J. Austral. Math. Soc. (*Series A*) **56** (1994), 403–414

MÖBIUS COVARIANCE OF ITERATED DIRAC OPERATORS

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(Received 17 July 1991; revised 3 March 1992)

Communicated by P. G. Dodds

Abstract

Using Fourier transforms, we give a new proof of certain identities for the fundamental solutions of the iterated Dirac operators $\underline{D}^\ell = (\sum_{i=1}^n e_i\ \partial/\partial x_i)^\ell$, $\ell \in \mathbf{Z}_+$ and $D^\ell = (\partial/\partial x_0 + \underline{D})^\ell$. Based on the close relationship between the fundamental solutions and the conformal weights we then give a simple proof of B. Bojarski's results on the conformal covariance of $\underline{D}^\ell$. We also prove a new conformal covariance result of D.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 30 G 35.

1. Introduction

This paper gives an alternative proof of B. Bojarski's results [1, 2] on Möbius covariance of the iterated Dirac operators $\underline{D}^\ell = \left(\sum_{i=1}^n e_i\ \partial/\partial x_i\right)^\ell$, $\ell \in \mathbf{Z}_+$, by recognizing the close relationship between the conformal weights and the fundamental solutions of the operators. We also give a covariance result on the Dirac operator $D = \partial/\partial x_0 + \underline{D}$, with some observations concerning non-existence of Möbius covariance of its iterations D^ℓ, $\ell > 1$ (see also [11]).

We begin by recalling basic knowledge related to Clifford algebras ([1, 2, 4, 3, 6]). The Clifford algebra $\mathscr{A}_n$ shall be the associative algebra over the real number system $\mathbb{R}$ generated by n elements $e_1, e_2, \ldots, e_n$ subject to the relations $e_ie_j = -e_je_i$, $i \neq j$, and $e_i^2 = -1$. Each element $a \in \mathscr{A}_n$ has a unique representation in the form $a = \sum a_s e_s$, where $a_s \in \mathbb{R}$ and the summation is over all ordered subsets $s = \{0 < i_1 < \cdots < i_\ell\} \subset \{1, 2, \cdots, n\}$, and we identify

e_s with $e_{i_1}\cdots e_{i_\ell}$. For the empty set $\emptyset$, $e_\emptyset$ is interpreted as the real number 1. $\mathscr{A}_0$, $\mathscr{A}_1$, $\mathscr{A}_2$ can be identified with $\mathbb{R}$, the complex field $\mathbf{C}$ and the quaternions, respectively.

$\mathscr{A}_n$ is a vector space of real dimension 2^n. In the literature there are two ways to identify $\mathbb{R}^n$ with certain linear subspaces of $\mathscr{A}_n$. In this paper we identify $\mathbb{R}^n$ with $\mathrm{span}\{e_1,\cdots,e_n\}$. For any element $x = x_0 + x_1e_1 + \cdots + x_ne_n$, we denote $x = x_0 + \underline{x}$ with $\underline{x} = x_1e_1 + \cdots + x_ne_n \in \mathbb{R}^n$. Define two operations on the basic elements: $(e_{i_1}\cdots e_{i_\ell})^* = e_{i_\ell}\cdots e_{i_1}$, $(e_{i_1}\cdots e_{i_\ell})' = (-1)^\ell(e_{i_1}\cdots e_{i_\ell})$ etc., and extend them by linearity to two corresponding operations on $\mathscr{A}_n$, still denoted by * and $'$. By combining them we define the third operation $^-$ by $\bar{x} = (x^*)'$; it is easy to see that $\bar{x} = x_0 - \underline{x}$ for $x = x_0 + \underline{x}$. The natural inner product between a and b, denoted by $\langle a, b\rangle$, is the number $\sum_s a_sb_s$ and the norm of a associated with this inner product is $|a| = (\sum_s |a_s|^2)^{\frac{1}{2}}$. We recall that the Clifford group Γ_n is defined as the multiplicative group of all elements in the Clifford algebra which can be written as products of non-zero vectors in $\mathbb{R}^n$. For elements a, b in $\Gamma_n \cup \{0\}$, $\bar{a}a = |a|^2$ and $|ab| = |a|\cdot|b|$ (see [1, 2, 4, 3, 6]).

If $a \in \Gamma_n$, then it has a representation $a = \prod_{j=1}^{M(a)} a_j$, where $a_j \in \mathbb{R}^n$. Generally, such a representation is not unique, and neither is the related integer $M(a)$. We let $m(a)$ be the minimum of $M(a)$ over all such representations. If $a \in \mathbb{R}\backslash\{0\}$, then we set $m(a) = 0$. So, $m(\underline{x}) = 1$, and for $a \in \Gamma_n$ it follows that $aa^* = a^*a = (-1)^{m(a)}|a|^2$.

By the Möbius group we mean the group of orientation preserving transformations acting in the Euclidean space $\mathbb{R}^n$, generated by rigid motions, dilations and inversions ([1, 2, 4, 3, 6]).

According to a theorem of Ahlfors ([1, 2, 4, 3]), all Möbius transforms from $\mathbb{R}^n \cup \{\infty\}$ to $\mathbb{R}^n \cup \{\infty\}$ are exactly those of form

$$\varphi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1}$$

where $a, b, c, d \in \Gamma_n \cup \{0\}$ and

$$ad^* - bc^* \in \mathbb{R} \setminus \{0\}, \qquad a^*c,\ cd^*,\ d^*b,\ ba^* \in \mathbb{R}^n.$$

See [2], for example. Furthermore, the identification between the φ's and the Clifford matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives a homomorphism under 2×2 block matrix multiplication.

Since we are interested in differentiation of transformations of functions, we can assume without loss of generality that the functions under consideration are defined in $\mathbb{R}^n$ or its one-point compactification, with compact supports and having as many orders of differentiability as we need in our argument; the same applies to functions defined in $\mathbb{R}^n_1 = \mathbb{R} \oplus \mathbb{R}^n = \{x = x_0 + \underline{x};\ x_0 \in \mathbb{R},\ \underline{x} \in \mathbb{R}^n\}$.

For Clifford number-valued functions defined in $\mathbb{R}^n$ belonging to the above mentioned nice classes we introduce the following Fourier transform:

$$\hat{f}(\underline{\xi}) = \int_{\mathbb{R}^n} e^{i\langle \underline{x}, \underline{\xi}\rangle} f(\underline{x}) d\underline{x}.$$

The associated symbol of the Dirac operator $\underline{D}$ is $i\underline{\xi}$; and, accordingly, those of $\underline{D}^\ell$ and $\underline{D}^{-\ell}$ are $(i\underline{\xi})^\ell$ and $(i\underline{\xi})^{-\ell}$, $\ell \in \mathbf{Z}_+$, respectively.

For functions defined in $\mathbb{R}^n_1 = \{x_0 + \underline{x} : x_0 \in \mathbb{R},\ \underline{x} \in \mathbb{R}^n\}$ we use the following definition

$$\hat{f}(\xi) = \int_{\mathbb{R}^n_1} e^{i\langle x, \xi\rangle} f(x) dx$$

where $\xi, x \in \mathbb{R}^n_1$.

With this definition the associated symbols of the iterated Dirac operator D^ℓ and $D^{-\ell}$, $\ell \in \mathbf{Z}_+$, where $D = \partial/\partial x_0 + \underline{D}$, are $(i\xi)^\ell$ and $(i\xi)^{-\ell}$, respectively.

It would be helpful to mention that Hans Jakobson and Michelle Vergne have established an analogue of Theorem 1 for the group $SU(2,2)$ ([10]); and Hans Jakobsen has established related results for other Lie groups ([9]).

2. Fundamental solutions of $\underline{D}^\ell$ and D^ℓ

First we deduce the fundamental solutions of $\underline{D}^\ell$. We prefer an approach that will not concentrate on integers ℓ at the beginning. It is consistent with the above if we define $\underline{D}^{-\alpha}$, $\alpha > 0$, by

$$\underline{D}^{-\alpha} f(\underline{x}) = c_n \int_{\mathbb{R}^n} e^{-i\langle \underline{x}, \underline{\xi}\rangle} (i\underline{\xi})^{-\alpha} \hat{f}(\underline{\xi}) d\underline{\xi},$$

where $(i\underline{\xi})^{-\alpha}$ is defined by

$$(i\underline{\xi})^{-\alpha} = |\underline{\xi}|^{-\alpha} \chi_+(\underline{\xi}) + (-|\underline{\xi}|)^{-\alpha} \chi_-(\underline{\xi})$$

and

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}\left(1 \pm \frac{i\underline{\xi}}{|\underline{\xi}|}\right).$$

https://doi.org/10.1017/S1446788700035576 Published online by Cambridge University Press

Thus, if $\alpha = \ell$ is a positive integer, we have

$$(i\underline{\xi})^{-\ell} = \begin{cases} \dfrac{1}{|\underline{\xi}|^{\ell}} & \text{if } \ell \text{ is even;} \\ \dfrac{i\underline{\xi}}{|\underline{\xi}|^{\ell+1}} & \text{if } \ell \text{ is odd.} \end{cases}$$

Therefore

$$\begin{aligned} \underline{D}^{-\alpha} f(\underline{x}) = \frac{c_n}{2}\Bigg[& \int_{\mathbb{R}^n} e^{-i\langle \underline{x},\underline{\xi}\rangle} |\underline{\xi}|^{-\alpha} \hat{f}(\underline{\xi}) d\underline{\xi} + \underline{D} \int_{\mathbb{R}^n} e^{-i\langle \underline{x},\underline{\xi}\rangle} |\underline{\xi}|^{-\alpha-1} \hat{f}(\underline{\xi}) d\underline{\xi} \\ & + \int_{\mathbb{R}^n} e^{-i\langle \underline{x},\underline{\xi}\rangle} (-|\underline{\xi}|)^{-\alpha} \hat{f}(\underline{\xi}) d\underline{\xi} + \underline{D} \int_{\mathbb{R}^n} e^{-i\langle \underline{x},\underline{\xi}\rangle} (-|\underline{\xi}|)^{-\alpha-1} \hat{f}(\underline{\xi}) d\underline{\xi} \Bigg]. \end{aligned}$$

If $0 < \alpha$, $\alpha + 1 < n$, by invoking the following formula (see [14, p. 117] for example) for $0 < \beta < n$,

$$\left(\frac{1}{|\underline{\xi}|^{\beta}}\right)^{\vee} = c_{n,\beta} \frac{1}{|\underline{x}|^{n-\beta}},$$

we conclude

$$\underline{D}^{-\alpha} f(\underline{x}) = K_{n,\alpha} * f(\underline{x})$$

where

$$K_{n,\alpha}(\underline{x}) = c_{n,\alpha}\left(1 + e^{-i\alpha\pi}\right) \cdot \frac{1}{|\underline{x}|^{n-\alpha}} + d_{n,\alpha}\left(1 - e^{-i\alpha\pi}\right) \underline{D}\left(\frac{1}{|\underline{x}|^{n-\alpha-1}}\right).$$

For general $\alpha > 0$, the same criterion gives

$$K_{n,\alpha}(\underline{x}) = c_{n,\alpha}\left(1 + e^{-i\alpha\pi}\right) G_{n,\alpha}(\underline{x}) + d_{n,\alpha}\left(1 - e^{-i\alpha\pi}\right) \underline{D} G_{n,\alpha+1}(\underline{x}),$$

where $G_{n,\beta}$ is the fundamental solution of $|\underline{D}|^{\beta}$ that is the operator associated with symbol $|\underline{\xi}|^{\beta}$.

Accordingly, we conclude that, for n odd,

$$K_{n,\ell}(\underline{x}) = \begin{cases} c_{n,\ell} \dfrac{\underline{x}}{|\underline{x}|^{n-\ell+1}}, & \ell \text{ odd;} \\ c_{n,\ell} \dfrac{1}{|\underline{x}|^{n-\ell}}, & \ell \text{ even,} \end{cases} \tag{1}$$

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and for n even

$$(2)\qquad K_{n,\ell}(\underline{x}) = \begin{cases} c_{n,\ell}\,\dfrac{\underline{x}}{|\underline{x}|^{n-\ell+1}}, & \ell \text{ odd and } \ell < n; \\ c_{n,\ell}\,\dfrac{1}{|\underline{x}|^{n-\ell}}, & \ell \text{ even and } \ell < n; \\ \left(c_{n,\ell}\log|\underline{x}| + d_{n,\ell}\right)\dfrac{\underline{x}}{|\underline{x}|^{n-\ell+1}}, & \ell \text{ odd and } \ell > n; \\ \left(c_{n,\ell}\log|\underline{x}| + d_{n,\ell}\right)\dfrac{1}{|\underline{x}|^{n-\ell}}, & \ell \text{ even and } \ell \geq n. \end{cases}$$

The formulae in (2) can also be derived using other methods. For example, they follow from the results in [12]. Our method by using Fourier transform, however, gives the formulae for $K_{n,\alpha}$ when α is not an integer. The formulae and the proof of Theorem 1 indicate the reason why $\underline{D}_\alpha$ has no conformal invariance when α is not an integer.

Now we turn to the fundamental solutions of D^ℓ, $\ell \in \mathbf{Z}_+$. In fact, writing $D_0 = \partial/\partial x_0$, it follows that $D^{-\ell} = (D_0 + \underline{D})^{-\ell} = (D_0 - \underline{D})^\ell (D_0^2 - \underline{D}^2)^{-\ell}$. According to the Fourier transform defined at the end of the introductory section, the symbol of $(D_0^2 - \underline{D}^2)^{-\ell}$ is $|\xi|^{-2\ell}$, while its inverse Fourier transform, for $0 < 2\ell < n+1$, is $c_{n,\ell}|x|^{-(n+1-2\ell)}$. This shows that the kernel of the operator $D^{-\ell}$ is

$$L_{n,\ell}(x) = c_{n,\ell}(D_0 - \underline{D})^\ell\left(\frac{1}{|x|^{n+1-2\ell}}\right), \qquad 0 < 2\ell < n+1.$$

A direct computation then gives

$$(3)\qquad L_{n,\ell}(x) = c_{n,\ell}\frac{x_0^{\ell-1}\bar{x}}{|x|^{n+1}}, \qquad \ell \in \mathbf{Z}_+.$$

This result is due to Delanghe and Brackx [7] with a different proof.

3. Conformal Covariance

We normalize the Möbius transformation defined in the introduction by adding the condition

$$ad^* - bc^* = 1,$$

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and consider the multiplier representation

$$T_\ell(\phi)f(\underline{x}) = J_{\ell,\phi} \cdot f\,(\phi(\underline{x}))\,,$$

where

$$J_{\ell,\phi}(\underline{x}) = \begin{cases} \dfrac{(c\underline{x}+d)^*}{|c\underline{x}+d|^{n-\ell+1}}, & \ell \text{ odd;} \\ \dfrac{1}{|c\underline{x}+d|^{n-\ell}}, & \ell \text{ even,} \end{cases} \tag{4}$$

and $\ell \in \mathbf{Z}$.

The following theorem is the main result of [1, 2].

THEOREM 1. *For $\ell \in \mathbf{Z}_+$, the iterated Dirac operator $\underline{D}^\ell$ intertwines the representations T_ℓ, $T_{-\ell}$ of the Möbius transformation group, that is, for $c \neq 0$*

$$\underline{D}^\ell\,(T_\ell f) = \begin{cases} (-1)^{m(c)+1} T_{-\ell}\left(\underline{D}^\ell f\right), & \ell \textit{ odd;} \\ T_{-\ell}\left(\underline{D}^\ell f\right), & \ell \textit{ even,} \end{cases} \tag{5}$$

and, if $c = 0$, then it must be the case that $d \neq 0$, and the factor $(-1)^{m(c)+1}$ in the last formula should be replaced by $(-1)^{m(d)}$.

REMARK. Note that the constant factors for ℓ odd in the theorem seem to be missing in [6].

The following proof of the theorem based on the close relationship between $K_{n,\ell}$, the fundamental solutions of $\underline{D}^\ell$, and the conformal weights $J_{\ell,\phi}$. Note that it is thus akin to a known proof of 'Bol's lemma' (cf. [8]).

PROOF. We only give the proof for the case $c \neq 0$. The proof for $c = 0$ is omitted, since it is similar, and even simpler. We are going to show

$$T_\ell f = \begin{cases} (-1)^{m(c)+1} \underline{D}^{-\ell} T_{-\ell}\left(\underline{D}^\ell f\right), & \ell \text{ odd;} \\ \underline{D}^{-\ell} T_{-\ell}\left(\underline{D}^\ell f\right), & \ell \text{ even.} \end{cases} \tag{5'}$$

First, let us assume that n is odd or that $\ell < n$ and n is even. Denote by ψ the inverse of ϕ. If $\underline{y} = \phi(\underline{x}) = (a\underline{x}+b)(c\underline{x}+d)^{-1} \in \mathbb{R}^n$, then $\underline{y}(c\underline{x}+d) = a\underline{x}+b$ and so, $\underline{x} = \psi(\underline{y}) = (\underline{y}c - a)^{-1}(-\underline{y}d + b)$. Let $z = z(\underline{y}) = \underline{y}c - a$, $A = b - ac^{-1}d$, it follows that

$$\underline{x} = z^{-1}A - c^{-1}d. \tag{6}$$

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On the other hand, since $\underline{x} = \underline{x}^*$, $\underline{y} = \underline{y}^*$, (6) is equivalent to

$$\underline{x} = A^* z^{*-1} - d^* c^{*-1}. \tag{6'}$$

It can be observed, from the properties of Möbius transformation and formula (6), that $c \neq 0$ implies $A \neq 0$. We have

$$\begin{aligned}
&\underline{D}^{-\ell}\left(T_{-\ell}\left(\underline{D}^\ell f\right)\right)(\psi(\underline{x}))\\
&\quad = c_{n,\ell}\int K_{n,\ell}\big(\psi(\underline{x}) - \underline{y}\big)\cdot J_{-\ell,\phi}(\underline{y})\left(\underline{D}^\ell f\right)(\phi(\underline{y}))d\underline{y}\\
&\quad = c_{n,\ell}\int K_{n,\ell}\big(\psi(\underline{x}) - \psi(\underline{y})\big)\cdot J_{-\ell,\phi}\big(\psi(\underline{y})\big)\left(\underline{D}^\ell f\right)(\underline{y})\left|\frac{d\psi(\underline{y})}{d\underline{y}}\right| d\underline{y},
\end{aligned}$$

where $|d\psi(y)/dy|$ is the Jacobian. Noticing that $\underline{x} = \psi(\underline{y})$ is also a Möbius transformation, by using formula (2.4) of [2] and the condition $ad^* - bc^* = 1$, we see that the Jacobian equals $|z(y)|^{-2n}$. In view of the identities (2), (4) and

$$\begin{aligned}
\psi(\underline{x}) - \psi(\underline{y}) &= \big(z^{-1}(\underline{x}) - z^{-1}(\underline{y})\big)A,\\
z^{-1}(\underline{x}) - z^{-1}(\underline{y}) &= -z^{-1}(\underline{x})\big(z(\underline{x}) - z(\underline{y})\big)z^{-1}(\underline{y}),\\
z(\underline{x}) - z(\underline{y}) &= (\underline{x} - \underline{y})c
\end{aligned}$$

and

$$J_{-\ell,\phi}(\psi(\underline{y})) = \begin{cases} \dfrac{A^*}{|A|^{n+\ell+1}}\dfrac{z^{*-1}(\underline{y})}{|z^{-1}(\underline{y})|^{n+\ell+1}}\dfrac{c^*}{|c|^{n+\ell+1}}, & \ell \text{ odd};\\[2ex] \dfrac{1}{|A|^{n+\ell}}\dfrac{1}{|z^{-1}(\underline{y})|^{n+\ell}}\dfrac{1}{|c|^{n+\ell}}, & \ell \text{ even}, \end{cases}$$

for ℓ odd, the above equals

$$\begin{aligned}
&-c_{n,\ell}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-\ell+1}}\int \frac{(\underline{x} - \underline{y})}{|\underline{x} - \underline{y}|^{n-\ell+1}}\frac{c}{|c|^{n-\ell+1}}\frac{z^{-1}(\underline{y})}{|z^{-1}(\underline{y})|^{n-\ell+1}}\frac{A}{|A|^{n-\ell+1}}\\
&\qquad\qquad\cdot\frac{A^*}{|A^*|^{n+\ell+1}}\frac{z^{-1*}(\underline{y})}{|z^{-1}(\underline{y})|^{n+\ell+1}}\frac{c^*}{|c|^{n+\ell+1}}\left(\underline{D}^\ell f\right)(\underline{y})\frac{1}{|z(\underline{y})|^{2n}}d\underline{y}\\
&\qquad = c_{n,\ell}\frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-\ell+1}}\int\frac{(\underline{x} - \underline{y})}{|\underline{x} - \underline{y}|^{n-\ell+1}}\left(\underline{D}^\ell f\right)(\underline{y})d\underline{y}\\
&\qquad = \frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-\ell+1}}\int K_{n,\ell}(\underline{x} - \underline{y})\left(\underline{D}^\ell f\right)(\underline{y})d\underline{y}\\
&\qquad = \frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-\ell+1}}f(\underline{x}),
\end{aligned}$$

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where we used $m(z^{-1}A) = 1$, which follows from the fact that $z^{-1}A = \underset{\sim}{x} + c^{-1}d$ and $c^{-1}d \in \mathbb{R}^n$ (see [2, Lemma 1.4]), and the properties of the fundamental solution. Replacing $\underset{\sim}{x}$ by $\phi(\underset{\sim}{x})$ in the above and noticing that $(\underset{\sim}{x} + d^*c^{*-1}) = z^{-1}(\phi(\underset{\sim}{x}))A$, we obtain

$$\underset{\sim}{D}^{-\ell}\left(T_{-\ell}\left(\underset{\sim}{D}^{\ell}f\right)\right)(\underset{\sim}{x}) = \frac{(-1)^{m(A)}cA^*}{|cA|^{n+\ell+1}} \cdot \frac{(cx+d)^*}{|cx+d|^{n-\ell+1}} f(\phi(\underset{\sim}{x})).$$

Now $bc^* = ad^* - 1$ together with $c^{-1}d \in \mathbb{R}^n$ implies $b = -c^{*-1} + ac^{-1}d$ and hence $A = -c^{*-1}$. Using this relation, the above immediately gives (5′). The proof for ℓ even is similar. The only difference is that this time we should use the expressions for the case ℓ even in formula (1), (2) and (4), respectively. We consider the case $\ell \geq n$ when n is even. Using (2) and the proceeding as above in the case ℓ is odd, it follows that

$$\begin{aligned}
&\underset{\sim}{D}^{-\ell}\left(T_{-\ell}\left(\underset{\sim}{D}^{\ell}f\right)\right)(\psi(\underset{\sim}{x})) \\
&= \frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underset{\sim}{x})}{|z^{-1}(\underset{\sim}{x})|^{n-\ell+1}}\int\Big[(-c_{n,\ell})\log|z(\underset{\sim}{x})| + (c_{n,\ell}\log|\underset{\sim}{x}-\underset{\sim}{y}|+d_{n,\ell}) \\
&\qquad + c_{n,\ell}\log|c| + (-c_{n,\ell})\log|z(\underset{\sim}{y})|\Big]\frac{(\underset{\sim}{x}-\underset{\sim}{y})}{|\underset{\sim}{x}-\underset{\sim}{y}|^{n-\ell+1}}(\underset{\sim}{D}^{\ell}f)(\underset{\sim}{y})d\underset{\sim}{y} \\
&= \sum_{1}^{4} I_i.
\end{aligned}$$

Since in the case $\ell \geq n$ when n is even and ℓ odd $(\underset{\sim}{x}-\underset{\sim}{y})/|\underset{\sim}{x}-\underset{\sim}{y}|^{n-\ell+1} = \pm(\underset{\sim}{x}-\underset{\sim}{y})^{\ell-n}$, I_1 and I_3 are obviously zero. The inequality

$$I_2 = \frac{1}{|A|^{2n}}\frac{(-1)^{m(c)}}{|c|^{2n}}\frac{z^{-1}(\underset{\sim}{x})}{|z^{-1}(\underset{\sim}{x})|^{n-\ell+1}}f(\underset{\sim}{x})$$

follows from the basic property of fundamental solutions. To conclude we only need to show $I_4 = 0$. In fact, since

$$\frac{(\underset{\sim}{x}-\underset{\sim}{y})}{|\underset{\sim}{x}-\underset{\sim}{y}|^{n-\ell+1}} = \pm\left[(\underset{\sim}{x}-ac^{-1})-(\underset{\sim}{y}-ac^{-1})\right]^{\ell-n} = \sum_{k+j=\ell-n} h_{kj}(\underset{\sim}{x}-ac^{-1})^k(\underset{\sim}{y}-ac^{-1})^j,$$

by integration by parts, we have

$$
\begin{aligned}
I_4 &= -c_{n,\ell} \sum_{k+j=\ell-n} h_{kj}(\underline{x}-ac^{-1})^k \int \left(\log|\underline{y}-ac^{-1}| + \log|c|\right)(\underline{y}-ac^{-1})^j \left(\underline{D}^\ell f\right)(\underline{y})d\underline{y} \\
&= -c_{n,\ell} \sum_{k+j=\ell-n} h_{kj}(\underline{x}-ac^{-1})^k \int \left(\log|\underline{y}-ac^{-1}|\right)(\underline{y}-ac^{-1})^j \left(\underline{D}^\ell f\right)(\underline{y})d\underline{y} \\
&= -c_{n,\ell} \sum_{k+j=\ell-n,\, j<\ell-n} h_{kj}(\underline{x}-ac^{-1})^k \int \log|\underline{y}-ac^{-1}|(\underline{y}-ac^{-1})^j (\underline{D})^\ell f(\underline{y})d\underline{y} \\
&\quad -h_{0,\ell-n} \int c_{n,\ell} \log|\underline{y}-ac^{-1}|(\underline{y}-ac^{-1})^{\ell-n} \left(\underline{D}^\ell f\right)(\underline{y})d\underline{y} \\
&= \pm h_{0,\ell-n} \int c_{n,\ell}\left(\log|\underline{y}-ac^{-1}| + d_{n,\ell}\right) \frac{(\underline{y}-ac^{-1})}{|\underline{y}-ac^{-1}|^{n-\ell+1}} \left(\underline{D}^\ell f\right)(\underline{y})d\underline{y} \\
&= \pm h_{0,\ell-n} f(ac^{-1}) \\
&= 0,
\end{aligned}
$$

where the last step used the convention that function $f \circ \phi$ has compact support. In fact, $f(ac^{-1}) = f \circ \phi \circ \psi(ac^{-1}) = f \circ \phi(\infty) = 0$.

So, we still have, as in the previous case

$$
\underline{D}^{-\ell}\left(T_{-\ell}\left(\underline{D}^\ell f\right)\right)(\psi(\underline{x})) = \frac{1}{|A|^{2n}} \frac{(-1)^{m(c)}}{|c|^{2n}} \frac{z^{-1}(\underline{x})}{|z^{-1}(\underline{x})|^{n-\ell+1}} f(\underline{x}).
$$

Replacing $\underline{x}$ by $\phi(\underline{x})$ and proceeding as before, we obtain the desired formula in (5′) for the case ℓ odd when $\ell \geq n$ and n is even.

The case ℓ even can be treated similarly, and hence the theorem is proved.

Now we turn to the operator D^ℓ, and ask if we have similar conformal covariance for these operators. The answer is *yes* for $\ell = 1$; and *no* for $\ell > 1$ (see remark below). First, notice that if we change $\mathbb{R}^n$ to $\mathbb{R}^n_1$ in the definition of Möbius transformations and the identification relationship between them and the certain Clifford matrices described in the introduction, then all the conclusions still hold. In fact, it then becomes L. Ahlfors's original result (see [1, 2, 4, 3]). Now denote by φ a Möbius transformation from $\mathbb{R}^n_1 \cup \{\infty\}$ into $\mathbb{R}^n_1 \cup \{\infty\}$ and let g be the fixed function from $\mathbb{R}^n_1 \cup \{\infty\}$ into $\mathbb{R}^n_1 \cup \{\infty\}$ defined by

$$
g(x) = \frac{x^*}{|x|^{n+1}},
$$

where $x = x_0 + \underline{x}$. Define the representations

$$
\begin{aligned}
S_1(\varphi)f(x) &= L_{n,1}\left((cx+d)^*\right) f\left(\varphi(x)\right), \\
S_{-1}(\varphi)f(x) &= g(cx+d) f\left(\varphi(x)\right).
\end{aligned}
$$

The following result holds.

THEOREM 2.

$$D(S_1 f) = S_{-1}(Df). \tag{7}$$

PROOF. By using the fundamental solution obtained in Section 1, formula (3), and the same notation and relations established in the proof of Theorem 1, with $\underline{x}$ and $\underline{y}$ are replaced by x and y, respectively, we have

$$\begin{aligned}
&D^{-1}\left(S_{-1}(Df)\right)(\psi(x)) \\
&= \int L_{n,1}\left(\psi(x) - \psi(y)\right) g(cz^{-1}(y)A)(Df)(y) \frac{1}{|z(y)|^{2(n+1)}} dy \\
&= c_{n,1} \frac{-\overline{z^{-1}(x)}}{|z^{-1}(x)|^{n+1}} \\
&\quad \times \int \frac{\overline{(x-y)cz^{-1}(y)A}}{|x-y|^{n+1}|c|^{n+1}|z^{-1}(y)|^{n+1}|A|^{n+1}} \frac{A^*(z^{-1}(y))^* c^*}{|A|^{n+3}|z^{-1}(y)|^{n+3}|c|^{n+3}} \frac{1}{|z(y)|^{2(n+1)}} dy \\
&= \frac{-1}{|Ac^*|^{2n+2}} \frac{-\overline{z^{-1}(x)}}{|z^{-1}(x)|^{n+1}} \int L_{n,1}(x-y)(Df)(y)dy \\
&= \frac{-1}{|Ac^*|^{2n+2}} \frac{-\overline{z^{-1}(x)}}{|z^{-1}(x)|^{n+1}} f(x).
\end{aligned}$$

Replacing x by $\phi(x)$ and using $Ac^* = -1$, we obtain (7).

The above proof is not applicable when $\ell > 1$. (For a different proof of Theorem 2, see [11].) It is caused by the presence of a positive power of the single component x_0, as indicated by formula (3).

REMARK. Now we indicate why a result such as Theorem 2 cannot be true for $\ell > 1$. Rather than continuing the rather tedious computation in [11]—in principle this is always possible—we supply the following argument which was suggested to us by Professor John Ryan (personal communication). Consider the Clifford algebra $\mathscr{A}_{n+1}$ with $n+1$ generators $e_1, \cdots, e_{n+1}$ and let $\mathscr{A}^+_{n+1}$ be the subalgebra of even elements of $\mathscr{A}_{n+1}$. Then there is an isomorphism $\rho : \mathscr{A}_n \to \mathscr{A}^+_{n+1}$ such that $e_j \mapsto e^{-1}_{n+1}e_j = -e_{n+1}e_j$ $(j = 1, \cdots, n)$. (This can be seen by observing that $\mathscr{A}^+_{n+1}$ is generated by the elements $e^{-1}_{n+1}e_j$ $(j = 1, \cdots, n)$, which again follows from the relation

$$e^{-1}_{n+1}x \cdot e^{-1}_{n+1}y = xy \qquad (x, y \in \mathscr{A}_n).)$$

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Consider now the equation

$$Df = \frac{\partial f}{\partial x_0} + \sum_{j=1}^{n} e_j \frac{\partial f}{\partial x_j}.$$

Applying the isomorphism ρ we obtain

$$\begin{aligned}(Df)^\rho &= \left(\frac{\partial}{\partial x_0} + \sum_{j=1}^{n} e_{n+1}^{-1} e_j \frac{\partial}{\partial x_j}\right) f^\rho \\ &= e_{n+1}^{-1}\left(e_{n+1}\frac{\partial}{\partial x_0} + \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}\right) f^\rho \\ &= e_{n+1}^{-1} \underset{\sim}{D} f^\rho.\end{aligned}$$

In condensed form, we may write this as

$$\rho D = e_{n+1}^{-1} \underset{\sim}{D} \rho.$$

This gives us a possibility of deriving Theorem 2 from Bojarski's Theorem 1 (the case $\ell = 1$). The point is that such a relation cannot be true if $\ell > 1$ and so the argument breaks down. For iterating gives

$$\rho D^\ell = \left(e_{n+1}^{-1} \underset{\sim}{D}\right)^\ell \rho.$$

That, is ρ intertwines D^ℓ and $(e_{n+1}^{-1} \underset{\sim}{D})^\ell$, not D^ℓ and $\underset{\sim}{D}^\ell$.

4. Acknowledgement

Our thanks are due to Professor John Ryan. A communication by him helped us to better understand some points in this work. We also thank Professor Bogdan Bojarski for pointing out a gross error in an earlier version of the paper.

The second author would like to thank for the invitation and the financial support received from the Mittag-Leffler Institute within the program *Operator theory and complex analysis*, and also acknowledge the *Civilingenjör Gustaf Sigurd Magnusons fond för främjande av vetenskapen inom ämnet matematik* of the Royal Swedish Academy of Sciences for further financial support.

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