\infty$ and thus $\Delta(1\otimes x)=0$.

Suppose $n>1$. Consider the exact sequence $0\to\mathrm{Ker}(p)\to\mathfrak{N}\xrightarrow{p}p\mathfrak{N}\to 0$ of φ-modules over $\mathfrak{S}$. It is not difficult to check that $\mathfrak{N}' := \mathrm{Ker}(p)$ and $\mathfrak{N}'' := p\mathfrak{N}$ are torsion Kisin modules of height $\leqslant r'$ over $\mathfrak{S}$ (cf. [Li1, Lemma 2.3.1]). Moreover, we can check that $\mathfrak{N}'$ and $\mathfrak{N}''$ have natural structures of objects of $\widetilde{\mathrm{Mod}}^{r',\hat{G}_s}_{/\mathfrak{S}_\infty}$ (which are denoted by $\hat{\mathfrak{N}}'$ and $\hat{\mathfrak{N}}''$, respectively) such that the sequence $0\to\mathfrak{N}'\to\mathfrak{N}\xrightarrow{p}\mathfrak{N}''\to 0$ induces an exact sequence $0\to\hat{\mathfrak{N}}'\to\hat{\mathfrak{N}}\to\hat{\mathfrak{N}}''\to 0$. By the lemma below, we know that $\hat{\mathfrak{N}}'$ and $\hat{\mathfrak{N}}''$ are in fact contained in $\widetilde{\mathrm{Mod}}^{r',\hat{G}_s,J}_{/\mathfrak{S}_\infty}$. By the induction hypothesis, we see that $\Delta(1\otimes x)$ has values in $(W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{N}')\cap(JW(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{N})$. By [Oz2, Lemma 6] and the assumption that $J\not\subset pW(R)$ is principal, we obtain that $\Delta(1\otimes x)\in JW(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{N}'$. Since $p\mathfrak{N}'=0$, an analogous argument in the case $n=1$ proceeds and we have $\Delta(1\otimes x)=0$ as desired. □

LEMMA 4.3. *Let $0\to\hat{\mathfrak{M}}'\to\hat{\mathfrak{M}}\to\hat{\mathfrak{M}}''\to 0$ be an exact sequence in $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}$. Suppose that $\hat{\mathfrak{M}}$ is an object of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$. Then $\hat{\mathfrak{M}}'$ and $\hat{\mathfrak{M}}''$ are also objects of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$.*

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

Proof. The fact $\hat{\mathfrak{M}}'' \in \widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}_s,J}$ is clear. Take any $x \in \mathfrak{M}'$. Then we have $\tau^{p^s}(x) - x \in (JW(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \cap (W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}')$. Since J is a principal ideal which is not contained in $pW(R)$, we obtain $\tau^{p^s}(x) - x \in JW(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'$ by [Oz2, Lemma 6]. This implies $\hat{\mathfrak{M}}' \in \widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}_s,J}$. □

4.2 The category $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$

In this subsection, we study some categorical properties of $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$.

Let $\hat{\mathfrak{M}}$ be an object of $\widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}_s}$. Following [Li2, Section 3.2] (note that arguments in [Li2] is the "free case"), we construct a map $\hat{\iota}_s$ which connects $\hat{\mathfrak{M}}$ and $\hat{T}_s(\hat{\mathfrak{M}})$ as follows. Observe that there exists a natural isomorphism of $\mathbb{Z}_p[G_s]$-modules

$$\hat{T}_s(\hat{\mathfrak{M}}) \simeq \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))$$

where G_s acts on $\mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))$ by $(\sigma.f)(x) = \sigma(f(\sigma^{-1}(x)))$ for $\sigma \in G_s$,

$$f \in \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)),$$
$$x \in W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = W(R) \otimes_{\widehat{\mathcal{R}}_s} (\widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}).$$

Thus we can define a morphism

$$\hat{\iota}'_s : W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}_s(\hat{\mathfrak{M}}), \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))$$

by

$$x \mapsto (f \mapsto f(x)), \quad x \in W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, f \in \hat{T}_s(\hat{\mathfrak{M}}).$$

Since $\hat{T}_s(\hat{\mathfrak{M}}) \simeq \bigoplus_{i\in I} \mathbb{Z}_p/p^{n_i}\mathbb{Z}_p$ as $\mathbb{Z}_p$-modules, we have a natural isomorphism $\mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}_s(\hat{\mathfrak{M}}), \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)) \simeq W(R) \otimes_{\mathbb{Z}_p} \hat{T}_s^\vee(\hat{\mathfrak{M}})$ where $\hat{T}_s^\vee(\hat{\mathfrak{M}}) = \mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}_s(\hat{\mathfrak{M}}), \mathbb{Q}_p/\mathbb{Z}_p)$ is the dual representation of $\hat{T}_s(\hat{\mathfrak{M}})$. Composing this isomorphism with $\hat{\iota}'_s$, we obtain the desired map

$$\hat{\iota}_s : W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to W(R) \otimes_{\mathbb{Z}_p} \hat{T}_s^\vee(\hat{\mathfrak{M}}).$$

It follows from a direct calculation that $\hat{\iota}_s$ is φ-equivariant and G_s-equivariant. If we denote by $\hat{\mathfrak{M}}_s$ the image of $\hat{\mathfrak{M}}$ for the functor $\widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}_s} \to \mathrm{Mod}_{/\mathfrak{S}_{s,\infty}}^{r,\hat{G}_s}$ (cf. Section 3.1), then the above $\hat{\iota}_s$ is isomorphic to "$\hat{\iota}$ for $\hat{\mathfrak{M}}_s$

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

in [Oz1, Section 4.1]". Hence [Oz1, Lemma 4.2(4)] implies that

$$W(\mathrm{Fr}\ R)\otimes\hat{\iota}_s : W(\mathrm{Fr}\ R)\otimes_{W(R)}(W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M})$$
$$\to W(\mathrm{Fr}\ R)\otimes_{W(R)}(W(R)\otimes_{\mathbb{Z}_p}\hat{T}_s^{\vee}(\hat{\mathfrak{M}}))$$

is bijective.

PROPOSITION 4.4. *Let $(R): 0\to T'\to T\to T''\to 0$ be an exact sequence in $\mathrm{Rep}_{\mathrm{tor}}(G_s)$. Assume that there exists $\hat{\mathfrak{M}}\in\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$ such that $\hat{T}_s(\hat{\mathfrak{M}})\simeq T$. Then there exists an exact sequence $(M): 0\to\hat{\mathfrak{M}}''\to\hat{\mathfrak{M}}\to\hat{\mathfrak{M}}'\to 0$ in $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$ such that $\hat{T}_s((M))\simeq(R)$.*

Proof. The same proof as [Oz1, Theorem 4.5], except using not $\hat{\iota}$ in the proof of [Oz1, Theorem 4.5] but $\hat{\iota}_s$ as above, gives an exact sequence (M): $0\to\hat{\mathfrak{M}}''\to\hat{\mathfrak{M}}\to\hat{\mathfrak{M}}'\to 0$ in $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}$ such that $\hat{T}_s((M))\simeq(R)$. Therefore, Lemma 4.3, gives the desired result. ∎

COROLLARY 4.5. *The full subcategory $\widetilde{\mathrm{Rep}}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s)$ of $\mathrm{Rep}_{\mathrm{tor}}(G_s)$ is stable under subquotients.*

Let L be as in Section 2, that is, the completion of an unramified algebraic extension of K with residue field k_L. We prove the following base change lemma.

LEMMA 4.6. *Assume that $J\supset u^pI^{[1]}W(R)$ or L is a finite unramified extension of K. If T is an object of $\widetilde{\mathrm{Rep}}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s)$, then $T|_{G_{L,s}}$ is an object of $\widetilde{\mathrm{Rep}}^{r,\hat{G}_{L,s},J}_{\mathrm{tor}}(G_{L,s})$.*

By an obvious way, we define a functor $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}\to\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,\infty}}$. The underlying Kisin module of the image of $\hat{\mathfrak{M}}\in\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}$ for this functor is $\mathfrak{M}_L=\mathfrak{S}_L\otimes_{\mathfrak{S}}\mathfrak{M}$. Lemma 4.6 immediately follows from the lemma below.

LEMMA 4.7. *Assume that $J\supset u^pI^{[1]}W(R)$ or L is a finite unramified extension of K. Then the functor $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}\to\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,\infty}}$ induces a functor $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}\to\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s},J}_{/\mathfrak{S}_{L,\infty}}$.*

Proof. Let $\hat{\mathfrak{M}}$ be an object of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}$ and let $\hat{\mathfrak{M}}_L$ be the image of $\hat{\mathfrak{M}}$ for the functor $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}\to\widetilde{\mathrm{Mod}}^{r,\hat{G}_{L,s}}_{/\mathfrak{S}_{L,\infty}}$. In the rest of this proof,

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

to avoid confusions, we denote the image of $x \in \mathfrak{M}_L$ in $W(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_L$ by $1 \otimes x$. Recall that we abuse notations by writing τ for the preimage of $\tau \in G_{K,p^\infty}$ under the bijection $G_{L,p^\infty} \simeq G_{K,p^\infty}$ of Lemma 2.2. Then τ^{p^s} is a topological generator of G_{L,s,p^∞}. It suffices to show the following: if $\hat{\mathfrak{M}}$ is an object of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$, then we have $\tau^{p^s}(1 \otimes x) - (1 \otimes x) \in JW(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_L$ for any $x \in \mathfrak{M}_L$. Now we suppose $\hat{\mathfrak{M}} \in \widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$. Take any $a \in \mathfrak{S}_L$ and $x \in \mathfrak{M}$. Note that we have $\tau^{p^s}(1 \otimes ax) - (1 \otimes ax) = \tau^{p^s}(\varphi(a))(\tau^{p^s}(1 \otimes x) - (1 \otimes x)) + (\tau^{p^s}(\varphi(a)) - \varphi(a))(1 \otimes x)$ in $W(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_L$. Since $\hat{\mathfrak{M}}$ is an object of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$, we have $\tau^{p^s}(\varphi(a))(\tau^{p^s}(1 \otimes x) - (1 \otimes x)) \in JW(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_L$. Therefore, it is enough to show $(\tau^{p^s}(\varphi(a)) - \varphi(a))(1 \otimes x) \in JW(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_L$. This follows from Lemma 3.6 immediately in the case where $J \supset u^p I^{[1]} W(R)$. Next we consider the case where L is a finite unramified extension of K. Let $c_1, \ldots, c_\ell \in W(k_L)$ be generators of $W(k_L)$ as a $W(k)$-module. Then we have $\mathfrak{S}_L = \sum_{j=1}^{\ell} c_j \mathfrak{S}$ and thus we can write $a = \sum_{j=1}^{\ell} a_j c_j$ for some $a_j \in \mathfrak{S}$. Hence it suffices to show $(\tau^{p^s}(\varphi(a_j)) - \varphi(a_j))(1 \otimes x) \in JW(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}_L$ but this in fact immediately follows from the equation $(\tau^{p^s}(\varphi(a_j)) - \varphi(a_j))(1 \otimes x) = (\tau^{p^s}(1 \otimes a_j x) - (1 \otimes a_j x)) - (\tau^{p^s}(\varphi(a_j))(\tau^{p^s}(1 \otimes x) - (1 \otimes x)))$. $\square$

REMARK 4.8. For a general L, the author does not know whether the statement of the above lemma is true or not.

4.3 Full faithfulness theorem for $\widetilde{\mathrm{Rep}}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s)$

Our goal in this subsection is to prove the following full faithfulness theorem, which plays an important role in our proofs of main theorems.

THEOREM 4.9. *Assume that $J \supset u^p I^{[1]} W(R)$ or k is algebraically closed. If $p^{s+2}/(p-1) \geqslant c_J > pr/(p-1)$, then the restriction functor $\widetilde{\mathrm{Rep}}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s) \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ is fully faithful.*

First we give a very rough sketch of the theory of maximal models for Kisin modules (cf. [CL1]). For any $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$, put $\mathfrak{M}[1/u] = \mathfrak{S}[1/u] \otimes_{\mathfrak{S}} \mathfrak{M}$ and denote by $F^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$ the (partially) ordered set (by inclusion) of torsion Kisin modules $\mathfrak{N}$ of height $\leqslant r$ which are contained in $\mathfrak{M}[1/u]$ and $\mathfrak{N}[1/u] = \mathfrak{M}[1/u]$ as φ-modules. The set $F^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$ has a greatest element (cf. [CL1, Corollary 3.2.6]). We denote this element by $\mathrm{Max}^r(\mathfrak{M})$. We say that $\mathfrak{M}$ is *maximal of height* $\leqslant r$ (or, *maximal* for simplicity) if it is the greatest element of $F^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$. The association $\mathfrak{M} \mapsto \mathrm{Max}^r(\mathfrak{M})$ defines

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

a functor "Max^r" from the category $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ of torsion Kisin modules of height $\leqslant r$ into the category $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ of maximal Kisin modules of height $\leqslant r$. The category $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ is abelian (cf. [CL1, Theorem 3.3.8]). Furthermore, the functor $T_{\mathfrak{S}}: \mathrm{Max}^r_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$, defined by $T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))$, is exact and fully faithful (cf. [CL1, Corollary 3.3.10]). It is not difficult to check that $T_{\mathfrak{S}}(\mathrm{Max}^r(\mathfrak{M}))$ is canonically isomorphic to $T_{\mathfrak{S}}(\mathfrak{M})$ as representations of G_∞ for any torsion Kisin module $\mathfrak{M}$ of height $\leqslant r$.

DEFINITION 4.10. [CL1, Section 3.6.1] Let d be a positive integer. Let $\mathfrak{n} = (n_i)_{i\in\mathbb{Z}/d\mathbb{Z}}$ be a sequence of nonnegative integers of smallest period d. We define a torsion Kisin module $\mathfrak{M}(\mathfrak{n})$ as below:

- as a $k[u]$-module, $\mathfrak{M}(\mathfrak{n}) = \bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}} k[u]e_i$;
- for all $i \in \mathbb{Z}/d\mathbb{Z}$, $\varphi(e_i) = u^{n_i}e_{i+1}$.

We denote by $\mathcal{S}^r_{\max}$ the set of sequences $\mathfrak{n} = (n_i)_{i\in\mathbb{Z}/d\mathbb{Z}}$ of integers $0 \leqslant n_i \leqslant \min\{er, p-1\}$ with smallest period d for some integer d except the constant sequence with value $p-1$ (if necessary). By definition, we see that $\mathfrak{M}(\mathfrak{n})$ is of height $\leqslant r$ for any $\mathfrak{n} \in \mathcal{S}^r_{\max}$. Putting $r_0 = \max\{r' \in \mathbb{Z}_{\geqslant 0}; e(r'-1) < p-1\}$, we also see that $\mathfrak{M}(\mathfrak{n})$ is of height $\leqslant r_0$ for any $\mathfrak{n} \in \mathcal{S}^r_{\max}$. It is known that $\mathfrak{M}(\mathfrak{n})$ is maximal for any $\mathfrak{n} \in \mathcal{S}^r_{\max}$ [CL1, Proposition 3.6.7]. If k is algebraically closed, then $\mathfrak{M}(\mathfrak{n})$ is simple in $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ for any $\mathfrak{n} \in \mathcal{S}^r_{\max}$ (cf. [CL1, Propositions 3.6.7 and 3.6.12]) and furthermore, the converse holds; any simple object in $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ is of the form $\mathfrak{M}(\mathfrak{n})$ for some $\mathfrak{n} \in \mathcal{S}^r_{\max}$ (cf. [CL1, Propositions 3.6.8 and 3.6.12]).

LEMMA 4.11. *Assume that $p^{s+2}/(p-1) \geqslant c_J$. Let d be a positive integer. Let $\mathfrak{n} = (n_i)_{i\in\mathbb{Z}/d\mathbb{Z}}$ be a sequence of nonnegative integers of smallest period d. If $\mathfrak{M}(\mathfrak{n})$ is of height $\leqslant r$, then $\mathfrak{M}(\mathfrak{n})$ has a structure of an object of $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$.*

Proof. Choose any (p^d-1)th root $\eta \in R$ of $\underline{\varepsilon}$. Since $[\eta]\cdot\exp(t/(p^d-1))$ is a (p^d-1)th root of unity, it is of the form $[a]$ for some $a \in \mathbb{F}^\times_{p^d}$. Replacing ηa^{-1} with η, we obtain $[\eta] = \exp(-t/(p^d-1)) \in \widehat{\mathcal{R}}^\times$. Put $x_i = [\eta]^{m_i} \in \widehat{\mathcal{R}}^\times$ and $\bar{x}_i = \eta^{m_i} \in (\widehat{\mathcal{R}}/p\widehat{\mathcal{R}})^\times \subset R^\times$ for any $i \in \mathbb{Z}/d\mathbb{Z}$, where $m_i = \sum_{j=0}^{d-1} n_{i+j}p^{d-j}$. We see that $x_i - 1$ is contained in $I_+\widehat{\mathcal{R}}$. In the rest of this proof, to avoid confusions, we denote the image of $x \in \mathfrak{M}(\mathfrak{n})$ in $\widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}(\mathfrak{n}) \subset R \otimes_{\varphi,k[u]} \mathfrak{M}(\mathfrak{n})$ by $1 \otimes x$. Now we define a $\hat{G}_s$-action on $\widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}(\mathfrak{n})$ by $\tau^{p^s}(1 \otimes e_i) := \bar{x}_i^{p^s}(1 \otimes e_i)$ for the basis $\{e_i\}_{i\in\mathbb{Z}/d\mathbb{Z}}$ of $\mathfrak{M}(\mathfrak{n})$ as in

Definition 4.10. We claim that $g\varphi=\varphi g$ on $\widehat{\mathcal{R}}_s\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}(\mathfrak{n})$ for any $g\in G_s$. For this, it suffices to check that the equality $\tau^{p^s}\varphi(1\otimes e_i)=\varphi\tau^{p^s}(1\otimes e_i)$ holds for any i. Note that we have

$$\tau^{p^s}\varphi(1\otimes e_i)=\tau^{p^s}(u^{pn_i}(1\otimes e_{i+1}))=\bar{x}_{i+1}^{p^s}(\underline{\varepsilon}^{p^s}u)^{pn_i}(1\otimes e_{i+1})$$

and

$$\varphi\tau^{p^s}(1\otimes e_i)=\varphi(\bar{x}_i^{p^s}(1\otimes e_i))=\bar{x}_i^{p^{s+1}}u^{pn_i}(1\otimes e_{i+1}).$$

Hence it is enough to check $x_i^{p^{s+1}}=x_{i+1}^{p^s}[\underline{\varepsilon}]^{p^{s+1}n_i}$ but we can show this equality without difficulty. In fact, we have equivalences

$$\begin{aligned}x_i^{p^{s+1}}&=x_{i+1}^{p^s}[\underline{\varepsilon}]^{p^{s+1}n_i}\Leftrightarrow\exp\Big(-p^{s+1}m_i\frac{t}{p^d-1}\Big)\\&=\exp\Big(-p^sm_{i+1}\frac{t}{p^d-1}-p^{s+1}n_it\Big)\Leftrightarrow pm_i\\&=m_{i+1}+(p^{d+1}-p)n_i\end{aligned}$$

and the last equality can be checked immediately by definition of m_i.

By the claim above, we see that $\mathfrak{M}(\mathfrak{n})$ has a structure of an object of $\widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}_s}$ via this $\hat{G}_s$-action; we denote it by $\hat{\mathfrak{M}}(\mathfrak{n})$. It suffices to prove that $\hat{\mathfrak{M}}(\mathfrak{n})$ is in fact an object of $\widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}_s,J}$. Recall that v_R is the valuation of R normalized such that $v_R(\underline{\pi})=1/e$. Define $\tilde{\mathfrak{t}}=\mathfrak{t}\bmod pW(R)$ an element of R. We denote by v_p the usual p-adic valuation normalized by $v_p(p)=1$. Note that we have $v_R(\underline{\varepsilon}-1)=p/(p-1)$ and $v_R(\tilde{\mathfrak{t}})=1/(p-1)$ (here, the latter equation follows from the relation $\varphi(\mathfrak{t})=pE(0)^{-1}E(u)\mathfrak{t}$). Moreover, we have $v_R(\underline{\varepsilon}^m-1)=v_R(\eta^m-1)=p^{v_p(m)+1}/(p-1)$ for any $m\in\mathbb{Z}_p$ by [GLS, Lemma 6.6(1)]. Thus we have

$$v_R(\bar{x}_i^{p^s}-1)=v_R(\eta^{p^sm_i}-1)=\frac{p^{s+v_p(m_i)+1}}{p-1}\geqslant\frac{p^{s+2}}{p-1}.$$

Since $p^{s+2}/(p-1)\geqslant c_J$ and the image of J in R is $\mathfrak{m}_R^{\geqslant c_J}$, we obtain

$$\tau^{p^s}(1\otimes e_i)-(1\otimes e_i)\in\mathfrak{m}_R^{\geqslant c_J}R\otimes_{\varphi,k[u]}\mathfrak{M}(\mathfrak{n})\simeq JW(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}(\mathfrak{n}).$$

Finally we have to show that $\tau^{p^s}(1\otimes ae_i)-(1\otimes ae_i)\in\mathfrak{m}_R^{\geqslant c_J}R\otimes_{\varphi,k[u]}\mathfrak{M}(\mathfrak{n})$ for any $a\in k[\![u]\!]$. Since $\tau^{p^s}(1\otimes ae_i)-(1\otimes ae_i)=\tau^{p^s}(\varphi(a))(\tau^{p^s}(1\otimes e_i)-(1\otimes e_i))+(\tau^{p^s}(\varphi(a))-\varphi(a))(1\otimes e_i)$, it suffices to show

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

$\tau^{p^s}(\varphi(a))-\varphi(a)\in\mathfrak{m}_R^{\geqslant c_J}$. Write $\varphi(a)=\sum_{i\geqslant 0}a_iu^{pi}$ for some $a_i\in k$. Then we have $\tau^{p^s}(\varphi(a))-\varphi(a)=\sum_{i\geqslant 1}a_i(\underline{\varepsilon}^{p^{s+1}i}-1)u^{pi}$. Since we have

$$v_R((\underline{\varepsilon}^{p^{s+1}i}-1)u^{pi})=p^{s+1}v_R(\underline{\varepsilon}^i-1)+v_R(u^{pi})>\frac{p^{s+2}}{p-1}\geqslant c_J$$

for any $i\geqslant 1$, we have done. ∎

Recall that $r_0=\max\{r'\in\mathbb{Z}_{\geqslant 0};\, e(r'-1)<p-1\}$. Put $r_1:=\min\{r,r_0\}$.

Corollary 4.12. *Assume that $p^{s+2}/(p-1)\geqslant c_J$. If $\mathfrak{n}\in\mathcal{S}^r_{\max}$, then $\mathfrak{M}(\mathfrak{n})$ has a structure of an object of $\widetilde{\mathrm{Mod}}^{r',\hat{G}_s,J}_{/\mathfrak{S}_\infty}$ for any $r'\geqslant r_1$. Furthermore, if $c_J>pr_1/(p-1)$, it is uniquely determined. We denote this object by $\hat{\mathfrak{M}}(\mathfrak{n})$.*

Proof. We should remark that $\mathfrak{M}(\mathfrak{n})$ is of height $\leqslant r_1$ for any $\mathfrak{n}\in\mathcal{S}^r_{\max}$. The uniqueness assertion follows from Proposition 4.2. ∎

Before the lemma below, we remark that any semisimple $\mathbb{F}_p$-representation of G_K is automatically tame.

Lemma 4.13. (1) *The restriction functor from the category of tamely ramified torsion $\mathbb{Z}_p$-representations of G_K to the category of torsion $\mathbb{Z}_p$-representations of G_∞ is fully faithful.*

(2) *The restriction functor in* (1) *induces an equivalence between the category of semisimple (resp. irreducible) $\mathbb{F}_p$-representations of G_K and the category of semisimple (resp. irreducible) $\mathbb{F}_p$-representations of G_∞.*

Proof. (1) The result immediately follows from the fact that G_K is topologically generated by G_∞ and the wild inertia subgroup of G_K.

(2) It suffices to show the assertion for irreducible representations. Denote by $\mathrm{Rep}^{\mathrm{irr}}_{\mathbb{F}_p}(G_K)$ and $\mathrm{Rep}^{\mathrm{irr}}_{\mathbb{F}_p}(G_\infty)$ the category of irreducible $\mathbb{F}_p$-representations of G_K and G_∞, respectively. First we show that the restriction of the action of G_K to G_∞ induces a functor $\mathrm{Rep}^{\mathrm{irr}}_{\mathbb{F}_p}(G_K)\to\mathrm{Rep}^{\mathrm{irr}}_{\mathbb{F}_p}(G_\infty)$. Let T be an irreducible $\mathbb{F}_p$-representation of G_K. Take a G_∞-stable submodule T' of T. Let K^{t} be the maximal tamely ramified extension of K and $I_p=\mathrm{Gal}(\overline{K}/K^{\mathrm{t}})$ the wild inertia subgroup of G_K. Then I_p acts on T trivially. In particular, T' is stable under I_p-action. Since G_K is topologically generated by G_∞ and I_p, we know that T' is a G_K-stable submodule of T. Hence $T'=0$ or T and this implies that $T|_{G_\infty}$ is irreducible. Thus the restriction functor $\mathrm{Rep}^{\mathrm{irr}}_{\mathbb{F}_p}(G_K)\to\mathrm{Rep}^{\mathrm{irr}}_{\mathbb{F}_p}(G_\infty)$

is well defined. This is fully faithful by (1). It is enough to show that this functor is essentially surjective. Let T be an irreducible $\mathbb{F}_p$-representation of G_∞. Since $G_\infty \cap I_p$ acts on T trivially, the G_∞-action on T factors through $G_\infty/G_\infty \cap I_p$. We define a G_K-action on T via natural maps $G_K \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{t}}/K) \simeq \mathrm{Gal}(K_\infty K^{\mathrm{t}}/K_\infty) \simeq G_\infty/G_\infty \cap I_p$. The restriction of this G_K-action on T to G_∞ coincides with the original G_∞-action on T and thus we finish a proof. ∎

LEMMA 4.14. *Assume that $J \supset u^p I^{[1]}W(R)$ or k is algebraically closed. Let $T \in \mathrm{Rep}_{\mathrm{tor}}(G_s)$ and $T' \in \widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$. Suppose that T is tame, $pT = 0$ and $T|_{G_\infty} \simeq T_{\mathfrak{S}}(\mathfrak{M})$ for some $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. Furthermore, we suppose $p^{s+2}/(p-1) \geqslant c_J > pr/(p-1)$. Then all G_∞-equivariant homomorphisms $T \to T'$ are G_s-equivariant.*

Proof. Let L be the completion of the maximal unramified extension K^{ur} of K. By identifying G_L with $G_{K^{\mathrm{ur}}}$, we may regard G_L as a subgroup of G_K. Note that $L_{(s)} = K_{(s)}L$ is the completion of the maximal unramified extension of $K_{(s)}$, and G_s is topologically generated by $G_{L,s}$ and G_∞. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{G_{L,s}}(T,T') & \hookrightarrow & \mathrm{Hom}_{G_{L,\infty}}(T,T') \\ \uparrow & & \uparrow \\ \mathrm{Hom}_{G_s}(T,T') & \hookrightarrow & \mathrm{Hom}_{G_\infty}(T,T'). \end{array}$$

Since $T'|_{G_{L,s}}$ is contained in $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_{L,s},J}(G_{L,s})$ if $J \supset u^p I^{[1]}W(R)$ (cf. Lemma 4.6), the above diagram allows us to reduce a proof to the case where k is algebraically closed. In the rest of this proof, we assume that k is algebraically closed. Under this assumption, an $\mathbb{F}_p$-representation of G_s is tame if and only if it is semisimple by Maschke's theorem. Thus we may also assume that T is irreducible (here, we remark that any subquotient of T is tame and, also remark that the essential image of $T_{\mathfrak{S}} : \mathrm{Mod}^r_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ is stable under subquotients in $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$). By the assumption on T, we have $T|_{G_\infty} \simeq T_{\mathfrak{S}}(\mathfrak{M}) \simeq T_{\mathfrak{S}}(\mathrm{Max}^r(\mathfrak{M}))$ for some $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. Since $T|_{G_\infty}$ is irreducible (cf. By Lemma 4.13(2)) and $T_{\mathfrak{S}} : \mathrm{Max}^r_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ is exact and fully faithful, we know that $\mathrm{Max}^r(\mathfrak{M})$ is a simple object in the abelian category $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$. Therefore, since k is algebraically closed, we have $\mathrm{Max}^r(\mathfrak{M}) \simeq \mathfrak{M}(\mathfrak{n})$ for some $\mathfrak{n} \in \mathcal{S}^r_{\max}$

(cf. [CL1, Propositions 3.6.8 and 3.6.12]). Let $\hat{\mathfrak{M}}(\mathfrak{n})$ be the object of $\widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}_s,J}$ as in Corollary 4.12. We recall that $T_{\mathfrak{S}}(\mathfrak{M}(\mathfrak{n}))$ is isomorphic to $\hat{T}_s(\hat{\mathfrak{M}}(\mathfrak{n}))|_{G_\infty}$ (see Theorem 2.5(1)), and hence we have an isomorphism $T|_{G_\infty} \simeq \hat{T}_s(\hat{\mathfrak{M}}(\mathfrak{n}))|_{G_\infty}$. Here, we note that T and $\hat{T}_s(\hat{\mathfrak{M}}(\mathfrak{n}))$ are irreducible as representations of G_s (cf. [CL1, Theorem 3.6.11]). Applying Lemma 4.13 again, we obtain an isomorphism $T \simeq \hat{T}_s(\hat{\mathfrak{M}}(\mathfrak{n}))$ as representations of G_s. On the other hand, we can take $\hat{\mathfrak{M}}' = (\mathfrak{M}', \varphi, \hat{G}_s) \in \widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}_s,J}$ such that $T' \simeq \hat{T}_s(\hat{\mathfrak{M}}')$. We consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{G_s}(T, T') \arrow[rr, hook] & & \mathrm{Hom}_{G_\infty}(T, T') \\
\mathrm{Hom}(\hat{\mathfrak{M}}', \hat{\mathfrak{M}}(\mathfrak{n})) \arrow[u, "\hat{T}_s"] \arrow[r, "\text{forgetful}"] & \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}', \mathfrak{M}(\mathfrak{n})) \arrow[r, "\mathrm{Max}^r"] & \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathrm{Max}^r(\mathfrak{M}'), \mathfrak{M}(\mathfrak{n})) \arrow[u, "T_{\mathfrak{S}}"']
\end{tikzcd}$$

Here, $\mathrm{Hom}(\hat{\mathfrak{M}}', \hat{\mathfrak{M}}(\mathfrak{n}))$ is the set of morphisms $\hat{\mathfrak{M}}' \to \hat{\mathfrak{M}}(\mathfrak{n})$ in the category $\widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}_s,J}$. The first arrow in the bottom line is bijective by Proposition 4.2 and so is the second (this follows from the fact that $\mathfrak{M}(\mathfrak{n})$ is maximal by [CL1, Proposition 3.6.7]). Since the right vertical arrow is bijective, the top horizontal arrow must be bijective. $\square$

Now we are ready to prove Theorem 4.9.

Proof of Theorem 4.9. At first, we note that the category $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$ is an exact category in the sense of Quillen [Qu, Section 2] by Corollary 4.5. Hence short exact sequences in $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$ give rise to exact sequences of Hom's and Ext^1's in the usual way. (This property holds for any exact category.) Let T and T' be objects of $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$. Take any Jordan–Hölder sequence $0 = T_0 \subset T_1 \subset \cdots \subset T_n = T$ of T in $\mathrm{Rep}_{\mathrm{tor}}(G_s)$. By Corollary 4.5 again, we know that T_i and $T_{i,i-1} := T_i/T_{i-1}$ are contained in $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$ for any i. By Lemma 4.14, if an exact sequence $0 \to T' \to V \to T_{i,i-1} \to 0$ in $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$ splits as representations of G_∞, then it splits as a sequence of representations of G_s. This shows that the fourth column in the diagram

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

below is injective:

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Hom}_{G_s}(T_{i,i-1},T') & \to & \mathrm{Hom}_{G_s}(T_i,T') & \to & \mathrm{Hom}_{G_s}(T_{i-1},T') & \to & \mathrm{Ext}^1(T_{i,i-1},T') \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathrm{Hom}_{G_\infty}(T_{i,i-1},T') & \to & \mathrm{Hom}_{G_\infty}(T_i,T') & \to & \mathrm{Hom}_{G_\infty}(T_{i-1},T') & \to & \mathrm{Ext}^1_{G_\infty}(T_{i,i-1},T')
\end{array}$$

Here, the extension $\mathrm{Ext}^1(T_{i,i-1},T')$ in the above diagram is taken in the category $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$. In addition, it follows from Lemma 4.14 that the first column is an isomorphism. Therefore, we obtain an implication that, if the third column is an isomorphism, then the second one is an isomorphism. Hence a dévissage argument works and the desired full faithfulness follows. ∎

4.4 Proof of Theorem 1.2

Now we are ready to prove our main theorems. First we prove Theorem 1.2. Recall that a torsion $\mathbb{Z}_p$-representation T of G_K is torsion crystalline with Hodge–Tate weights in $[0,r]$ if it can be written as the quotient of lattices in some crystalline $\mathbb{Q}_p$-representation of G_K with Hodge–Tate weights in $[0,r]$. Let $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ be the category of them. We apply our arguments given in previous subsections with the following J:

$$J = u^p I^{[1]} W(R) = u^p \varphi(\mathfrak{t}) W(R).$$

Then we have $c_J = p/e + p/(p-1)$ and thus the inequalities $p^{s+2}/(p-1) \geqslant c_J > pr/(p-1)$ are satisfied if $e(r-1) < p-1$. Therefore, Theorem 1.2 is an easy consequence of the following proposition and Theorem 4.9.

Proposition 4.15. *The category* $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ *is a subcategory of the category* $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$ *when* $s=0$.

Proof. In this proof, we put $s=0$. So we omit subscript s in various notations (e.g., $\hat{G}_s = \hat{G}$, $\widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}_s} = \widetilde{\mathrm{Mod}}_{/\mathfrak{S}_\infty}^{r,\hat{G}}$). Let T be an object of $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ and let $L \subset L'$ be lattices in a crystalline $\mathbb{Q}_p$-representation with Hodge–Tate weights in $[0,r]$ such that $L'/L \simeq T$. By Theorem 2.5(1), there exists an injection $\hat{\mathfrak{L}}' \hookrightarrow \hat{\mathfrak{L}}$ of $(\varphi,\hat{G})$-modules over $\mathfrak{S}$ which corresponds to the injection $L \hookrightarrow L'$. Now we put $\mathfrak{M} = \mathfrak{L}/\mathfrak{L}'$. Since L'/L is killed by a power of p, $\mathfrak{M}$ is an object of $\mathrm{Mod}_{/\mathfrak{S}_\infty}^r$. We equip a $\hat{G}$-action with $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

by a natural isomorphism $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \simeq (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{L})/(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{L}')$. Then we see that $\mathfrak{M}$ has a structure of an object of $\widetilde{\mathrm{Mod}}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$; denote it by $\hat{\mathfrak{M}}$. Moreover, Theorem 3.5 implies that $\hat{\mathfrak{M}}$ is in fact contained in $\widetilde{\mathrm{Mod}}^{r,\hat{G},J}_{/\mathfrak{S}_\infty}$. By a similar argument to the proof of [CL2, Lemma 3.1.4], we have an exact sequence $0 \to \hat{T}(\hat{\mathfrak{L}}) \to \hat{T}(\hat{\mathfrak{L}}') \to \hat{T}(\hat{\mathfrak{M}}) \to 0$ of representations of G_K which is isomorphic to $0 \to L \to L' \to T \to 0$. This finishes a proof. ∎

4.5 Proof of Theorem 1.3

We give a proof of Theorem 1.3. If $s \geqslant n-1$, then we put

$$J = u^p I^{[p^{s-n+1}]} W(R) = u^p \varphi(\mathfrak{t})^{p^{s-n+1}} W(R).$$

Note that we have $c_J = p/e + p^{s-n+2}/(p-1)$ and thus the inequalities $p^{s+2}/(p-1) \geqslant c_J > pr/(p-1)$ are satisfied if

$$s > n - 1 + \log_p(r - (p-1)/e).$$

PROPOSITION 4.16. *Suppose $s \geqslant n-1$. If T is an object of $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathrm{tor}}(G_K)$ which is killed by p^n, then $T|_{G_s}$ is contained in $\widetilde{\mathrm{Rep}}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s)$.*

Proof. Let L be an object of $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathbb{Z}_p}(G_K)$. Take a $(\varphi, \hat{G})$-module $\hat{\mathfrak{L}}$ over $\mathfrak{S}$ such that $L \simeq \hat{T}(\hat{\mathfrak{L}})$. It is known that $(\tau-1)^i(x) \in u^p I^{[i]} W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{L}$ for any $i \geqslant 1$ and any $x \in \mathfrak{L}$ (cf. the latter half part of the proof of [GLS, Proposition 4.7]). Take any $x \in \mathfrak{L}$. Since $(\tau^{p^s} - 1)(x) = \sum_{i=1}^{p^s} \binom{p^s}{i} (\tau-1)^i(x)$, we obtain that

$$(\tau^{p^s} - 1)(x) \in \sum_{i=1}^{p^s} p^{s-v_p(i)} u^p I^{[i]} W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{L}. \tag{7}$$

Now let T be an object of $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathrm{tor}}(G_K)$ which is killed by p^n. Take an exact sequence $(R): 0 \to L_1 \to L_2 \to T \to 0$ of $\mathbb{Z}_p$-representations of G_K with $L_1, L_2 \in \mathrm{Rep}^{r,\mathrm{cris}}_{\mathbb{Z}_p}(G_K)$. By [CL2, Theorem 3.1.3 and Lemma 3.1.4], there exists an exact sequence $(M): 0 \to \hat{\mathfrak{L}}_2 \to \hat{\mathfrak{L}}_1 \to \hat{\mathfrak{M}} \to 0$ of $(\varphi, \hat{G})$-modules over $\mathfrak{S}$ such that $\hat{T}((M)) \simeq (R)$. By (7), we see that

$$(\tau^{p^s} - 1)(x) \in \sum_{i=1}^{p^s} p^{s-v_p(i)} u^p I^{[i]} W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$$

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

for any $x \in \mathfrak{M}$. Since $\mathfrak{M}$ is killed by p^n and $s \geqslant n-1$, we have

$$\begin{aligned}\sum_{i=1}^{p^s} p^{s-v_p(i)} u^p I^{[i]} W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} &= \sum_{\substack{i=1,\ldots,p^s \\ s-v_p(i)<n}} p^{s-v_p(i)} u^p I^{[i]} W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \\ &= \sum_{\ell=0}^{n-1} p^{\ell} u^p I^{[p^{s-\ell}]} W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \\ &\subset u^p I^{[p^{s-n+1}]} W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}.\end{aligned}$$

Therefore, we obtained the desired result. □

Proof of Theorem 1.3. By Theorem 1.2, we may suppose $\log_p(r-(p-1)/e) \geqslant 0$, that is, $e(r-1) \geqslant p-1$. Suppose $s > n-1+\log_p(r-(p-1)/e)$. Note that the condition $s \geqslant n-1$ is now satisfied. Let T and T' be as in the statement of Theorem 1.3. Let $f : T \to T'$ be a G_∞-equivariant homomorphism. Denote by L the completion of K^{ur} and identify G_L with the inertia subgroup of G_K. We note that $T|_{G_L}$ and $T'|_{G_L}$ are objects of $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_L)$. By Proposition 4.16, $T|_{G_{L,s}}$ and $T'|_{G_{L,s}}$ are objects of $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_{L,s},J}(G_{L,s})$. Hence we have that f is $G_{L,s}$-equivariant by Theorem 4.9. Since G_s is topologically generated by $G_{L,s}$ and G_∞, we see that f is G_s-equivariant. □

4.6 Galois equivariance for torsion semistable representations

In this subsection, we prove a Galois equivariance theorem for torsion semistable representations. A torsion $\mathbb{Z}_p$-representation T of G_K is *torsion semistable with Hodge–Tate weights in* $[0, r]$ if it can be written as the quotient of lattices in some semistable $\mathbb{Q}_p$-representation of G_K with Hodge–Tate weights in $[0, r]$. We denote by $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{st}}(G_K)$ the category of them. Note that $\mathrm{Rep}_{\mathrm{tor}}^{0,\mathrm{st}}(G_K) = \mathrm{Rep}_{\mathrm{tor}}^{0,\mathrm{cris}}(G_K)$. Similar to Theorem 1.3, we show the following, which is the main result of this subsection.

THEOREM 4.17. *Suppose that $s > n-1+\log_p r$. Let T and T' be objects of $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{st}}(G_K)$ which are killed by p^n. Then any G_∞-equivariant homomorphism $T \to T'$ is G_s-equivariant.*

If $s \geqslant n-1$, then we put

$$J = I^{[p^{s-n+1}]} W(R) = \varphi(\mathfrak{t})^{p^{s-n+1}} W(R).$$

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

Then we have $c_J = p^{s-n+2}/(p-1)$. To show Theorem 4.17, we use similar arguments to those in the proof of Theorem 1.3.

PROPOSITION 4.18. *Suppose $s \geqslant n-1$. If T is an object of $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{st}}(G_K)$ which is killed by p^n, then $T|_{G_s}$ is contained in $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$.*

Proof. Let L be a lattice in a semistable $\mathbb{Q}_p$-representation of G_K with Hodge–Tate weights in $[0,r]$. Take a $(\varphi,\hat{G})$-module $\hat{\mathfrak{L}}$ over $\mathfrak{S}$ such that $L \simeq \hat{T}(\hat{\mathfrak{L}})$. It is known that $(\tau-1)^i(x) \in I^{[i]}W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{L}$ for any $i \geqslant 1$ and any $x \in \mathfrak{L}$ (cf. the proof of [Li3, Proposition 2.4.1]). Thus the same proof proceeds as that of Proposition 4.16. $\square$

Proof of Theorem 4.17. We have $\mathrm{Rep}_{\mathrm{tor}}^{0,\mathrm{st}}(G_K) = \mathrm{Rep}_{\mathrm{tor}}^{0,\mathrm{cris}}(G_K)$ and thus Theorem 1.3 for $r=0$ is an easy consequence of Theorem 1.2. Hence we may assume $r \geqslant 1$. The rest of a proof is similar to the proof of Theorem 1.3. $\square$

4.7 Some consequences

In this subsection, we generalize some results proved in [Br3, Section 3.4]. First of all, we show the following elementary lemma, which should be well known to experts, but we include a proof here for the sake of completeness.

LEMMA 4.19. *The full subcategories $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ and $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{st}}(G_K)$ of $\mathrm{Rep}_{\mathrm{tor}}(G_K)$ are stable under formation of subquotients, direct sums and the association $T \mapsto T^\vee(r)$. Here $T^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Q}_p/\mathbb{Z}_p)$ is the dual representation of T.*

Proof. We prove the statement only for the category $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$. Let $T \in \mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ be killed by p^n for some $n>0$. Assertions for quotients and direct sums are clear. We prove that $T^\vee(r)$ is contained in $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$. There exist lattices $L_1 \subset L_2$ in some crystalline $\mathbb{Q}_p$-representation of G_K and an exact sequence $0 \to L_1 \to L_2 \to T \to 0$ of $\mathbb{Z}_p[G_K]$-modules. This exact sequence induces an exact sequence $0 \to T \to L_1/p^nL_1 \to L_2/p^nL_2 \to T \to 0$ of finite $\mathbb{Z}_p[G_K]$-modules. By duality, we obtain an exact sequence $0 \to T^\vee \to (L_2/p^nL_2)^\vee \to (L_1/p^nL_1)^\vee \to T^\vee \to 0$ of finite $\mathbb{Z}_p[G_K]$-modules. Then we obtain a G_K-equivariant surjection $L_1^\vee \twoheadrightarrow T^\vee$ by the composite $L_1^\vee \twoheadrightarrow L_1^\vee/p^nL_1^\vee \xrightarrow{\sim} (L_1/p^nL_1)^\vee \twoheadrightarrow T^\vee$ of natural maps (here, for any free $\mathbb{Z}_p$-representation L of G_K, $L^\vee := \mathrm{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p)$ stands for the dual of L). Therefore, we obtain $L_1^\vee(r) \twoheadrightarrow T^\vee(r)$ and thus $T^\vee(r) \in \mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$. Finally, we prove the stability assertion for subobjects. Let T' be a G_K-stable submodule of T. We have a G_K-equivariant surjection $f : L_1^\vee \twoheadrightarrow T^\vee \twoheadrightarrow (T')^\vee$. Let L_2' be a free $\mathbb{Z}_p$-representation of G_K such that its dual is the kernel of f. We have an exact sequence $0 \to (L_2')^\vee \to L_1^\vee \xrightarrow{f} (T')^\vee \to 0$ of $\mathbb{Z}_p[G_K]$-modules. Repeating the construction of the surjection $L_1^\vee \twoheadrightarrow T^\vee$, we obtain a G_K-equivariant surjection $L_2' = (L_2')^{\vee\vee} \twoheadrightarrow (T')^{\vee\vee} = T'$ and thus we have $T' \in \mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$. □

In the case where $r = 1$, the assertion (1) of the following corollary was shown in [Br3, Theorem 3.4.3].

Corollary 4.20. *Let T be an object of $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ which is killed by p^n for some $n > 0$. Let T' be a G_∞-stable subquotient of T.*

(1) If $e(r-1) < p-1$, then T' is G_K-stable (with respect to T).

(2) If $s > n - 1 + \log_p(r - (p-1)/e)$, then T' is G_s-stable (with respect to T).

Proof. By the duality assertion of Lemma 4.19, it is enough to show the case where T' is a G_∞-stable submodule of T. Take any sequence $T' = T_0 \subset T_1 \subset \cdots \subset T_m = T$ of torsion G_∞-stable submodules of T such that T_i/T_{i-1} is irreducible for any i. As explained in the proof of Proposition 4.14, the G_∞-action on T_i/T_{i-1} can be (uniquely) extended to G_K. By Theorem 5.3 given in the next section, we know that T_i/T_{i-1} is an object of $\mathrm{Rep}_{\mathrm{tor}}^{r_0,\mathrm{cris}}(G_K)$ where $r_0 := \max\{r' \in \mathbb{Z}_{\geqslant 0}; e(r'-1) < p-1\}$.

(1) We may suppose $r = r_0$. The G_∞-equivariant projection $T = T_m \twoheadrightarrow T_m/T_{m-1}$ is G_K-equivariant by the full faithfulness theorem (= Theorem 1.2). Thus we know that T_{m-1} is G_K-stable in T, and also know that T_{m-1} is contained in $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ by Lemma 4.19. By the same argument for the G_∞-equivariant projection $T_{m-1} \twoheadrightarrow T_{m-1}/T_{m-2}$, we know that T_{m-2} is G_K-stable in T, and also know that T_{m-2} is contained in $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$. Repeating this argument, we have that $T' = T_0$ is G_K-stable in T.

(2) Put $J = u^p I^{[p^{s-n+1}]} W(R)$. By (1) we may assume $e(r-1) \geqslant p-1$. Under this assumption we have $r \geqslant r_0$ and $s > n - 1 + \log_p(r - (p-1)/e) \geqslant n-1$. In particular, $T|_{G_s}$ and $(T_i/T_{i-1})|_{G_s}$, for any i, are contained in $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$ by Proposition 4.16. First we consider the case where k is algebraically closed. By Theorem 4.9, the G_∞-equivariant projection $T = T_m \twoheadrightarrow T_m/T_{m-1}$ is G_s-stable. Thus we know that T_{m-1} is G_s-stable in T, and also know that T_{m-1} is contained in $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$ by Corollary 4.5. By the same argument for the G_∞-equivariant projection $T_{m-1} \twoheadrightarrow T_{m-1}/T_{m-2}$, we know that T_{m-2} is G_s-stable in T, and also know that T_{m-2} is contained in $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r,\hat{G}_s,J}(G_s)$. Repeating this argument, we have that $T' = T_0$ is G_s-stable

in T. Next we consider the case where k is not necessary algebraically closed. Let L be the completion of the maximal unramified extension K^{ur} of K, and we identify G_L with the inertia subgroup of G_K. Clearly $T|_{G_L}$ is contained in $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_L)$ and T' is G_{L_∞}-stable submodule of T. We have already shown that T' is $G_{L,s}$-stable in T. Since G_s is topologically generated by $G_{L,s}$ and G_∞, we conclude that T' is G_s-stable in T. $\square$

Now let V be a $\mathbb{Q}_p$-representation of G_K and T a $\mathbb{Z}_p$-lattice of V which is stable under G_∞. Then we know that T is automatically G_s-stable for some $s \geqslant 0$. Indeed we can check this as follows. Take any G_K-stable $\mathbb{Z}_p$-lattice T' of V which contains T, and take an integer $n > 0$ with the property that $p^nT' \subset T$. Furthermore, we take a finite extension K' of K such that $G_{K'}$ acts trivially on T'/p^nT'. Then T/p^nT' is G_∞-stable and also $G_{K'}$-stable in T'/p^nT'. If we take any integer $s \geqslant 0$ with the property $K' \cap K_\infty \subset K_{(s)}$, we know that T/p^nT' is G_s-stable. This implies that T is G_s-stable in T'.

The following corollary, which was shown in [Br3, Corollary 3.4.4] in the case where $r = 1$, is related with the above property.

COROLLARY 4.21. *Let V be a crystalline $\mathbb{Q}_p$-representation of G_K with Hodge–Tate weights in $[0, r]$ and T a $\mathbb{Z}_p$-submodule of V which is stable under G_∞. If $e(r-1) < p-1$, then T is stable under G_K.*

Proof. We follow the method of the proof of [Br3, Corollary 3.4.4]. First we suppose that T is finitely generated over $\mathbb{Z}_p$. Take any G_K-stable $\mathbb{Z}_p$-lattice T' of V which contains T. Since T'/p^nT' is contained in $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ for any $n > 0$, Corollary 4.20(1) implies that any G_∞-stable submodule of T'/p^nT' is in fact G_K-stable. Thus $(T + p^nT')/p^nT'$ is G_K-stable in T'/p^nT'. Therefore, we obtain $g(T) \subset \bigcap_{n>0} (T + p^nT') = T$ for any $g \in G_K$. Next we consider general case; so T is not necessary finitely generated over $\mathbb{Z}_p$. We may suppose $T \neq 0$. Denote by T_x the smallest $\mathbb{Z}_p$-submodule of T which contains x and is stable under G_∞. Since T_x is contained in some (G_K-stable) $\mathbb{Z}_p$-lattice of V, we see that T_x is finitely generated over $\mathbb{Z}_p$, and hence it is stable under G_K. Then the relation $T = \bigcup_{x\in T} T_x$ gives the desired result. $\square$

§5. Crystalline lifts and c-weights

We continue to use the same notation except for that we may allow $p = 2$. We remark that a torsion $\mathbb{Z}_p$-representation of G_K is torsion crystalline with Hodge–Tate weights in $[0, r]$ if there exists a lattice L in some crystalline

$\mathbb{Q}_p$-representation of G_K with Hodge–Tate weights in $[0,r]$ and a G_K-equivariant surjection $f: L \twoheadrightarrow T$. We call f a *crystalline lift* (*of* T) *of weight* $\leqslant r$. Our interest in this section is to determine the minimum integer r (if it exists) such that T admits crystalline lifts of weight $\leqslant r$. We call this minimum integer the *c-weight of* T and denote it by $w_c(T)$. If T does not have crystalline lifts of weight $\leqslant r$ for any integer r, then we define the c-weight $w_c(T)$ of T to be ∞. For the existence of crystalline lifts of various torsion representations, for example, it is useful for the readers to refer the Muller's PhD Thesis [Mu]. Motivated by [CL2, Question 5.5], we pose the following question.

QUESTION 5.1. For a torsion $\mathbb{Z}_p$-representation T of G_K, is the c-weight $w_c(T)$ of T finite? Furthermore, can we calculate $w_c(T)$?

This question strongly related to the weight part of Serre's conjecture. It is dated to Serre, when raised Serre's conjecture over $\mathbb{Q}$, he had already considered the question to lift a 2-dimensional mod p representation of $G_{\mathbb{Q}_p}$ to a 2-dimensional crystalline representation with "optimal" weights (which is very close to minimum weights considered here). He obtained some partial results that contained in Proposition 5.6 and Corollary 5.7. We do not go into details here but the recent developments of the weight part of Serre's conjecture (e.g., [GLS]) also contribute (explicitly or implicitly) partial results in this section.

5.1 General properties of c-weights

We study general properties of c-weights. At first, by ramification estimates, it is known that c-weights may have infinitely large values [CL2, Theorem 5.4]; for any $c > 0$, there exists a torsion $\mathbb{Z}_p$-extension T of G_K with $w_c(T) > c$. In this paper, we mainly consider representations with "small" c-weights. If c-weights are "small", they are closely related with *tame inertia weights*. Now we recall the definition of tame inertia weights. Let I_K be the inertia subgroup of G_K. Let T be a d-dimensional irreducible $\mathbb{F}_p$-representation of I_K. Then T is isomorphic to

$$\mathbb{F}_{p^d}(\theta_{d,1}^{n_1} \cdots \theta_{d,d}^{n_d})$$

for one sequence of integers between 0 and $p-1$, periodic of period d. Here, $\theta_{d,1}, \ldots, \theta_{d,d}$ are the fundamental characters of level d. The integers $n_1/e, \ldots, n_d/e$ are called the tame inertia weights of T. For any $\mathbb{F}_p$-representation T of G_K, the tame inertia weights of T are the tame inertia weights of the Jordan–Hölder quotients of $T|_{I_K}$.

Let $\chi_p : G_K \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character and $\bar{\chi}_p : G_K \to \mathbb{F}_p^\times$ the mod p cyclotomic character. It is well known that $\bar{\chi}_p|_{I_K} = \theta_1^e$ where $\theta_1 : I_K \twoheadrightarrow \mathbb{F}_p^\times$ is the fundamental character of level 1. In particular, denoting by K^{ur} the maximal unramified extension of K, we have $[K^{\mathrm{ur}}(\mu_p) : K^{\mathrm{ur}}] = (p-1)/\gcd(e, p-1)$.

PROPOSITION 5.2. (1) *Minimum c-weights are invariant under finite unramified extensions of the base field K.*

(2) *The c-weight of an unramified torsion $\mathbb{Z}_p$-representation of G_K is* 0.

(3) *Put $\nu = (p-1)/\gcd(e, p-1)$. Let s be an integer such that $\nu(s-1) < w_c(T) \leqslant \nu s$. Then we have $\nu(s-1) < w_c(T^\vee) \leqslant \nu s$. In particular, if $(p-1) \mid e$, then we have $w_c(T) = w_c(T^\vee)$.*

(4) *Let T be an $\mathbb{F}_p$-representation of G_K and i the largest tame inertia weight of T. Then we have $w_c(T) \geqslant i$.*

Proof. (1) Let T be a torsion $\mathbb{Z}_p$-representation of G_K. Let K' be a finite unramified extension of K. It suffices to prove that T has crystalline lifts of weight $\leqslant r$ if and only if $T|_{G_{K'}}$ has crystalline lifts of weight $\leqslant r$. The "only if" assertion is clear and thus it is enough to prove the "if" assertion. Let $f : L \twoheadrightarrow T|_{G_{K'}}$ be a crystalline lift of $T|_{G_{K'}}$ of weight $\leqslant r$. Since K'/K is unramified, $\mathrm{Ind}_{G_{K'}}^{G_K} L$ is a lattice in some crystalline $\mathbb{Q}_p$-representation of G_K with Hodge–Tate weights in $[0, r]$. Furthermore, the map

$$\mathrm{Ind}_{G_{K'}}^{G_K} L = \mathbb{Z}_p[G_K] \otimes_{\mathbb{Z}_p[G_{K'}]} L \to T, \quad \sigma \otimes x \mapsto \sigma(f(x))$$

is a G_K-equivariant surjection and hence we have done.

(2) The result follows from (1) immediately.

(3) Taking a finite unramified extension K' of K with the property $[K^{\mathrm{ur}}(\mu_p) : K^{\mathrm{ur}}] = [K'(\mu_p) : K']$, it follows from Lemma 4.19 that we have $\nu(s-1) < w_c(T|_{G'_K}) \leqslant \nu s$ if and only if we have $\nu(s-1) < w_c((T^\vee)|_{G'_K}) \leqslant \nu s$. Thus the result follows from the assertion (1).

(4) If $ew_c(T) \geqslant p-1$, then there is nothing to prove, and thus we may suppose that $ew_c(T) < p-1$. Let $L \twoheadrightarrow T$ be a crystalline lift of T of weight $\leqslant w_c(T)$. Since the tame inertia polygon of L lies on the Hodge polygon of L [CS, Théorème 1], the largest slope of the former polygon is less than or equal to that of the latter polygon. This implies $w_c(T) \geqslant i$. ∎

THEOREM 5.3. *Let T be a tamely ramified $\mathbb{F}_p$-representation of G_K. Let i be the largest tame inertia weight of T. Then we have $w_c(T) = \min\{h \in \mathbb{Z}_{\geqslant 0}; h \geqslant i\}$.*

Proof. The proof below is essentially due to Caruso and Liu [CL2, Theorem 5.7], but we give a proof here for the sake of completeness. Put $i_0 = \min\{h \in \mathbb{Z}_{\geqslant 0}; h \geqslant i\}$. By Proposition 5.2(4), we have $w_c(T) \geqslant i_0$. Thus it suffices to show $w_c(T) \leqslant i_0$. We note that $T|_{I_K}$ is semisimple. Any irreducible component T_0 of $T|_{I_K}$ is of the form $\mathbb{F}_{p^d}(\theta_{d,1}^{n_1} \cdots \theta_{d,d}^{n_d})$ for one sequence of integers between 0 and $p-1$, periodic of period d. We decompose $n_j = em_j + n'_j$ by integers $0 \leqslant m_j \leqslant i_0$ and $0 \leqslant n'_j < e$. Now we define an integer $k_{j,\ell}$ by

$$k_{j,\ell} := \begin{cases} e & \text{if } 1 \leqslant \ell \leqslant m_j, \\ n'_j & \text{if } \ell = m_j + 1, \\ 0 & \text{if } \ell > m_j + 1. \end{cases}$$

Note that we have $n_j = \sum_{\ell=1}^{i_0} k_{j,\ell}$, and also have an I_K-equivariant surjection

$$T_0 \simeq \bigotimes_{\ell=1,\ldots,i_0,\mathbb{F}_{p^d}} \mathbb{F}_{p^d}(\theta_{d,1}^{k_{1,\ell}} \cdots \theta_{d,d}^{k_{d,\ell}}) \twoheadleftarrow \bigotimes_{\ell=1,\ldots,i_0,\mathbb{F}_p} \mathbb{F}_{p^d}(\theta_{d,1}^{k_{1,\ell}} \cdots \theta_{d,d}^{k_{d,\ell}}).$$

By a classical result of Raynaud, each $\mathbb{F}_{p^d}(\theta_{d,1}^{k_{1,\ell}} \cdots \theta_{d,d}^{k_{d,\ell}})$ comes from a finite flat group scheme defined over K^{ur}. We should remark that such a finite flat group scheme is in fact defined over a finite unramified extension of K. Since any finite flat group scheme can be embedded in a p-divisible group, the above observation implies the following: there exist a finite unramified extension K' over K, a lattice L in some crystalline $\mathbb{Q}_p$-representation of $G_{K'}$ with Hodge–Tate weights in $[0, i_0]$ and an I_K-equivariant surjection $f : L \twoheadrightarrow T$. The map f induces an I_K-equivariant surjection $\tilde{f} : L/pL \twoheadrightarrow T$. Since L/pL and T is finite, we see that $\tilde{f}$ is in fact $G_{K''}$-equivariant for some finite unramified extension K'' over K', and then so is f. Therefore, we obtain $w_c(T|_{G_{K''}}) \leqslant i_0$. By Proposition 5.2(1), we obtain $w_c(T) \leqslant i_0$. $\square$

5.2 Rank 2 cases

We give some computations of c-weights related with torsion representations of rank 2. We prove the following lemma by an almost identical method with [GLS, Lemma 9.4].

LEMMA 5.4. *Let K be a finite extension of $\mathbb{Q}_p$. Let E be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$. Let i and ν be integers such that ν is divisible by $[K(\mu_p) : K]$. Suppose that T is an $\mathbb{F}$-representation of G_K which* *sits in an exact sequence* $(*) : 0 \to \mathbb{F}(i) \to T \to \mathbb{F} \to 0$ *of* $\mathbb{F}$*-representations of* G_K*. Then there exist a ramified degree at most* 2 *extension* E' *over* E*, with integer ring* $\mathcal{O}_{E'}$*, and an unramified continuous character* $\chi : G_K \to \mathbb{F}^\times$ *with trivial reduction such that* $(*)$ *is the reduction of some exact sequence* $0 \to \mathcal{O}_{E'}(\chi\chi_p^{i+\nu}) \to \Lambda \to \mathcal{O}_{E'} \to 0$ *of free* $\mathcal{O}_{E'}$*-representations of* G_K*. Furthermore, we have the followings:*

(1) *If* $i+\nu = 1$ *or* $\bar{\chi}_p^{1-i} \neq 1$*, then we can take* $E' = E$ *and* $\chi = 1$*.*

(2) *If* $i+\nu = 0$ *and* T *is unramified, then we can take* $E' = E$*,* $\chi = 1$ *and* Λ *to be unramified.*

Proof. Suppose $i+\nu = 1$ (resp. $\bar{\chi}_p^{1-i} \neq 1$). Then the map $H^1(K, \mathcal{O}_E(i+\nu)) \to H^1(K, \mathbb{F}(i))$ arising from the exact sequence $0 \to \mathcal{O}_E(i+\nu) \stackrel{\varpi}{\to} \mathcal{O}_E(i+\nu) \to \mathbb{F}(i) \to 0$ is surjective since $H^2(K, \mathcal{O}_E(1)) \simeq \mathcal{O}_E$ (resp. $H^2(K, \mathcal{O}_E(i+\nu)) = 0$), where ϖ is a uniformizer of E. Hence we obtained a proof of (1). The assertion (2) follows immediately from the fact that the natural map $H^1(G_K/I_K, \mathcal{O}_E) \to H^1(G_K/I_K, \mathbb{F})$ is surjective.

In the rest of this proof, we always assume that $i+\nu \neq 1$ and $\bar{\chi}_p^{1-i} = 1$. Let $L \in H^1(K, \mathbb{F}(i))$ be a 1-cocycle corresponding to $(*)$. We may suppose $L \neq 0$. For any unramified continuous character $\chi : G_K \to \mathbb{F}^\times$ with trivial reduction, we denote by

$$\delta^1_\chi : H^1(K, \mathbb{F}(i)) \to H^2(K, \mathcal{O}_E(\chi\chi_p^{i+\nu}))$$
$$(\text{resp. } \delta^0_\chi : H^0(K, E/\mathcal{O}_E(\chi^{-1}\chi_p^{1-i-\nu})) \to H^1(K, \mathbb{F}))$$

the connection map arising from the exact sequence

$$0 \to \mathcal{O}_E(\chi\chi_p^{i+\nu}) \stackrel{\varpi}{\to} \mathcal{O}_E(\chi\chi_p^{i+\nu}) \to \mathbb{F}(i) \to 0$$
$$(\text{resp. } 0 \to \mathbb{F} \to E/\mathcal{O}_E(\chi^{-1}\chi_p^{1-i-\nu}) \stackrel{\varpi}{\to} E/\mathcal{O}_E(\chi^{-1}\chi_p^{1-i-\nu}) \to 0)$$

of $\mathcal{O}_E[G_K]$-modules. Consider the following commutative diagram:

$$\begin{tikzcd}
H^1(K, \mathbb{F}(i)) \arrow[d, "\delta^1_\chi"'] & \times & H^1(K, \mathbb{F}) \arrow[r] & E/\mathcal{O}_E \arrow[d, equal] \\
H^2(K, \mathcal{O}_E(\chi\chi_p^{i+\nu})) & \times & H^0(K, E/\mathcal{O}_E(\chi^{-1}\chi_p^{1-i-\nu})) \arrow[u, "\delta^0_\chi"] \arrow[r] & E/\mathcal{O}_E
\end{tikzcd}$$

Since we know that the above two pairings are perfect, we see that L lifts to $H^1(G_K, \mathcal{O}_E(\chi\chi_p^{i+\nu}))$ if and only if H is contained in the image of

δ^0_χ. Here, $H \subset H^1(K,\mathbb{F})$ is the annihilator of L under the local Tate pairing $H^1(K,\mathbb{F}(i)) \times H^1(K,\mathbb{F}) \to E/\mathcal{O}_E$. Let $n \geqslant 1$ be the largest integer with the property that $\chi^{-1}\chi_p^{1-i-\nu} \equiv 1 \bmod \varpi^n$ (such n exists since $\bar{\chi}_p^{1-i} = 1$ and $1 - i - \nu \neq 0$). We define $\alpha_\chi : G_K \to \mathcal{O}_E$ by the relation $\chi^{-1}\chi_p^{1-i-\nu} = 1 + \varpi^n \alpha_\chi$, and denote $(\alpha_\chi \bmod \varpi) : G_K \to \mathbb{F}$ by $\bar{\alpha}_\chi$. By definition, $\bar{\alpha}_\chi$ is a nonzero element of $H^1(K,\mathbb{F})$, and it is not difficult to check that the image of δ^0_χ is generated by $\bar{\alpha}_\chi$. If $\bar{\alpha}_\chi$ is contained in H for some χ, we are done. Suppose this is not the case.

Suppose that H is not contained in the unramified line in $H^1(K,\mathbb{F})$. We claim that we can choose χ such that $\bar{\alpha}_\chi$ is ramified. Let m be the largest integer with the property that $(\chi^{-1}\chi_p^{1-i-\nu})|_{I_K} \equiv 1 \bmod \varpi^n$. Clearly, we have $m \geqslant n$. If $m = n$, then we are done and thus we may assume $m > n$. Fix a lift $g \in G_K$ of the Frobenius of K. We see that $\bar{\alpha}_\chi(g) \neq 0$. Let χ' be the unramified character sending g to $1 + \varpi^n \alpha_\chi(g)$. Then χ' has trivial reduction. After replacing χ with $\chi\chi'$, we reduce the case where $m = n$ and thus the claim follows. Suppose $\bar{\alpha}_\chi$ is ramified. Then there exists a unique $\bar{x} \in \mathbb{F}^\times$ such that $\bar{\alpha}_\chi + u_{\bar{x}} \in H$ where $u_{\bar{x}} : G_K \to \mathbb{F}$ is the unramified character sending g to $\bar{x}$. Denote by χ'' the unramified character sending g to $1 + \varpi^n \alpha_\chi(g)$. Replacing χ with $\chi\chi''$, we have done.

Suppose that H is contained in the unramified line in $H^1(K,\mathbb{F})$ (thus H and the unramified line coincide with each other). By replacing E with $E(\sqrt{\varpi})$, we may assume that $n > 1$. Let χ_0 be a character defined by χ times the unramified character sending our fixed g to $1 + \varpi$. Since $n > 1$, we see that $\chi_0^{-1}\chi_p^{1-i-\nu} \equiv 1 \bmod \varpi$ and $\chi_0^{-1}\chi_p^{1-i-\nu} \not\equiv 1 \bmod \varpi^2$. We define $\alpha_{\chi_0} : G_K \to \mathcal{O}_E$ by the relation $\chi_0^{-1}\chi_p^{1-i-\nu} = 1 + \varpi\alpha_{\chi_0}$, and denote $(\alpha_{\chi_0} \bmod \varpi) : G_K \to \mathbb{F}$ by $\bar{\alpha}_{\chi_0}$. By definition and the assumption $n > 1$, $\bar{\alpha}_{\chi_0}$ is a nonzero unramified element of $H^1(K,\mathbb{F})$, hence it is contained in H. Therefore, we have done. ∎

LEMMA 5.5. *Let K be a finite extension of $\mathbb{Q}_p$, $n \geqslant 2$ an integer and $\chi : G_K \to E^\times$ an unramified character. Then any E-representation of G_K which is an extension of E by $E(\chi\chi_p^n)$ is crystalline.*

Proof. This is well known; for example, see the argument of [BK, Section 3]. ∎

PROPOSITION 5.6. *Suppose $p > 2$. Let K be a finite unramified extension of $\mathbb{Q}_p$. Let $T \in \mathrm{Rep}_{\mathrm{tor}}(G_K)$ be killed by p and sit in an exact sequence $0 \to \mathbb{F}_p(i) \to T \to \mathbb{F}_p \to 0$ of $\mathbb{F}_p$-representations of G_K. Then we have the followings:*

(1) *If* $i=0$ *and* T *is unramified, then we have* $w_c(T)=0$.
(2) *If* $i=0$ *and* T *is not unramified, then we have* $w_c(T)=p-1$.
(3) *If* $i=2,\ldots,p-2$, *then we have* $w_c(T)=i$.

Proof. (1) By Lemma 5.4(2), we know that T has unramified (and thus crystalline) lift, which implies $w_c(T)=0$.

(2) By Lemmas 5.4 and 5.5, it suffices to prove that T is not torsion crystalline with Hodge–Tate weights in $[0,p-2]$ if T is not unramified. Let K_T be the definition field of the representation T of G_K and put $G=\mathrm{Gal}(K_T/K)$. Let G^j be the upper numbering jth ramification subgroup of G (in the sense of [Se]). Since T is not unramified and killed by p, we see that K_T is a totally ramified degree p extension over K. Thus G^1 is the wild inertia subgroup of G and $G^1=G$, which does not act on T trivial by the definition of G. Thus we obtain the desired result by ramification estimates of [Fo1] (or [Ab1]) for torsion crystalline representations with Hodge–Tate weights in $[0,p-2]$: if T is torsion crystalline with Hodge–Tate weights in $[0,p-2]$, then G^j acts on T trivial for any $j>(p-2)/(p-1)$.

(3) The result follows immediately from Proposition 5.2(4), Lemmas 5.4 and 5.5. ∎

COROLLARY 5.7. *Let* K *be a finite unramified extension of* $\mathbb{Q}_p$. *Then any 2-dimensional* $\mathbb{F}_p$*-representation of* G_K *is torsion crystalline with Hodge–Tate weights in* $[0,2p-2]$.

Proof. If T is irreducible, the result follows from Theorem 5.3. Assume that T is reducible. Since K is unramified over $\mathbb{Q}_p$, any continuous character $G_K\to\mathbb{F}_p^{\times}$ is of the form $\chi\bar{\chi}_p^i$ for some unramified character χ and some integer i. Replacing K with its finite unramified extension, we may assume that T sits in an exact sequence $0\to\mathbb{F}_p(i)\to T\to\mathbb{F}_p(j)\to 0$ of $\mathbb{F}_p$-representations of G_K, where i and j are integers in the range $[0,p-2]$ (we remark that $w_c(T)$ is invariant under unramified extensions of K by Proposition 5.2(1)). It follows from Lemmas 5.4 and 5.5 that $w_c(T(-j))\leqslant p$. Therefore, we obtain $w_c(T)=w_c(T(-j)\otimes_{\mathbb{F}_p}\mathbb{F}_p(j))\leqslant w_c(T(-j))+w_c(\mathbb{F}_p(j))\leqslant p+(p-2)=2p-2$. ∎

REMARK 5.8. The author does not know whether $2p-2$ in the statement of Corollary 5.7 is optimal or not.

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

5.3 Extensions of $\mathbb{F}_p$ by $\mathbb{F}_p(1)$ and nonfullness theorems

By Lemma 5.4, we know that the c-weight $w_c(T)$ of an $\mathbb{F}_p$-representation T of G_K which sits in an exact sequence $0 \to \mathbb{F}_p(1) \to T \to \mathbb{F}_p \to 0$ of $\mathbb{F}_p$-representations of G_K, is less than or equal to p. Let us calculate $w_c(T)$ for such T more precisely. We should remark that such T is written as p-torsion points of a Tate curve. Hence we consider torsion representations coming from Tate curves.

Let v_K be the valuation of K normalized such that $v_K(K^\times) = \mathbb{Z}$, and take any $q \in K^\times$ with $v_K(q) > 0$. Let E_q be the Tate curve over K associated with q and $E_q[p^n]$ the module of p^n-torsion points of E_q for any integer $n > 0$. It is well known that there exists an exact sequence

$$(\#) \quad 0 \to \mu_{p^n} \to E_q[p^n] \to \mathbb{Z}/p^n\mathbb{Z} \to 0$$

of $\mathbb{Z}_p[G_K]$-modules. Here, μ_{p^n} is the group of p^nth roots of unity in $\overline{K}$. Let $x_n : G_K \to \mu_{p^n}$ be the 1-cocycle defined to be the image of 1 for the connection map $H^0(K, \mathbb{Z}/p^n\mathbb{Z}) \to H^1(K, \mu_{p^n})$ arising from the exact sequence (#). Then x_n corresponds to q mod $(K^\times)^{p^n}$ via the isomorphism $K^\times/(K^\times)^{p^n} \simeq H^1(K, \mu_{p^n})$ of Kummer theory. Thus the exact sequence (#) splits if and only if $q \in (K^\times)^{p^n}$.

First we consider the case $p \mid v_K(q)$ (i.e., *peu ramifié* case).

Lemma 5.9. *Let K be a finite extension of $\mathbb{Q}_p$. If $p \mid v_K(q)$, then $E_q[p]$ is the reduction modulo p of a lattice in some 2-dimensional crystalline $\mathbb{Q}_p$-representation with Hodge–Tate weights in $[0, 1]$.*

Proof. Since $p \mid v_K(q)$, there exists $q' \in K^\times$ such that $v_K(q' - 1) > 0$ and $q \equiv q'$ mod $(K^\times)^p$. Consider the exact sequence $0 \to \mathbb{Z}_p(1) \to L \to \mathbb{Z}_p \to 0$ of $\mathbb{Z}_p$-representations of G_K corresponding to q' via the isomorphism $H^1(K, \mathbb{Z}_p(1)) \simeq \varprojlim_n K^\times/(K^\times)^{p^n}$ of Kummer theory. By the condition $q \equiv q'$ mod $(K^\times)^p$, the reduction modulo p of L is $E_q[p]$. Thus it suffices to show that $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L$ is crystalline. Take a system $(q'_n)_{n \geqslant 0}$ of p-power roots of q' in $\mathcal{O}_{\overline{K}}$ such that $q'_0 = q'$ and $(q'_{n+1})^p = q'_n$ for any $n \geqslant 0$. We also take a system $(\varepsilon'_n)_{n \geqslant 0}$ of p-power roots of unity in $\mathcal{O}_{\overline{K}}$ such that $\varepsilon'_0 = 1$, $\varepsilon'_1 \neq 1$ and $(\varepsilon'_{n+1})^p = \varepsilon'_n$ for any $n \geqslant 0$. We define a map $c : G_K \to \mathbb{Z}_p$ by $g(q_n) = (\varepsilon'_n)^{c(g)} q_n$ for any $n \geqslant 0$. Then we can choose a basis $\mathbf{e}, \mathbf{f}$ of V such that $g(\mathbf{e}) = \chi_p(g)\mathbf{e}$ and $g(\mathbf{f}) = c(g)\mathbf{e} + \mathbf{f}$ for any $g \in G_K$. Put $\underline{q}' = (q'_n \bmod p)_{n \geqslant 0} \in R$, $\underline{\varepsilon}' = (\varepsilon'_n \bmod p)_{n \geqslant 0} \in R$ and $t' = -\log[\underline{\varepsilon}'] \in A_{\mathrm{cris}}$. By the condition $v_K(q' - 1) > 0$, we see $([\underline{q}'] - 1)^e \in \mathrm{Fil}^1 W(R) + pW(R)$ and thus

$\log[\underline{q}']$ converges in B^+_{cris}. With these notations, we see that the $W(k)[1/p]$-vector space $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is of dimension 2 with basis $e_1 := t^{-1}\mathbf{e}$ and $e_2 := \log[\underline{q}'] \cdot t^{-1}\mathbf{e} + \mathbf{f}$. Therefore, V is crystalline. □

COROLLARY 5.10. *Suppose that K is a finite extension of $\mathbb{Q}_p$, $(p-1) \nmid e$ and $p \mid v_K(q)$. Then we have $w_c(E_q[p]) = 1$.*

Proof. By the assumption $(p-1) \nmid e$, we know that the largest tame inertia weight of $E_q[p]$ is positive. Thus Proposition 5.2(4) shows $w_c(E_q[p]) \geqslant 1$. The inequality $w_c(E_q[p]) \leqslant 1$ follows from Lemma 5.9. □

Next we consider the case $p \nmid v_K(q)$ (i.e., *très ramifié* case).

PROPOSITION 5.11. *If $e(r-1) < p-1$ and $p \nmid v_K(q)$, then $E_q[p^n]$ is not torsion crystalline with Hodge–Tate weights in $[0, r]$ for any $n > 0$.*

REMARK 5.12. If $e = 1$, the fact that $E_\pi[p^n]$ is not torsion crystalline with Hodge–Tate weights in $[0, p-1]$ immediately follows from the theory of ramification bound as below. We may suppose $n = 1$. Suppose $E_\pi[p]$ is torsion crystalline with Hodge–Tate weights in $[0, p-1]$. Then the upper numbering jth ramification subgroup G^j_K of G_K (in the sense of [Se]) acts trivially on $E_\pi[p]$ for any $j > 1$ [Ab1, Section 6, Theorem 3.1]. However, this contradicts the fact that the upper bound of the ramification of $E_\pi[p]$ is $1 + 1/(p-1)$.

Proof of Proposition 5.11. We may suppose $n = 1$. We choose any uniformizer π' of K. Putting $v_K(q) = m$, we can write $q = (\pi')^m x$ with some unit x of the integer ring of K. Since m is prime to p, we have a decomposition $x = \zeta_\ell y^m$ in $K^\times$ for some $\ell > 0$ prime to p and $y \in K$ with $v_K(y-1) > 0$. Here ζ_ℓ is a (not necessary primitive) ℓth root of unity. Since ℓ is prime to p, we have $\zeta_\ell = \zeta_\ell^{ps}$ for some integer s. We put $\pi = \pi' y$. This is a uniformizer of K. Choose any pth root π_1 of π and put $q_1 = \zeta_\ell^s \pi_1^m \in K(\pi_1)^\times$. Then we have $q = q_1^p \in (K(\pi_1)^\times)^p$ and in particular, the exact sequence (#) (for $n = 1$) splits as representations of $\mathrm{Gal}(\overline{K}/K(\pi_1))$. Now assume that $E_q[p]$ is torsion crystalline with Hodge–Tate weights in $[0, r]$. Then (#) (for $n = 1$) splits as representations of G_K by Theorem 1.2. This contradicts the assumption $p \nmid v_K(q)$ (and hence $q \notin (K^\times)^p$). □

Now we put $r_0' = \min\{r \in \mathbb{Z}_{\geqslant 0}; e(r-1) \geqslant p-1\}$. Recall that we have $[K^{\mathrm{ur}}(\mu_p) : K^{\mathrm{ur}}] = (p-1)/\gcd(e, p-1)$.

LEMMA 5.13. *Let K be a finite extension of $\mathbb{Q}_p$. Then $E_q[p]$ is torsion crystalline with Hodge–Tate weights in $[0, 1 + (p-1)/\gcd(e, p-1)]$.*

Proof. Taking a finite unramified extension K' of K such that $[K^{\mathrm{ur}}(\mu_p):K^{\mathrm{ur}}]=[K'(\mu_p):K']$, we obtain $w_c((E_q[p])|_{G_{K'}})\leqslant 1+(p-1)/\gcd(e,p-1)$ by Lemma 5.4. Thus we have $w_c(E_q[p])\leqslant 1+(p-1)/\gcd(e,p-1)$ by Proposition 5.2(1). ∎

COROLLARY 5.14. *Suppose that K is a finite extension of $\mathbb{Q}_p$, and also suppose $e\mid(p-1)$ or $(p-1)\mid e$. We further suppose that $p\nmid v_K(q)$. Then we have $w_c(E_q[p])=r'_0$.*

Proof. We have $w_c(E_q[p])\leqslant r'_0$ by Lemma 5.13. In addition, we also have $w_c(E_q[p])\geqslant r'_0$ by Proposition 5.11. ∎

Lemma 5.13 gives some nonfullness results on torsion crystalline representations.

COROLLARY 5.15. *Suppose that K is a finite extension of $\mathbb{Q}_p$. If $r\geqslant 1+(p-1)/\gcd(e,p-1)$, then the restriction functor $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathrm{tor}}(G_K)\to\mathrm{Rep}_{\mathrm{tor}}(G_1)$ is not full.*

Proof. We consider two representations $E_\pi[p]$ and $\mathbb{F}_p(1)\oplus\mathbb{F}_p$, which are objects of $\mathrm{Rep}^{r}_{\mathrm{tor}}(G_K)$ by Lemma 5.13. They are not isomorphic as representations of G_K but isomorphic as representations of G_1. Thus the desired nonfullness follows. ∎

COROLLARY 5.16. *Suppose that any one of the following holds:*

- *$p=2$ and K is a finite extension of $\mathbb{Q}_2$ (in this case $r'_0=2$);*
- *K is a finite unramified extension of $\mathbb{Q}_p$ (in this case $r'_0=p$);*
- *K is a finite extension of $\mathbb{Q}_p(\mu_p)$ (in this case $r'_0=2$).*

Then the restriction functor $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathrm{tor}}(G_K)\to\mathrm{Rep}_{\mathrm{tor}}(G_1)$ is not full.

Acknowledgments. The author would like to thank Shin Hattori, Naoki Imai and Yuichiro Taguchi who gave him many valuable advice. The author is grateful to the anonymous referee for his/her comments. This work was supported by JSPS KAKENHI Grant Number 25·173.

REFERENCES

[Ab1] V. Abrashkin, *Modular representations of the Galois group of a local field and a generalization of a conjecture of Shafarevich*, Izv. Akad. Nauk SSSR Ser. Mat. **53**(6) (1989), 1135–1182 (Russian); Math. USSR-Izv. **35**(3) 469–518 (English).

[Ab2] V. Abrashkin, *Group schemes of period $p>2$*, Proc. Lond. Math. Soc. (3) **101** (2010), 207–259.

[BK] S. Bloch and K. Kato, "*L-functions and Tamagawa numbers of motives*", in *The Grothendieck Festschrift, vol. I*, Progr. Math. **86**, Birkhäuser Boston, Boston, MA, 1990, 333–400.

[Br1] C. Breuil, *Représentations p-adiques semi-stables et transversalité de Griffiths*, Math. Ann. **307** (1997), 191–224.

[Br2] C. Breuil, *Une application du corps des normes*, Compos. Math. **117** (1999), 189–203.

[Br3] C. Breuil, "*Integral p-adic Hodge theory*", in *Algebraic Geometry 2000, Azumino (Hotaka)*, Adv. Stud. Pure Math. **36**, Math. Soc. Japan, 2002, 51–80.

[Ca] X. Caruso, *Représentations semi-stables de torsion dans le case $er < p - 1$*, J. Reine Angew. Math. **594** (2006), 35–92.

[CL1] X. Caruso and T. Liu, *Quasi-semi-stable representations*, Bull. Soc. Math. France **137**(2) (2009), 185–223.

[CL2] X. Caruso and T. Liu, *Some bounds for ramification of p^n-torsion semi-stable representations*, J. Algebra **325** (2011), 70–96.

[CS] X. Caruso and D. Savitt, *Polygons de Hodge, de Newton et de línertie moderee des representations semi-stables*, Math. Ann. **343** (2009), 773–789.

[Fo1] J.-M. Fontaine, "*Schémas propres et lisses sur $\mathbb{Z}$*", in *Proceedings of the Indo-French Conference on Geometry Bombay, 1989*, Hindustan Book Agency, 1993, 43–56.

[Fo2] J.-M. Fontaine, *Le corps des périodes p-adiques*, Astérisque (223) (1994), 59–111; With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).

[GLS] T. Gee, T. Liu and D. Savitt, *The Buzzard–Diamond–Jarvis conjecture for unitary groups*, J. Amer. Math. Soc. **27** (2014), 389–435.

[Kim] W. Kim, *The classification of p-divisible groups over 2-adic discrete valuation rings*, Math. Res. Lett. **19**(1) (2012), 121–141.

[Kis] M. Kisin, "*Crystalline representations and F-crystals*", in *Algebraic Geometry and Number Theory*, Progr. Math. **253**, Birkhäuser Boston, Boston, MA, 2006, 459–496.

[La] E. Lau, *Relation between Dieudonné displays and crystalline Dieudonné theory*, Algebra Number Theory **8** (2014), 2201–2262.

[Li1] T. Liu, *Torsion p-adic Galois representations and a conjecture of Fontaine*, Ann. Sci. Éc. Norm. Supér. (4) **40**(4) (2007), 633–674.

[Li2] T. Liu, *A note on lattices in semi-stable representations*, Math. Ann. **346** (2010), 117–138.

[Li3] T. Liu, *Lattices in filtered (φ, N)-modules*, J. Inst. Math. Jussieu 2 **11**(03) (2012), 659–693.

[Li4] T. Liu, *The correspondence between Barsotti–Tate groups and Kisin modules when* $p = 2$, J. Théor. Nombres Bordeaux **25**(3) (2013), 661–676.

[Mu] A. Muller, *Relèvemens cristallins de représentations galoisiennes*, Ph.D. thesis, Strasbourg University (2013).

[Oz1] Y. Ozeki, *Torsion representations arising from $(\varphi, \hat{G})$-modules*, J. Number Theory **133** (2013), 3810–3861.

[Oz2] Y. Ozeki, *Full faithfulness theorem for torsion crystalline representations*, New York J. Math. **20** (2014), 1043–1061.

[Qu] D. Quillen, "*Higher algebraic K-theory: I*", in *Algebraic K-Theory, I: Higher K-Theories (Seattle, 1972)*, Lecture Notes in Math., **341**, Springer-Verlag, New York, 1973, 85–147.

[Se] J.-P. Serre, *Corps Locaux*, Hermann, Paris, 1968.

https://doi.org/10.1017/nmj.2016.68 Published online by Cambridge University Press

Kanagawa University
Kanagawa 259-1293
Japan
ft101992yp@kanagawa-u.ac.jp