BULL. AUSTRAL. MATH. SOC.
VOL. 36 (1987) 153-160

20E26, 20E06

A NOTE ON GROUPS WITH SEPARABLE FINITELY GENERATED SUBGROUPS

R.G. BURNS, A. KARRASS AND D. SOLITAR

An example is given of an infinite cyclic extension of a free group of finite rank in which not every finitely generated subgroup is finitely separable. This answers negatively the question of Peter Scott as to whether in all finitely generated 3-manifold groups the finitely generated subgroups are finitely separable. In the positive direction it is shown that in knot groups and one-relator groups with centre, the finitely generated normal subgroups are finitely separable.

1.

Following Mal'cev we say that a group G has *finitely separable finitely generated subgroups* (or, briefly, is *subgroup separable*) if every finitely generated subgroup is the intersection of the subgroups of finite index containing it. (An equivalent condition is that the finitely generated subgroups be closed in the profinite topology on G.) References to results concerning this property may be found in, for example [1], [3].

Received 19 September, 1986.

In an invited lecture at the 1983 meeting of the Australian Mathematical Society in Brisbane, Peter Scott asked if infinite cyclic extensions of finitely generated free groups are necessarily subgroup separable. The following example shows that the answer to this is in the negative. (See Section 2 for the verification.)

Example. The group K with presentation

(1) $$K = \langle y, \alpha, \beta \mid y^{-1}\alpha y = \alpha\beta \,,\; y^{-1}\beta y = \beta\rangle$$

is not subgroup separable.

Now it is not difficult to see that K is the fundamental group of a punctured-torus bundle over the circle, namely that obtained by identifying the lid and base of $[0,1] \times T$, where T is the punctured torus, by means of a homeomorphism inducing the automorphism given by $\alpha \to \alpha\beta$, $\beta \to \beta$, of the (free) fundamental group $\langle\alpha,\beta\rangle$ of T. Hence our example also provides a negative answer to the question of main interest to Scott, namely whether the (finitely generated) fundamental groups of 3-manifolds are subgroup separable.

COROLLARY. *Not all finitely generated 3-manifold groups are subgroup separable.*

On the other hand it is not difficult to verify that K is not a knot group (since $K/K' \cong \mathbb{Z} \oplus \mathbb{Z}$), and in fact not even the group of a link. Since one of the simplest knot groups whose subgroup separability is unknown is the figure-eight knot group

$$\langle y, \alpha, \beta \mid y^{-1}\alpha y = \beta\alpha \,,\; y^{-1}\beta y = \beta\alpha\beta\rangle$$

it is therefore natural to ask the following

Question. Are knot groups, in particular the figure-eight knot group, subgroup separable?

If a group is not subgroup separable one might still ask if at least certain of its subgroups can be separated, in particular its cyclic subgroups (this was considered in [1]), or its finitely generated normal subgroups. In this direction we note the following fact, essentially a consequence of results in Bieri [2]:

PROPOSITION. *In every knot group and one-relator group with centre the finitely generated normal subgroups are finitely separable.*

(One may ask whether in fact in every finitely generated 3-manifold group and one-relator group the (nontrivial) finitely generated normal subgroups are separable. Note that in, for instance, free products and Fuchsian groups such subgroups are trivially separable, being of finite index.)

Proof. According to Bieri [2, Corollaries 8.6, 8.7], if G is a finitely generated group of co(homological)-dimension ≤ 2, and N is a non-trivial finitely presented normal subgroup, then the quotient G/N is a finite extension of a free group. Hence G/N is certainly residually finite, or, equivalently, N is finitely separable in G.

Now since knot groups have codimension ≤ 2 (Papakyriakopoulos [7]), and since their finitely generated subgroups are finitely presented (Scott [8] has shown that all 3-manifold groups have this property), the proposition follows for these. (The separability of the identity subgroup is equivalent to residual finiteness, which has been established for knot groups on the basis of Thurston's work.)

Pietrowski [6] has shown that one-relator groups with centre are treed HNN groups with infinite cyclic vertex groups. Hence for one-relator groups with centre the proposition is a consequence of the following more general

LEMMA. *A finitely generated treed HNN group G (that is fundamental group of a graph of groups) with free vertex groups and cyclic amalgamated and associated subgroups, has separable non-trivial, finitely generated normal subgroups.*

This is in turn a consequence of the fact that every finitely generated subgroup of such a group G is finitely presented (see [4] and [5]), and the result in Bieri [2, Propositions 6.1, 6.2] that the class of groups of codimension ≤ 2 is closed under forming amalgamated products and HNN extensions with free amalgamated and associated subgroups. (Here the separability of the identity subgroup, or, equivalently, the residual finiteness of one-relator groups with centre, follows from Murasugi's result that such groups are infinite cyclic extensions of finitely generated free groups.)

2.

Verification of the example. Note first that, eliminating β from the presentation (1), we have, since $\beta = \alpha^{-1}y^{-1}\alpha y$,

$$K = \langle y,\alpha \mid y^{-1}(\alpha^{-1}y^{-1}\alpha y)y = \alpha^{-1}y^{-1}\alpha y\rangle$$

$$= \langle y,\alpha \mid y^{-1}\alpha^{-1}y^{-1}\alpha y\alpha^{-1}y\alpha = 1\rangle$$

$$= \langle y,\alpha \mid [y,\alpha^{-1}y\alpha] = 1\rangle \ .$$

Thus putting $x = \alpha^{-1}$, $a = y$, we have

$$K = \langle x,a \mid [xax^{-1},a] = 1\rangle \ ,$$

and it is in this form that we shall work with K .

Write

$$A = \langle a^G\rangle = \langle a_i \mid i \in \mathbb{Z}\rangle \ , \text{ where } \ a_i = x^i a x^{-i} \ .$$

Then K is an extension of A by the infinite cycle $\langle x\rangle$, with the action of x on A given by $xa_ix^{-1} = a_{i+1}$. We shall show that the element $[a_0,a_2] = a_0^{-1}a_2^{-1}a_0a_2$ of A is not finitely separable in K from the subgroup $H = \langle x,a_0a_1^{-2}\rangle$.

We first verify that $[a_0,a_2] \notin H$. To begin with observe that A has presentation

$$A = \langle a_i \mid [a_i,a_{i+1}] = 1 \ , \ i \in \mathbb{Z}\rangle \ ,$$

https://doi.org/10.1017/S0004972700026393 Published online by Cambridge University Press

so that it has the structure of an iterated amalgamated product:

$$(2) \qquad A = \cdots \underset{a_{-1}=a_{-1}}{*} \langle a_{-1},a_0\rangle \underset{a_0=a_0}{*} \langle a_0,a_1\rangle \underset{a_1=a_1}{*} \langle a_1,a_2\rangle \underset{a_2=a_2}{*} \cdots ,$$

with the amalgamations as indicated and with each vertex group $\langle a_i,a_{i+1}\rangle$ free abelian of rank 2. Now it is a well-known fact, following essentially from the normal form for the elements of an amalgamated product $X \underset{U}{*} Y$, that if $X_1 \le X$ and $Y_1 \le Y$ intersect the amalgamated subgroup U trivially, then the subgroup $\langle X_1,Y_1\rangle$ they together generate is equal to $X_1 * Y_1$, and, more to the point here, satisfies

$$\langle X_1,Y_1\rangle \cap X = X_1 \ , \ \langle X_1,Y_1\rangle \cap Y = Y_1$$

Applying this to the subgroup

$$H \cap A = \langle a_i a_{i+1}^{-2} \mid i \in \mathbb{Z} \rangle$$

of A, with

$$X = \langle a_i \mid i \le 0\rangle \ , \ Y = \langle a_i \mid i \ge 0\rangle \ , \ U = \langle a_0\rangle$$

we infer that

$$H \cap Y(=(H \cap A)\cap Y) = \langle a_i a_{i+1}^{-2} \mid i \ge 0\rangle \ ,$$

A further application, this time to the group $\langle a_i \mid i \ge 0\rangle$, with

$$X = \langle a_0,a_1,a_2\rangle \ , \ Y = \langle a_i \mid i \ge 2\rangle \ , \ U = \langle a_2\rangle \ ,$$

yields

$$H \cap \langle a_0,a_1,a_2\rangle = \langle a_0a_1^{-2},a_1a_2^{-2}\rangle \ .$$

Hence if $[a_0,a_2]$ were in H, we should have $[a_0,a_2] \in \langle a_0a_1^{-2},a_1a_2^{-2}\rangle$, or, since a_1 centralizes this latter group, $[a_0,a_2] \in \langle a_0,a_2^2\rangle$ which is

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easily seen to be impossible.

We now show that $[a_0,a_2]$ lies in every finite-index subgroup L of K, containing H. Let $C = \mathrm{core}_K L$, the intersection of all conjugates of L in K. Then C has finite index in K (since L does), and $C \trianglelefteq K$. It follows that $N = C \cap A$ is also normal in K, and has finite index in A. Clearly $(H \cap A)N$ is contained in L. We shall show that in fact

$$(H \cap A)N \trianglelefteq A . \tag{3}$$

From this and the fact that $a_i a_{i+1}^{-2} \in H \cap A$ for all $i \in \mathbb{Z}$, it will then follow that $A/(H \cap A)N$ is abelian (in fact cyclic), whence $[a_0,a_2] \in (H \cap A)N$.

As the first step towards establishing (3) we show that

$$a_0^{\varepsilon}(a_i a_{i+1}^{-2})a_0^{-\varepsilon} \in H \cap A \text{ for } i > 0 \ (\varepsilon = \pm 1) . \tag{4}$$

Now since $a_0a_1^{-2}, (a_1a_2^{-2})^2 \left(= a_1^2a_2^{-4} \right) \in H$ (using $[a_1,a_2] = 1$), we have

$$a_0a_1^{-2}a_1^2a_2^{-4} = a_0a_2^{-4} \in H \cap A ,$$

and then since $(a_2a_3^{-2})^4 = a_2^4a_3^{-8} \in H \cap A$, we get in turn $a_0a_3^{-8} \in H \cap A$. Proceeding inductively we obtain

$$a_0a_i^{-2^i} \in H , \ i > 0 , \tag{5}$$

and, similarly, since $a_ia_{i+1}^{-2} = a_{i+1}^{-2}a_i$, we have

$$a_i^{-2^i}a_0 \in H , \ i > 0 \tag{6}$$

Now for $i > 0$

$$a_0(a_ia_{i+1}^{-2})a_0^{-1} = (a_0a_i^{-2^i})(a_ia_{i+1}^{-2})(a_0a_i^{-2^i})^{-1}$$

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since a_i centralizes $\langle a_i, a_{i+1}\rangle$, whence from (5) we have $a_0(a_i a_{i+1}^{-2})a_0^{-1} \in H$. That $a_0^{-1}(a_i a_{i+1}^{-2})a_0 \in H$ follows similarly from (6).

The next, and final, step is to show that

$$a_0^{\varepsilon}(a_i a_{i+1}^{-2})a_0^{-\varepsilon} \in (H \cap A)N \quad \text{for all } i \ (\varepsilon = \pm 1) . \tag{7}$$

The desired conclusion (3) will then follow since $(H \cap A)N$ is normalized by $\langle x\rangle$, so that from (7) one obtains

$$a_r^{\varepsilon}(a_s a_{s+1}^{-2})a_r^{-\varepsilon} \in (H \cap A)N \quad \text{for all } r, s \ (\varepsilon = \pm 1) .$$

Consider the infinite subset $\{a_i \mid i \in \mathbb{Z}\}$ of A. Since N has finite index in A, certainly some two of the elements of this subset are in the same coset of N in A, so that $a_i a_j^{-1} \in N$ for some pair $i, j,\ i < j$. It follows by means of iterated conjugation by x that for some fixed $t > 0$,

$$a_{nt}a_{(n+1)t}^{-1} \in N , \quad \text{for all } n \in \mathbb{Z} ,$$

and thence by suitable multiplication of these elements that

$$a_{nt}a_{(n+k)t}^{-1} \in N , \quad \text{for all } n, k \in \mathbb{Z} , \tag{8}$$

Returning now to (7), consider for instance $a_0(a_i a_{i+1}^{-2})a_0^{-1}$, where now $i < 0$. By (8) there exists $j < i$ such that $a_j a_0^{-1} \in N$, so that $a_0(a_i a_{i+1}^{-2})a_0^{-1} \in (H \cap A)N$ if and only if

$$(a_j a_0^{-1})a_0(a_i a_{i+1}^{-2})a_0^{-1}(a_j a_0^{-1})^{-1} = a_j(a_i a_{i+1}^{-2})a_j^{-1} \in (H \cap A)N ,$$

or, conjugating by a suitable power of x, if and only if

$$a_0(a_{i-j}a_{i-j+1}^{-2})a_0^{-1} \in (H \cap A)N .$$

Now since $i - j > 0$, this element does in fact lie in $(H \cap A)N$ by (4). A very similar argument establishes (7) with $\varepsilon = -1$.

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Department of Mathematics,
York University,
North York,
Ontario, M3J 1P3
Canada.