PERTURBATIONS DUE TO THE ASTEROID BELT

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ABSTRACT: A new expansion for the gravitational potential of the asteroid belt is derived in this paper on the basis of binomial expansions. Its advantages are: (1)the unified form both for the inner and for the outer regions of the belt; (2)suitability for discussing the motions of the celestial bodies with perihelions within but apohelions beyond the belt; (3)rapidity of convergence.

The perturbations due to the asteroid belt are studied by using our expansion.

1. INTRODUCTION

According to the present knowledge about the solar system, the total mass of the asteroid belt is believed to be about o.1% of that of the Earth and, generally speaking, there is no need to consider its influence when the motions of celestial bodies in the solar system are studied. However, with the development of planetary exploration and the increase in the accuracy of observations, the perturbations due to the asteroid belt have become a factor not to be ignored. The effect of the asteroid belt on the orbit of Mars has been studied by A.P.Mayo,[1] who obtained the perturbations of the order of several kms within 700 days. They are big enough to be detected by modern observational techniques.

Similarly to Liu et al[3] and Mayo,[1] who both studied the influence of the asteroid belt on Mars, Plakhov[2] discussed the effect of Saturn's ring on the satellite orbits about the planet. In these studies, the disturbing function is expanded into a Laurent series in the heliocentric radius r of the disturbed body. The forms of the expansion at radii within the ring and at those beyond it are quite different. In fact, the disturbing function has the form of a power series in r in the first case, while it can only be expanded into a power series in 1/r in the latter case. When r is close to the inner or outer radius of the ring, the convergence of the expansion will be broken. The most serious defect of the expansion is that it cannot be used to discuss the motions in those orbits with

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perihelions within but aphelions beyond the ring. But, as we know, that is precisely the form of motion in the case of the spacecraft exploring the outer solar system.

To avoid these drawbacks, another expansion of the disturbing function is provided in this paper on the basis of binomial expansion. Its advantages are: (1)the same form both for the inner and outer regions of the belt; (2)suitability for studying the motions of the celestial bodies with perihelions within but apohelions beyond the asteroid belt, such as Beira (1474) and Ganymed (1036); (3)rapidity of convergence.

Section 2 deals with the expansion of the disturbing function. Section 3 is devoted to decomposition of the disturbing function. In section 4 we will derive the short-period and secular (together with the long-period) perturbations respectively, being confined to the first nine terms of the expansion. Finally, some numerical results are given in the last section.

2. EXPANSION OF THE DISTURBING FUNCTION

For a point $P(r', \phi', \lambda')$, lying outside a belt with constant density and arbitrary thickness, the disturbing function by the belt can be written as

$$U = k^2 \rho \iiint \frac{1}{\Delta} r'^2 \cos\varphi' \, dr' d\varphi' d\lambda' . \tag{1}$$

Owing to the axisymmetry of the belt, the indirect part of the disturbing function vanishes.

According to the binomial formula, $1/\Delta$ can be expressed as

$$\frac{1}{\Delta} = (r^2 + r'^2 - 2rr'\cos H)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n (n!)^2} (r+r')^{-(2n+1)} r^n r'^n (1+\cos H)^n . \tag{2}$$

By introducing a constant $C \geqslant (r + r')$, $(r + r')^{-(2n+1)}$ can be expanded as

$$(r+r')^{-(2n+1)} = C^{-(2n+1)} \sum_{m=0}^{\infty} \frac{(2n+1)_m}{m!} [1-(r+r')/C]^m . \tag{3}$$

In fact, for any concrete problem only a finite number of terms in (3) are needed. The number M is determined according to the magnitude of $1-(r+r')/C$ and the accuracy desired. On the other hand, $(1+\cos H)$ can be expanded according to the cosine law in spherical trigonometry and the addition theorem for the Legendre polynomials, and then, all the odd zonal harmonics and the tesseral harmonics will vanish after integration because of the belt symmetry.

After some calculations and arrangements we obtain finally

$$U = \varepsilon \sum_{N=0}^{\infty} \sum_{q=0}^{[N/2]} (r/C)^N P_{2q}(\sin\varphi) K_{Nq} , \tag{4}$$

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where

$$\varepsilon = 4\pi k^2 \rho A^3 / C\,, \tag{5}$$

$$K_{Nq} = \frac{4q+1}{2^{2q}} \sum_{n=N_1}^{N} \sum_{m=N-n}^{M} \sum_{p=N-n}^{m} \frac{(-1)^p (2n+m)!}{(N-n)!(n-2q)!(n+2q+1)!(m-p)!(p-N+n)!} \times$$

$$\times \left(\frac{A}{C}\right)^{2n+p-N} K^*_{2n+p-N,q}\,, \tag{6}$$

$$N_1 = \max(2q, N-M)\,, \tag{7}$$

$$K^*_{\ell,q} = \sum_{j=0}^{q} (-1)^{q-j} \frac{(2q+2j)!}{(q+j)!(q-j)!(2j+1)!} \cdot \left(\frac{h}{2A}\right)^{2j+1} G_{\ell,j}\,, \tag{8}$$

$$G_{\ell,j} = \begin{cases} -\ln(B/A)\,, & \ell-2j+2=0\,, \\ \dfrac{1-(B/A)^{\ell-2j+2}}{\ell-2j+2}\,, & \ell-2j+2 \neq 0\,, \end{cases} \tag{9}$$

h, A, B denote the thickness, outer radius and inner radius of the belt, respectively.

The only limit during the process of expansion of the disturbing function is that P must be outside the belt.But there is no limit to its radius r. Therefore,(4) is a unified form of the disturbing function U for all the outside points, no matter $r>A$ or $r<B$.

To improve the convergence of (4), we arrange it as

$$U = \varepsilon \sum_{N=0}^{\infty} \sum_{\ell=0}^{[N/2]} F_{N\ell}\, (r/a)^N \sin^{2\ell} u\,, \tag{10}$$

where

$$F_{N\ell} = \sum_{q=\ell}^{[N/2]} \frac{(-1)^{q-\ell}}{2^{2q}} \binom{2q}{q+\ell} \binom{2q+2\ell}{2q} K_{Nq} \cdot \left(\frac{a}{C}\right)^N \sin^{2\ell} I\,, \tag{11}$$

$$\sin u = \sin\varphi / \sin I\,. \tag{12}$$

Expression (10) has a better feature of convergence than (4).

3. DECOMPOSITION OF THE DISTURBING FUNCTION

While using numerical method, we only need the partial derivatives $\partial U/\partial r$ and $\partial U/\partial \varphi$, both of which can easily be obtained from (10).

However, when the analytical method is used, the disturbing function must be expressed in terms of the orbital elements of the disturbed body and separated into secular, long- and short-periodic parts.

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For example, we take $(N,l) = (1,0),(2,0),(3,0),(4,0),(2,1),(3,1),(4,1),(5,1),(6,1)$ and use eccentric anomaly E as the independent variable, then U can be expressed in closed form

$$U = \varepsilon \sum_{i=0}^{6} (-1)^i \left[(P_i - Q_i \cos 2\omega) \cos iE + \sqrt{1-e^2}\, R_i \sin 2\omega \sin iE \right], \tag{13}$$

where

$$\begin{aligned} P_i &= e^i \left[\sum_{j=1}^{4} p_{j0}(e) F_{j0} + \sum_{k=2}^{6} p_{k1}(e) F_{k1} \right], \\ Q_i &= e^{|i-2|} \sum_{k=2}^{6} q_k(e) F_{k1}, \\ R_i &= e^{|i-2|} \sum_{k=2}^{6} r_k(e) F_{k1}, \end{aligned} \tag{14}$$

$p_{j0}(e)$, $p_{k1}(e)$, $q_k(e)$, $r_k(e)$ being polynomials in e.

Now, the Hamiltonian is

$$H = H_0 + \varepsilon H_1, \tag{15}$$

where

$$H_0 = \mu^2/2L^2, \qquad H_1 = U/\varepsilon. \tag{16}$$

The secular part of H_1 is given by

$$\langle H_1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} H_1 \, dl = \left(P_0 + \frac{e}{2} P_1\right) - \left(Q_0 + \frac{e}{2} Q_1\right) \cos 2\omega, \tag{17}$$

and the corresponding first-order generating function by

$$S_1 = \frac{L^3}{\mu^2} \int (H_1 - \langle H_1 \rangle) \, dl = \frac{L^3}{\mu^2} S_1^*, \tag{18}$$

$$\begin{aligned} S_1^* = {} & \frac{e^2}{2}(P_1 - Q_1 \cos 2\omega) \sin E + \sum_{i=1}^{7} \frac{(-1)^i}{i} \left[\frac{e}{2}(P_{i-1} + P_{i+1}) + P_i \right] \sin iE - \\ & - \cos 2\omega \sum_{i=1}^{7} \frac{(-1)^i}{i} \left[\frac{e}{2}(Q_{i-1} + Q_{i+1}) + Q_i \right] \sin iE - \\ & - \sqrt{1-e^2} \sin 2\omega \sum_{i=1}^{7} \frac{(-1)^i}{i} \left[\frac{e}{2}(R_{i-1} + R_{i+1}) + R_i \right] \cos iE, \\ & (P_i = Q_i = R_i \text{ if } i \bar{\in} [1,6]). \end{aligned} \tag{19}$$

Having eliminated the short-periodic terms, the new Hamiltonian can be written as

$$H^* = H_0 + \varepsilon \langle H_1 \rangle + O(\varepsilon^2). \tag{20}$$

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4. PERTURBATIONS

(a) Short-periodic perturbations

The short-periodic perturbations of the orbital elements can be obtained immediately from the generating function S_1. But in many cases the perturbations of the coordinates are required.

Let δr, $r\delta u$ and δz denote the components of the displacement of the celestial body due to the belt along the directions of radial, cross and perpendicular to the orbital plane, respectively. Then

$$\delta r = \varepsilon \frac{a^3}{\mu r}\Big[-e\sin E\,\Big(3S_1^* + 2a\frac{\partial S_1^*}{\partial a} + \frac{1-e^2}{e}\Big) + \frac{\sqrt{1-e^2}}{e}(e-\cos E)\Big(\sqrt{1-e^2}\,\widetilde{H}_1 - \frac{\partial S_1^*}{\partial \omega}\Big) + 2\Big(\frac{r}{a}\Big)^2 \widetilde{H}_1\Big], \tag{21}$$

$$r\delta u = \varepsilon \frac{a\sqrt{1-e^2}}{\mu e}\Big[\Big(r - \frac{a^2(1-e^2)}{r}\Big)\frac{\partial S_1^*}{\partial e} + a\sin E\frac{\partial S_1^*}{\partial E} - \frac{a^2 e}{r}\Big(3S_1^* + 2a\frac{\partial S_1^*}{\partial a}\Big) + a\sin E\,(2+e\cos f)\Big(\widetilde{H}_1 - \frac{1}{\sqrt{1-e^2}}\frac{\partial S_1^*}{\partial \omega}\Big)\Big], \tag{22}$$

$$\delta z = \varepsilon \frac{a r}{\mu\sqrt{1-e^2}}\Big[\sin(f+\omega)\operatorname{ctg} I\,\frac{\partial S_1^*}{\partial \omega} - \cos(f+\omega)\sin 2I\,\frac{\partial S_1^*}{\partial(\sin^2 I)}\Big], \tag{23}$$

where

$$\widetilde{H}_1 = H_1 - \langle H_1\rangle . \tag{24}$$

(b) Secular and long-periodic perturbations

Let $\bar{a}$, $\bar{e}$, $\bar{I}$, $\bar{\omega}$, $\bar{\Omega}$, $\bar{M}$ be the averaged keplerian elements. As H^* is independent of M and Ω, we have the integrals immediately

$$\bar{a} = \text{const}, \qquad \cos^2\bar{I}\,(1-\bar{e}^2) = \text{const}. \tag{25}$$

The second integral indicates that $\bar{I}$ well increases as $\bar{e}$ decreases, and vice versa.

If at the beginning we have

$$\bar{e} = \bar{I} = 0, \tag{26}$$

then the state will conserve forever.

The variations of the other elements are governed by the averaged equations of motion, especially we have

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$$\frac{d\bar{e}}{dt} = -\varepsilon\bar{e}\sqrt{\frac{1-\bar{e}^2}{\mu\bar{a}}}\left(\sum_{k=2}^{6} C_k(\bar{e})F_{k1}\right)\sin 2\bar{\omega}, \tag{27}$$

$$\frac{d\bar{\omega}}{dt} = \frac{\varepsilon}{\sqrt{\mu\bar{a}(1-\bar{e}^2)}}\left(\bar{A} - \bar{B}\cos 2\bar{\omega}\right), \tag{28}$$

$$\bar{A} = \sum_{j=1}^{4}\bar{A}_{j0}(\bar{e})F_{j0} + \sum_{k=2}^{6}\left[\bar{A}_{k1}(\bar{e}) - \frac{\bar{A}^*_{k1}(\bar{e})}{\sin^2\bar{I}}\right]F_{k1}, \tag{29}$$

$$\bar{B} = \sum_{k=2}^{6}\left[\bar{B}_k(\bar{e}) - \frac{\bar{e}^2}{\sin^2\bar{I}}\bar{B}^*_k(\bar{e})\right]F_{k1}, \tag{30}$$

where $C_k(\bar{e})$, $\bar{A}_{j0}(\bar{e})$, $\bar{A}_{k1}(\bar{e})$, $\bar{A}^*_{k1}(\bar{e})$, $\bar{B}_k(\bar{e})$, $\bar{B}^*_k(\bar{e})$ are all polynomials in e.

Finally we conclude that (1) if $\bar{\omega} = \frac{1}{2}n\pi$, $\bar{A}=(-1)^n\bar{B}$, then equations (27),(28) have stationary solution; (2) if $\bar{A}=0$, $\bar{\omega}$ will oscillate around a libration point.

5. NUMERICAL RESULTS

Assuming

$a = 2.7$, $e = 0.5$, $I = 26^\circ.5$, $\omega = 5^\circ$,

$A = 4$, $B = 2$, $h = 2/3$, $m = 0.001m_\oplus$,

we have derived the following numerical results by taking C = 11.23:

$\varepsilon = 2.53\times10^{-12}$;

Coefficients $F_{N\ell}$ in the expansion (10) (see Sec. 2):

N	l	F $(\times10^{-2})$
1	0	-1.06
2	0	0.26
3	0	-0.83
4	0	0.23
5	0	---
6	0	---

N	l	F $(\times10^{-2})$
2	1	-0.72
3	1	0.45
4	1	-0.23
5	1	0.14
6	1	-0.05

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Short-periodic perturbations (km):

E°	δr	$r\delta u$	δz
0	-0.10	0.28	0.034
10	-0.08	0.14	0.013
20	-0.11	0.00	-0.007
30	-0.15	-0.11	-0.024
40	-0.21	-0.20	-0.037
50	-0.26	-0.27	-0.044
60	-0.31	-0.32	-0.045
70	-0.34	-0.35	-0.042
80	-0.36	-0.37	-0.034
90	-0.37	-0.39	-0.026
100	-0.38	-0.39	-0.017
110	-0.38	-0.39	-0.010
120	-0.38	-0.38	-0.007
130	-0.38	-0.35	-0.009
140	-0.38	-0.32	-0.016
150	-0.38	-0.27	-0.028
160	-0.39	-0.20	-0.044
170	-0.40	-0.13	-0.062

E°	δr	$r\delta u$	δz
180	-0.41	-0.05	-0.080
190	-0.42	0.03	-0.096
200	-0.43	0.11	-0.108
210	-0.45	0.19	-0.114
220	-0.46	0.26	-0.113
230	-0.48	0.32	-0.104
240	-0.50	0.38	-0.089
250	-0.53	0.43	-0.067
260	-0.55	0.47	-0.040
270	-0.56	0.51	-0.012
280	-0.56	0.54	0.017
290	-0.55	0.57	0.042
300	-0.53	0.60	0.063
310	-0.48	0.61	0.076
320	-0.41	0.61	0.082
330	-0.32	0.58	0.079
340	-0.23	0.51	0.069
350	-0.15	0.41	0.053

Long-periodic perturbations:

$$d\bar{e}/dt = 1.45\times10^{-13}\sin2\bar{\omega}\ (\text{day}^{-1}),$$

$$d\bar{\omega}/dt = -(7.60 + 3.16\cos2\bar{\omega})\times10^{-8}\ (''/\text{day}).$$

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