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INDECOMPOSABLE VECTOR BUNDLES ON THE PROJECTIVE LINE

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1. Introduction. Let A be a commutative ring, and let $X = P_A{}^1 = \text{Proj } A[t_0, t_1]$. By a vector bundle on X we mean a locally free sheaf of finite rank on X. Set $t = t_1/t_0$. Then X is made up of two affine pieces $U_1 = \text{Spec } A[t]$, and $U_2 = \text{Spec } A[t^{-1}]$. Let $\mathscr{P}(R)$ denote the category of finitely generated projective modules over the ring R. The category of vector bundles on X is equivalent to the category of triples (P_1, f, P_2), where $P_1 \in \mathscr{P}(A[t])$, $P_2 \in \mathscr{P}(A[t^{-1}])$, and

$$f : P_1 \otimes_{A[t]} A[t, t^{-1}] \rightarrow P_2 \otimes_{A[t^{-1}]} A[t, t^{-1}]$$

is an $A[t, t^{-1}]$-isomorphism. In [**2**], the category of vector bundles on $P_A{}^1$ is defined directly in this manner, without first defining $P_A{}^1$ (so that one could work over a non-commutative ring). We prove that if A is a Krull ring (or a Noetherian ring with connected spec) of dimension > 0, then there is an indecomposable vector bundle of rank n on X, for every positive integer n. In [**3**], the question is raised as to whether or not every vector bundle $\mathscr{V}$ on $P_A{}^1$ is (stably) the direct sum of vector bundles of the form $P \otimes_A \mathscr{O}(n)$, where $P \in \mathscr{P}(A)$, and $\mathscr{O}(n)$ is a canonical line bundle on $P_A{}^1$ (defined below). "Stably", here, means that there exists an integer r such that $\mathscr{V} \oplus r\mathscr{O}$ is the direct sum of vector bundles of the form $P \otimes_A \mathscr{O}(n)$. Our vector bundles are easily seen not to be of this form.

The structure of vector bundles on $P_A{}^1$ is known if A is a field, and my proofs make use of these results.

2. The field case. In this section, assume that A is a field. Every projective $A[t]$-module of finite rank is free. Thus, if (P_1, f, P_2) is a vector bundle of rank n on X, we may choose bases for P_1 and P_2 and then f will be given by a matrix $a \in \text{GL}(n, A[t, t^{-1}])$. Two matrices a and a' give rise to isomorphic vector bundles if and only if there exist $b \in \text{GL}(n, k[t])$ and $c \in \text{GL}(n, k[t^{-1}])$ such that $cab = a'$. This description works over any ring if it is assumed that P_1 and P_2 are free.

The isomorphism classes of line bundles are clearly $(A[t], t^{-n}, A[t^{-1}])$, for $n \in \mathbf{Z}$. This line bundle is denoted $\mathscr{O}(n)$ (even if A is not a field).

The structure of vector bundles on $P_A{}^1$ follows from the following known lemma:

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LEMMA. *Let A be a field. Then, given a matrix $a \in \mathrm{GL}(n, A[t, t^{-1}])$, there exist matrices $b \in \mathrm{GL}(n, A[t])$ and $c \in \mathrm{GL}(n, A[t^{-1}])$ such that cab is diagonal.*

I learned this lemma from Tate in a course at Harvard University. It is more or less due to Grothendieck [**4**].

The above lemma shows that every vector bundle $\mathscr{V}$ on $P_A{}^1$ is the direct sum of line bundles. If $\mathscr{V} \cong \oplus_i m_i \mathscr{O}(n_i)$, where the m_i denotes the direct sum of m_i copies of $\mathscr{O}(n_i)$, then the n_i and the m_i are uniquely determined. This can be shown by appealing to the Krull-Schmidt theorem [**1**], or by a direct proof using the fact that if $m < n$, then there are no non-zero morphisms from $\mathscr{O}(n)$ to $\mathscr{O}(m)$.

3. The degree of a vector bundle. Let $\mathscr{V} = (P_1, f, P_2)$ be a vector bundle on $P_A{}^1$. Let $A \rightarrow B$ be a ring homomorphism. Then we can define a vector bundle $\mathscr{V} \otimes_A B$ on $P_B{}^1$, by

$$\mathscr{V} \otimes_A B = (P_1 \otimes_A B, f \otimes 1, P_2 \otimes_A B).$$

In scheme language, $P_B{}^1$ is obtained by completing the cartesian square

$$\begin{array}{ccc} P_B{}^1 & \xrightarrow{g} & P_A{}^1 \\ \downarrow & & \downarrow \\ \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \end{array}$$

and $\mathscr{V} \otimes_A B = g^*(\mathscr{V})$. Note that $g^*(\mathscr{O}(n)) = \mathscr{O}(n)$, so we will not specify the ring in the notation $\mathscr{O}(n)$.

If A is a field, the degree of a vector bundle $\mathscr{V}$ can be defined as the power of t^{-1} occurring in the determinant of the matrix a defining $\mathscr{V}$. Thus, $\mathscr{O}(n)$ is of degree n and the degree is additive over direct sums. Let I be a prime ideal in A, and let B be the quotient field of A/I. Let $\mathscr{V}$ be a vector bundle over $P_A{}^1$. Then we define the degree of $\mathscr{V}$ at I to be the degree of $\mathscr{V} \otimes_A B$. The decomposition of $\mathscr{V} \otimes_A B$ into line bundles will be referred to as the decomposition of $\mathscr{V}$ into line bundles at I.

If $\mathscr{V} = (P_1, f, P_2)$ has rank r, then the degree of $\mathscr{V}$ at I is the same as that of the line bundle $\Lambda^r \mathscr{V} = (\Lambda^r P_1, \Lambda^r f, \Lambda^r P_2)$. This observation may be used to prove the following:

THEOREM 1. *If A is a Krull ring, or noetherian with connected spec, and $\mathscr{V}$ is a vector bundle on $P_A{}^1$, then the degree of $\mathscr{V}$ is the same at all primes of A.*

Proof. The above observation shows that we need consider only the case where $\mathscr{V}$ is a line bundle. Suppose first that A is a Krull ring. Let

$$\mathscr{V} = (P_1, f, P_2)$$

be a line bundle on $P_A{}^1$, where $P_1 \in \mathscr{P}(A[t])$ and $P_2 \in \mathscr{P}(A[t^{-1}])$ are of rank one. Then $P_1 = P \otimes_A A[t]$, and $P_2 = Q \otimes_A A[t^{-1}]$, where P and Q are projective A modules of rank one [**2**, p. 147]. We have an isomorphism

$$f : P \otimes_A A[t, t^{-1}] \to Q \otimes_A A[t, t^{-1}],$$

and applying the retraction map $A[t, t^{-1}] \to A$ sending t to 1, we get $P \cong Q$. Thus,

$$\mathscr{V} \cong (P \otimes_A A[t], f, P \otimes_A A[t^{-1}]).$$

Here, f is multiplication by a unit in $A[t, t^{-1}]$. Since A is an integral domain, every unit of $A[t, t^{-1}]$ is of the form ut^n, u a unit in A. The degree of $\mathscr{V}$ at all primes will then be $-n$.

If A is noetherian with connected spec, then by [**5**, Remark 4.2.7, p. 75],

$$\mathscr{V} \cong (P \otimes_A A[t], t^{-n}, P \otimes_A A[t^{-1}]),$$

where P is a projective A-module of rank one. The degree will, thus, be n at all primes.

Note that if $\mathscr{V}$ is described by an invertible matrix over $\mathrm{GL}(n, A[t, t^{-1}])$, then the degree at all primes is just the power of t^{-1} appearing in the determinant of this matrix.

4. Indecomposable vector bundles. To illustrate the method of constructing indecomposable vector bundles, I will first give some simple examples where $A = \mathbf{Z}$, the integers. Every projective $Z[t]$ module is free by Seshadri's theorem [**2**, p. 212]; therefore, every vector bundle of rank n on $P_{\mathbf{Z}}{}^1$ is given (as in the field case) by an element of $\mathrm{GL}(n, \mathbf{Z}[t, t^{-1}])$, and every line bundle is isomorphic to $\mathscr{O}(n)$ for some n. Consider the vector bundle $\mathscr{V}$ of rank 2 given by the matrix

$$\begin{bmatrix} t^n & 2t^a \\ 0 & 1 \end{bmatrix}.$$

At the prime 2, this vector bundle clearly gives $\mathscr{O}(-n) \oplus \mathscr{O}$ for all a. We may reduce this matrix by elementary column transformations over $\mathbf{Z}[t]$, or elementary row transformations over $\mathbf{Z}[t^{-1}]$ without changing the isomorphism class. Thus, if $a \geqq n$, or if $a \leqq 0$, $\mathscr{V} \cong \mathscr{O} \oplus \mathscr{O}(-n)$ over $P_{\mathbf{Z}}{}^1$. But if $0 < a < n$ and we are at any prime $p \neq 2$, we diagonalize the matrix (with elementary column transformations over $(\mathbf{Z}/p\mathbf{Z})[t]$ and elementary row transformations over $(\mathbf{Z}/p\mathbf{Z})[t^{-1}]$) as follows:

$$\begin{bmatrix} t^n & 2t^a \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} t^n & 2t^a \\ -\frac{1}{2}t^{n-a} & 0 \end{bmatrix} \to \begin{bmatrix} 0 & t^a \\ t^{n-a} & 0 \end{bmatrix} \to \begin{bmatrix} t^{n-a} & 0 \\ 0 & t^a \end{bmatrix}.$$

Thus, at all primes but 2, $\mathscr{V}$ becomes $\mathscr{O}(a-n) \oplus \mathscr{O}(-a)$. This shows that $\mathscr{V}$ is indecomposable, since, if $\mathscr{V} = \mathscr{O}(n_1) \oplus \mathscr{O}(n_2)$ (over $\mathbf{Z}$), then the decomposition into the direct sum of line bundles would have to be the same (namely, $\mathscr{O}(n_1) \oplus \mathscr{O}(n_2)$) at all primes.

Note that in this example, the "spread" in the degrees of the line bundles is less at the generic point than at any other prime. In the next section, this will be shown to be always the case (at least, if A is a Dedekind domain).

We have quite a bit of choice as to how $\mathscr{V}$ will behave at the different primes. For example, let $p_1, \ldots p_{n-1}$ be distinct primes, and let q_i = product of all these primes but p_i. Let

$$f = q_1 t + q_2 t^2 + \ldots + q_{n-1} t^{n-1}.$$

Then the vector bundle on $P_Z{}^1$ given by the matrix

$$\begin{bmatrix} t^n & f \\ 0 & 1 \end{bmatrix}$$

has decomposition $\mathscr{O}(i - n) + \mathscr{O}(-i)$ at the prime p_i. However, at all but a finite number of primes the vector bundle will decompose into line bundles in the same manner as at the prime ideal zero.

Now, suppose that A is a Krull ring (or a noetherian ring with connected spec) of dimension > 0. Then there exist prime ideals I_1 and I_2, and $p \in A$, with $p \in I_1$, $p \notin I_2$. Consider the vector bundle over $P_A{}^1$ determined by the $n \times n$ matrix

$$\begin{bmatrix} t^n & pt & & & & \\ 0 & 1 & pt & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & 1 & pt \\ & & & & & 1 \end{bmatrix}$$

At the prime ideal I_1, this splits up as $\mathscr{O}(-n) \oplus (n - 1)\mathscr{O}$. However, at the prime ideal I_2, we diagonalize it (working over the quotient field of A/I_2) as follows (starting at the upper left hand corner):

$$\begin{bmatrix} t^n & pt & 0 \\ 0 & 1 & pt \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & pt & 0 \\ -\frac{1}{p} t^{n-1} & 1 & pt \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & t & 0 \\ t^{n-1} & 0 & pt \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} t & 0 & 0 \\ 0 & t^{n-1} & pt \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that this has not disturbed the rest of the matrix. Continuing down the diagonal, we get that at the prime ideal I_2, $\mathscr{V} = n\mathscr{O}(-1)$.

Now, suppose that $\mathscr{V} = \mathscr{V}_1 \oplus \mathscr{V}_2, \mathscr{V}_i \neq 0$. From the decomposition at I_2 (using Theorem 1 and the uniqueness of the decomposition into line bundles over $P_R{}^1$, if R is a field) we see that both $\mathscr{V}_1$ and $\mathscr{V}_2$ must have negative degree. However, from the decomposition at I_1, one must have degree 0, and the other, degree $-n$. This contradicts Theorem 1. Thus, we have proved:

THEOREM 2. *If A is a Krull ring (or a noetherian ring with connected spec) of dimension > 0, then for every positive integer n, there exists an indecomposable vector bundle of rank n on $P_A{}^1$.*

Note that the above vector bundles are stably indecomposable, in that one cannot have $\mathscr{V} \oplus \mathscr{W} = \mathscr{V}_1 \oplus \mathscr{V}_2 \oplus \mathscr{W}, \mathscr{V}_i \neq 0$ (same proof).

5. Further remarks. Assume in this section that A is a Dedekind domain. I do not know necessary and sufficient conditions for there to exist a vector bundle on $P_A{}^1$ with prescribed decomposition into line bundles at each prime of A. At all but a finite number of primes the decomposition into line bundles will be the same as at the prime ideal zero, since there will be only a finite number of denominators involved in the diagonalization over the quotient field of A. The rank is clearly constant, and Theorem 1 says that the degree is the same at all primes.

Another restriction is the following: If V is a vector bundle over $P_A{}^1$, I is a non-zero prime of A, n is the largest integer such that $\mathscr{O}(n)$ occurs in the decomposition of $\mathscr{V}$ into line bundles at 0, and m is the largest integer at I, then $n \leqq m$. Dually, if n is the smallest integer at 0, and m is the smallest at I, then $n \geqq m$.

In order to prove this, it is sufficient to consider the case where A is a discrete valuation ring, since, if B is the localization of A at I and $K = A/I$, then

$$(\mathscr{V} \otimes_A B) \otimes_B K = \mathscr{V} \otimes_A K,$$

and, if L is the quotient field of A, then

$$(\mathscr{V} \otimes_A B) \otimes_B L = \mathscr{V} \otimes_A L.$$

Thus, suppose that A is a discrete valuation ring, with quotient field L and maximal ideal M. Let $\mathscr{V} = (P_1, f, P_2)$ be a vector bundle of rank n over $P_A{}^1$ and assume that n is the largest integer occurring in the decomposition of $\mathscr{V} \otimes_A L$ into line bundles. Then there is a morphism

$$g : \mathscr{O}(n) \to (P_1 \otimes_A L, f, P_2 \otimes_A L).$$

More explicitly, $\mathscr{O}(n) = (L[t], t^{-n}, L[t^{-1}])$ and g consists of two homomorphisms

$$g_1 : L[t] \to P_1 \otimes_A L, \text{ and } g_2 : L[t] \to P_2 \otimes_A L,$$

such that the following diagram commutes:

$$\begin{array}{ccc} L[t, t^{-1}] & \xrightarrow{g_1} & (P_1 \otimes_{A[t]} A[t, t^{-1}]) \otimes_A L \\ \downarrow t^{-n} & & \downarrow f \\ L[t, t^{-1}] & \xrightarrow[g_2]{} & (P_2 \otimes_{A[t^{-1}]} A[t, t^{-1}]) \otimes_A L. \end{array}$$

By Seshadri's theorem, P_1 is a free $A[t]$-module, and P_2 is a free $A[t^{-1}]$-module. Choose a basis for P_1 and P_2, and express the elements of $P_1 \otimes_A L$ and

$P_2 \otimes_A L$ by their co-ordinates with regard to this basis. Then g is determined by

$$g_1(1) = (a_1, \ldots, a_n) \in P_1 \otimes_A L,$$

and

$$g_2(1) = (b_1, \ldots, b_n) \in P_2 \otimes_A L (a_i \in L[t], b_i \in L[t^{-1}]).$$

Let c be the least common multiple of the denominators occurring in the a_i, b_i. Then we can define a morphism $g' : \mathcal{O}(n) \to V$ over A by

$$g_1'(1) = (ca_1, \ldots, ca_n)$$

and $g_2'(1) = (cb_1, \ldots, cb_n)$. Then g' gives a non-zero homomorphism

$$g' \otimes 1 : \mathcal{O}(n) \to \mathscr{V} \otimes_A (A/M)$$

over A/M. Hence, there exists an integer $m \geqq n$ such that $\mathcal{O}(m)$ occurs in the decomposition of $\mathscr{V} \otimes_A (A/M)$ into line bundles. (Recall that there exist non-zero morphisms $\mathcal{O}(n) \to \mathcal{O}(m)$ if and only if $n \leqq m$.)

The corresponding statement with the least integers is proved by considering the dual of $\mathscr{V}$.

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