Canad. Math. Bull. Vol. **38** (4), 1995 pp. 429–433

WEAK NORMALIZATION OF POWER SERIES RINGS

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ABSTRACT. It is proved that if R^* is the weak normalization of an integral domain R, then the weak normalization of the power series ring $R[[X_1,\ldots,X_n]]$ is contained in $R^*[[X_1,\ldots,X_n]]$. Consequently, if R is a weakly normal integral domain, then $R[[X_1,\ldots,X_n]]$ is also weakly normal.

Let $A \subseteq B$ be (commutative) integral domains. We let A_B^+ and A_B^* denote the seminormalization of A and the weak normalization of A, respectively, in the integral closure of A in B, in the sense of [6] and [1] (see also [7]). As usual, we say that A is *seminormal* (resp., *weakly normal*) *in* B in case $A_B^+ = A$ (resp., $A_B^* = A$). When B is the quotient field of A, we use the notations A^+ and A^* instead of A_B^+ and A_B^*. The domain A is called *seminormal* (resp., *weakly normal*) if it is so in its quotient field. These concepts are related by the following criterion [7, Theorem 1]: A is weakly normal in B if and only if A is seminormal in B and, whenever an element u in B satisfies $u^p, pu \in A$ for some prime p, then $u \in A$.

We denote by $\mathbf{X}$ a finite nonempty set of indeterminates. It is known that if R is a seminormal integral domain, then the polynomial ring $R[\mathbf{X}]$ and the power series ring $R[[\mathbf{X}]]$ are seminormal (*cf.* [4, Theorem 1.6] and [3]). In this note, we use the criterion from [7] to establish the analogue of these results for weak normality.

We collect in Lemma 1 some basic properties of weak normalization.

LEMMA 1. *(i) For any integral domains $A \subseteq B$, we have $(A_B^*)_B^* = A_B^*$.*

(ii) For any integral domains $A \subseteq B$ and $C \subseteq D$ such that $A \subseteq C$ and $B \subseteq D$, we have $A_B^* \subseteq C_D^*$.

iii) For any integral domains $A \subseteq B \subseteq C$, we have $A_B^* = A_C^* \cap B$.

iv) The weak normalization of an integral domain A in a given extension domain B is the smallest ring S such that $A \subseteq S \subseteq B$ and S is weakly normal in B.

PROOF. (i) This assertion (that is, A_B^* is weakly normal in B) is obtained in [8, p. 91].

(ii) This follows from [8, Theorem 2]; A_B^* (resp., C_D^*) is the filtered union of all subrings of B (resp., D) obtained from A (resp., C) by finitely many elementary subintegral or elementary weakly subintegral extensions.

(iii) The inclusion $A_B^* \subseteq A_C^* \cap B$ follows from (ii). On the other hand, $A_C^* \cap B$ is a weakly subintegral extension of A and so $A_C^* \cap B$ is contained in A_B^*, which is the largest weakly subintegral extension of A contained in B [8, p. 90].

The first author was supported in part by the University of Tennessee Science Alliance and the University of Haifa. Prof. Dobbs gratefully acknowledges the warm hospitality shown during his visit to Haifa in 1994.
Received by the editors June 29, 1994.
AMS subject classification: Primary: 13G05; secondary: 13F25, 13B22, 13B25.
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https://doi.org/10.4153/CMB-1995-062-0 Published online by Cambridge University Press

(iv) By (i), A_B^* is weakly normal in B. If S is a domain such that $A \subseteq S \subseteq B$ and S is weakly normal in B, then, by (ii), $A_B^* \subseteq S_B^* = S$. ∎

The analogue of Lemma 1 for seminormalization can be established by appealing to [5]. Moreover, if $A \subseteq B$ are integral domains, then $A_B^+ = A^+ \cap B$. Indeed, the construction of seminormalization in [5, Theorem 2.8] implies that A_B^+ is contained in the quotient field of A and also in $A^+ \cap B$; and the reverse inclusion holds since $A^+ \cap B$ is a subintegral extension of A.

LEMMA 2. *Let $A \subseteq B$ be integral domains such that A is seminormal in B. Let $m \geq 2$ be an integer. Suppose that $a \in A$ and $b \in B$ satisfy $ab, ab^m \in A$. Then $ab^i \in A$ for all $1 \leq i \leq m$.*

PROOF. Fix i such that $1 \leq i < m$. Since A is seminormal in B, it suffices to show $(ab^i)^N \in A$ for all sufficiently large integers N(*cf.* [2]). For any positive integer N, the division algorithm gives $iN = qm + r$ for suitable integers q, r such that $q \geq 0$ and $0 \leq r \leq m - 1$.

We have

$$(ab^i)^N = (ab^m)^q (ab)^r a^{N-q-r} \in A$$

if $N \geq q + r = \frac{iN-r}{m} + r$; that is, if $N \geq \frac{r(m-1)}{m-i}$. Since $(m-1)^2 \geq r(m-1) \geq \frac{r(m-1)}{m-i}$, this holds for all $N \geq (m-1)^2$. ∎

THEOREM 3. *(i) Let $R \subseteq T$ be integral domains. Then $R[[\mathbf{X}]]^*_{T[[\mathbf{X}]]} \subseteq R^*_T[[\mathbf{X}]]$.*

(ii) If $R \subseteq T$ are integral domains such that R is weakly normal in T, then $R[[\mathbf{X}]]$ is weakly normal in $T[[\mathbf{X}]]$.

PROOF. We first prove (ii). By induction, we may assume that the set $\mathbf{X}$ contains just one indeterminate X.

We shall apply the criterion in [7, Theorem 1]. First, since R is weakly normal in T, R is seminormal in T. Thus, $R = R_T^+ = R^+ \cap T$. By [3],

$$R[[\mathbf{X}]]^+_{T[[\mathbf{X}]]} = R[[\mathbf{X}]]^+ \cap T[[\mathbf{X}]] \subseteq R^+[[\mathbf{X}]] \cap T[[\mathbf{X}]] = (R^+ \cap T)[[\mathbf{X}]] = R[[\mathbf{X}]].$$

Thus, $R[[\mathbf{X}]]$ is seminormal in $T[[\mathbf{X}]]$. Next suppose that an element f of $T[[X]]$ satisfies $f^p, pf \in R[[X]]$ for some prime p. Write $f(X) = \sum_{i=0}^{\infty} b_i X^i \in T[[X]]$. It suffices to prove that $b_i \in R$ for each i.

The conditions on f lead to $b_0^p, pb_0 \in R$, and so $b_0 \in R$ since R is weakly normal in T. Set $g = f - b_0$. Then $pg = pf - pb_0 \in R[[X]]$. It suffices to show that $g^p \in R[[X]]$; for then, by replacing f with (g/X) in the above argument, we have $b_1 \in R$, and the proof concludes by induction.

Since $f^p, pf \in R[[X]]$, applying Lemma 2 with $A = R[[X]]$ and $B = T[[X]]$, we obtain $pf^i \in R[[X]]$ for all $1 \leq i \leq p-1$. Moreover,

$$g^p = (f - b_0)^p = \sum_{i=0}^{p} \binom{p}{i} f^i (-b_0)^{p-i}.$$

https://doi.org/10.4153/CMB-1995-062-0 Published online by Cambridge University Press

As $\binom{p}{i}$ is an integral multiple of p for $1 \leq i < p$ and $f^p \in R[[X]]$, we conclude that $g^p \in R[[X]]$, and the proof of (ii) is completed.

To prove (i), note that by (ii), $R^*_T[[\mathbf{X}]]$ is weakly normal in $T[[\mathbf{X}]]$. Since $R[[\mathbf{X}]] \subseteq R^*_T[[\mathbf{X}]]$, we have $R[[\mathbf{X}]]^*_{T[[\mathbf{X}]]} \subseteq R^*_T[[\mathbf{X}]]$ by Lemma 1(iv). ∎

For a domain R, we let R' and R^c denote the integral closure and the complete integral closure, respectively, of R in its quotient field.

COROLLARY 4. *(i)* $R[[\mathbf{X}]]^* \subseteq R^*[[\mathbf{X}]]$ *for each integral domain R.*

(ii) If R is a weakly normal integral domain, then $R[[\mathbf{X}]]$ *is also weakly normal.*

PROOF. Let K be the quotient field of R. Since $K[[\mathbf{X}]]$ is normal, we have $R[[\mathbf{X}]]' \subseteq K[[\mathbf{X}]]$. By Lemma 1(ii),

$$R[[\mathbf{X}]]^* = R[[\mathbf{X}]]^*_{R[[\mathbf{X}]]'} \subseteq R[[\mathbf{X}]]^*_{K[[\mathbf{X}]]}.$$

Thus, the assertions follow by taking $T = K$ in Theorem 3. ∎

REMARK 5. In Corollary 4(i), we generally do not have equality: $R[[\mathbf{X}]]^*$ and $R^*[[\mathbf{X}]]$ may not even have the same quotient field. Moreover, even if these domains have the same quotient field, the equality might fail. All this is possible even if $R^* = R^c$ is a factorial domain (and so completely integrally closed) and $R[[\mathbf{X}]]^* = R[[\mathbf{X}]]^+ = R[[\mathbf{X}]]'$.

For example, let A be a factorial domain containing a field of characteristic zero and let p be either 0 or a prime element of A. Let $\mathbf{Y} = \left(Y_n \mid n \geq 1\right)$ be an infinite sequence of indeterminates over A. Let I be the ideal of $A[\mathbf{Y}]$ generated by $\{pY_n, Y_n^2, Y_n^3 \mid n \geq 1\}$. Set

$$R = A + I.$$

We claim the following.

(i) $R^* = R^+ = R' = R^c = A[\mathbf{Y}]$ is a factorial domain.

(ii) $R[[\mathbf{X}]]^* = R[[\mathbf{X}]]^+ = R[[\mathbf{X}]]' = \bigcup_{n=1}^{\infty} R[Y_1, \ldots, Y_n][[\mathbf{X}]]$.

(iii) $R^*[[\mathbf{X}]]$ and $R[[\mathbf{X}]]^*$ have the same quotient field $\Leftrightarrow p \neq 0$.

(iv) For $X \in \mathbf{X}$, we have $\sum_{n=1}^{\infty} Y_n X^n \in R^*[[\mathbf{X}]] \setminus R[[\mathbf{X}]]^*$.

PROOF. (i) This is straightforward.

Set $T = \bigcup_{n=1}^{\infty} R[Y_1, \ldots, Y_n][[\mathbf{X}]]$.

(ii) Since $Y_n^i \in R$ for all $n \geq 1$ and $i \geq 2$, we have $T \subseteq R[[\mathbf{X}]]^+$. Since $R[[\mathbf{X}]] \subseteq T \subseteq R[[\mathbf{X}]]^+$, it is enough to show that T is normal. Let F be an element in the quotient field of T which is integral over T:

$$F^m + t_{m-1}F^{m-1} + \cdots + t_0 = 0,$$

where $m \geq 1$ and $t_0, \ldots, t_{m-1}$ are elements of T. Since A is factorial, the domain $A[\mathbf{Y}][[\mathbf{X}]]$ is completely integrally closed; so, $F \in A[\mathbf{Y}][[\mathbf{X}]]$. Assume that $F \notin T$ and we will get a contradiction. Set $B = A/Ap$ and let f be the canonical image of F in $B[\mathbf{Y}][[\mathbf{X}]]$. We may assume that no Y_k^i occurs in f with $i \geq 2$. Since there are just finitely many Y_k's dividing

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f in $B[\mathbf{Y}][[\mathbf{X}]]$, we see that for infinitely many positive integers k, the element f is of the form $f = Y_k g_k + h_k$, where g_k and h_k are elements in $B[\mathbf{Y}][[\mathbf{X}]]$ not involving Y_k. Since $Y_k^i A[[\mathbf{Y}]][[\mathbf{X}]] \in R$ for all $k \geq 1$ and $i \geq 2$, we obtain that for k as above and $t_m = 1$, the element $Y_k \sum_{i=1}^{m} it_i g_k h_k^{i-1}$ is in the ring T_0, which is defined analogously to T, with A replaced by B and p replaced by 0. It follows that $\sum_{i=1}^{m-1} it_i h_k^{i-1} = 0$. Since char $T = 0$, we obtain that all such h_k are roots of the same nonzero (monic) polynomial over T_0. It follows that there is an element $h \in B[\mathbf{Y}][[\mathbf{X}]]$ such that $h_k = h$ for infinitely many k's as above. Hence $f - h$ is divisible in $B[\mathbf{Y}][[\mathbf{X}]]$ by infinitely many Y_k's, a contradiction.

(iii) $\Rightarrow$: Assume that $p = 0$, but $t := \sum_{n=1}^{\infty} Y_n X^n$ belongs to the quotient field of $R[[\mathbf{X}]]$ for some $X \in \mathbf{X}$. Thus, there is a nonzero element $g \in R[[\mathbf{X}]]$ such that $gt \in R[[\mathbf{X}]]$. There is an integer k such that Y_k does not divide g in $A[\mathbf{Y}][[\mathbf{X}]]$. Since $gt \in R[[\mathbf{X}]]$, we obtain that the only powers of Y_k that can occur in gt are ≥ 2, a contradiction.

$\Leftarrow$: Indeed, $pR^*[[\mathbf{X}]] \subseteq R[[\mathbf{X}]]$ since $pA[\mathbf{Y}] \subseteq R$.

(iv) Replacing A by A/Ap, we may assume that $p = 0$, since the assertion was already proved above in this case without using the assumption that A is factorial.

This finishes the proof of our claims.

Explicitly, let k be a field of characteristic zero; and set

$R_1 = k + (\{Y_n^2, Y_n^3 \mid n \geq 1\})k[\mathbf{Y}]$ if $p = 0$ and

$R_2 = k[Z] + (\{ZY_n, Y_n^2, Y_n^3 \mid n \geq 1\})k[Z, \mathbf{Y}]$ if $p \neq 0$

(Here, $c = Z$ is an indeterminate over $k[\mathbf{Y}]$.)

Note that in the proof of Remark 5, since $Y_n^i A[[\mathbf{Y}]] \subseteq R$ for all $n \geq 1$ and $i \geq 2$, we have

$$R[[\mathbf{X}]]^* = \bigcup_{n=1}^{\infty} R[Y_1, \ldots, Y_n][[\mathbf{X}]] = R[[\mathbf{X}]][\mathbf{Y}].$$

COROLLARY 6. *(i) Let $R \subseteq T$ be integral domains. Then* $R[\mathbf{X}]^*_{T[\mathbf{X}]} = R^*_T[\mathbf{X}]$.

(ii) If $R \subseteq T$ are integral domains such that R is weakly normal in T, then $R[\mathbf{X}]$ is weakly normal in $T[\mathbf{X}]$.

(iii) $R[\mathbf{X}]^* = R^*[\mathbf{X}]$ for each integral domain R.

(iv) If R is a weakly normal integral domain, then $R[\mathbf{X}]$ is weakly normal.

PROOF. (i) By Lemma 1 and Theorem 3,

$$R[\mathbf{X}]^*_{T[\mathbf{X}]} = R[\mathbf{X}]^*_{T[[\mathbf{X}]]} \cap T[\mathbf{X}] \subseteq R[[\mathbf{X}]]^*_{T[[\mathbf{X}]]} \cap T[\mathbf{X}] \subseteq R^*_T[[\mathbf{X}]] \cap T[\mathbf{X}] = R^*_T[\mathbf{X}].$$

Thus $R[\mathbf{X}]^*_{T[\mathbf{X}]} \subseteq R^*_T[\mathbf{X}]$. By Lemma 1(ii), $R^*_T \subseteq R[\mathbf{X}]^*_{T[\mathbf{X}]}$, so (i) holds.

Part (ii) follows from (i).

For (iii), note that if K denotes the quotient field of R, then $R[\mathbf{X}]^* = R[\mathbf{X}]^*_{K[\mathbf{X}]}$ since $K[\mathbf{X}]$ is normal. Finally, (iv) follows from (iii). ∎

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https://doi.org/10.4153/CMB-1995-062-0 Published online by Cambridge University Press